θ

Actions under Presuppositions

Albert Visser

π

▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼

Actions under Presuppositions

Albert Visser

▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼▼

ABSTRACT: This paper consists of two parts: the first is contained in section 1. It reviews, perhaps too briefly, some basic philosophy on meaning, information, information state, information ordering and the like. In the remaining sections two interwoven problems are considered: the first is how to view update functions as partial states (or more generally partial 'actions'). Partial states (actions) are viewed as states (actions) under a presupposition. The second problem is how the merger of meanings interacts with the synchronic information ordering. We explore some consequences of the hypothesis that this interaction is described by a Residuation Lattice.

Es kommt mir darauf an zu zeigen, daß das Argument nicht mit zur Funktion gehört, sondern mit der Funktion zusammen ein vollständiges Ganzes bildet; denn die Funktion für sich allein is unvollständig, ergänzungsbedürftig oder ungesättigt zu nennen. Und dadurch unterscheiden sich die Funktionen von den Zahlen von Grund aus. (Frege[75], 21-22).

Partial actions and states, that is what this paper is about. A partial action is viewed as something unsaturated or etwas ergänzungsbedürftiges. We model the partial objects as certain partial update functions which can in their turn be represented by pairs of total objects. The first component of such a pair can be seen as a test: if you satisfy the test you can plug the hole. Consequently you can benefit from the resulting total object. This total object is given by the second component. As we shall see, this construction is quite similar to the familiar construction of the integers

1

from the natural numbers.

1   Pictures & Updates

This section is an attempt to place the paper in the somewhat broader, but hazy context of research in dynamic and discourse semantics. Some notations and basic notions are introduced.

1.1   Setting the stage: Consider two simple-minded pictures of *meaning* or, perhaps, *information content*. The first is the DRT-view (DRT = Discourse Representation Theory; it is primarily due to Heim and Kamp): a meaning is like a picture, is like a structured database, is like a mental state. These meanings are called DRS's (= Discourse Representation Structures). The second is the imagery of Update Semantics (primarily due to Gardenförs and Landman & Veltman): a meaning or information item is (or can be represented as) an update function of mental states. The purpose of this paper is to study the relationship between these pictures (or more accurately: certain aspects of this relationship).

Prima facie these views are quite different. The DRT-view provides static objects, while the essence of the Update picture is meaning-as-something-dynamic. Also there must be far more updates of mental states, than there are mental states. Our basic idea to resolve the tension between the two views is (i) to consider only a restricted class of update functions and (ii) to represent these functions as *partial states* or *states under a presupposition*. The original states or total states are embedded in a natural way among their partial brethren.

1.2   Monoids for Merging Meanings: Databases or pictures can be put together or merged. Update functions can be composed. In both cases there is a fundamental operation: the *merger* respectively *function composition*. These operations are associative. We stipulate the presence of an unit element *1* for these operations. The identity is the empty database respectively the identity function. We assume that 1 is a (total) state: the state of absolute ignorance or *tabula rasa*.

We use the expression *merger* for whatever basic associative function glues meanings together, thus viewing function composition as a special case of the merger.

We take the merger as in some sense the basic or fundamental operation on meanings. Other operations are either defined in terms of it or in some wider sense

derived from it. Of course, in the light of the generality of our present discussion, taking the merger as fundamental is only a *schematic* step. Yet it serves already to distinguish the present approach from Montagovian Semantics, where the basic operation is Function Application.

In view of the foregoing discussion we see that meanings form a *monoid* $\mathfrak{M} = \langle M,\bullet,1 \rangle$. We take the mental states to be a subset of the meanings. Thus we define: a *merge algebra* $\mathfrak{M}$ is a structure $\langle M,S,\bullet,1 \rangle$, where $1 \in S \subseteq M$ and where $\langle M,\bullet,1 \rangle$ is a monoid.

1.2.1 Conventions: We let x,y,z,... range over M and s,t,... over S.

We use postfix notation for function application. Our notation for function composition is in line with this convention: $xF \circ G := (xF)G$.

If F and G are partial functions we write e.g. wF=uG for: either sF and uG are both defined and have the same value, or both are undefined. We write $sF \cong uG$ for sF and uG are both defined and their values are equal. $\bigcirc$

We associate update functions to our algebra in the obvious way. An update function on a merge algebra $\mathfrak{M}$ is a partial function from S to S. To each x in M we associate an update function Φ_x as follows:

$s\Phi_x := s \bullet x$ if $s \bullet x \in S$, $s\Phi_x$ is undefined otherwise.

We say that $\mathfrak{M}$ is an *update algebra* if the map Φ with $\Phi(x)=\Phi_x$ is a homomorphism from $\langle M,\bullet,1 \rangle$ to $\langle$ the update functions on M,o,ID$\rangle$, where ID is the identity function on S.

Under what conditions is a merge algebra an update algebra? The answer is that the algebra has to satisfy the OTAT-principle.

1.2.2 The OTAT-Principle: a merge algebra $\mathfrak{M}$ satisfies OTAT if:

for all $x,y \in M$: $x \bullet y \in S \Rightarrow x \in S$.

OTAT means: once a thief, always a thief. If something fails to be a state, then it will never become a state whatever happens afterwards.

We have also a local version of OTAT. An element y of M has the OTAT property iff for all $x \in M$: $x \bullet y \in S \Rightarrow x \in S$. $\bigcirc$

We have the following theorem.

1.2.3 Theorem: $\mathfrak{M}$ is an update algebra iff $\mathfrak{M}$ satisfies OTAT.

The theorem is an immediate corollary of:

1.2.4 Lemma: Consider $y \in M$

$$\forall s \in S \, \forall x \in M \; s\Phi_x\Phi_y = s\Phi_{x \cdot y} \; \Leftrightarrow \; y \text{ has the OTAT property.}$$

Proof of 1.2.4: "$\Rightarrow$" Suppose $\forall s \in S \, \forall x \in M \; s\Phi_x\Phi_y = s\Phi_{x \cdot y}$. Consider any x in M and suppose $x \cdot y \in S$. It follows that $1\Phi_{x \cdot y}$ is defined and hence (ex hypothesi) so is $1\Phi_x\Phi_y$. Thus $1\Phi_x$ must be defined, which means that $x \in S$.

"$\Leftarrow$" Suppose y has the OTAT-property. Consider any s and y. If $s\Phi_x\Phi_y$ is defined then $s\Phi_x\Phi_y = s \cdot x \cdot y$ and $s \cdot x \cdot y \in S$. Hence $s\Phi_{x \cdot y}$ is defined and $s\Phi_{x \cdot y} = s \cdot x \cdot y$. Conversely suppose $s\Phi_{x \cdot y}$ is defined. In this case $s\Phi_{x \cdot y} = s \cdot x \cdot y$ and $s \cdot x \cdot y \in S$. By the OTAT property we find that $s \cdot x \in S$ and hence $s\Phi_x$ is defined. Since $s\Phi_x = s \cdot x$ also $s\Phi_x\Phi_y$ is defined and $s\Phi_x\Phi_y = s \cdot x \cdot y$. $\square$

The OTAT-principle suggests that the partiality of the non-states is something backwards looking, a kind of lack on the input side rather than at the output side. In other words the OTAT principle suggests that meanings have *a presuppositional structure*.

1.3   The true and the proper nature of states: What is a state? The connotations of the word *state* suggest that a state is something static. Thus the notion of state would have its proper place in the static-dynamic opposition. A state would be something like a *test* or a *condition*.

I disagree with this idea. First etymology can be misleading. Think of e.g. *state of motion* versus *state of rest* to illustrate that even if a state is a something-a-thing-is-in-at-a-particular-moment, a state is not necessarily something that has no 'active' properties. A state may contain the germs of the next state and may even be said to be one of the causes of the next state. Secondly the OTAT-principle suggests that states find their natural home within the saturated-unsaturated distinction. A state is something that is saturated towards the past, i.o.w. something that carries no

4

presupposition.

Let's consider some examples. In our first batch of examples we choose to ignore all possible sources of unsaturatedness, other than those arising from anaphoric phenomena (at the surface level).

i)   *A man comes in.*

ii)   *He smiles.*

iii)   *All men smile.*

iv)   *A men comes in. He smiles at her.*

I would say that (the meaning of / the content of) (i) is a state. No referents need to be supplied from previous discourse.

Of course calling (i) a state, carries the suggestion that (i) could be the whole knowledge-content of an organism. I'm inclined to think that if *logical possibility* is intended here, this is true. On the other hand such radical claims -in spite of their exciting character- are not really at issue in the present discussion. We could easily stipulate that *state* here is intended as part or aspect of the total holistic (*pardonnez le mot*) state of an organism, that can be considered as standing on its own relative to a certain kind of analysis carrying its own degree of resolution (etc.).

Note that (i) is saturated when seen from the past, but not so when seen from the future -traveling for a short moment backwards in time-, since it exports a referent to later discourse. (i) is not a test or condition, since it exports a *new* referent to later discourse.

(ii) is not a state since it is unsaturated or, with Frege's beautiful expression, *Ergänzungsbedüftig* towards the past (and in this case the future too). On the other hand (ii) is static, since it neither creates nor destroys a referent. Thus we may say that (ii) is a test (for smiliness).

We leave it to the industrious reader to see that (iii) is both a state and a test and that (iv) is neither state nor test.

1.3.1 Excursion: Let's briefly glance at some possible paraphrases of (i)-(iv) in the language of DRT/DPL (DPL = Dynamic Predicate Logic, a variant of Predicate Logic introduced by Groenendijk & Stokhof):

pi) $\exists x.MAN(x).COME\text{-}IN(x)$

pii)   SMILE(x)

piii) $\forall x(MAN(x) \rightarrow SMILE(x))$ [or: $(\exists x.MAN(x) \rightarrow SMILE(x))$]

piv) $\exists x.MAN(x).COME\text{-}IN(x).FEMALE(y).SMILE\text{-}AT(x,y)$

In (pi) the quantifier $\exists x$ introduces a new variable, but unlike in ordinary Predicate Logic the (possible) scope of this variable is not constrained to the formula given. We can go on and 'merge' (the "." stands for the merge here) e.g. SMILE(x) with the formula (pi). The variable x will in this case get bound. The whole formula (pi) functions as a quantifier where values of the variable x are constrained to incoming men. Thus x is *bound*, but *active*. A symptom of this phenomenon is that α-conversion does not preserve meaning here. On the other hand x in (pii) is not bound, but still active: x 'asks' for a value to be imported from previous discourse and sends this value on to later discourse. The variable x in the formula (piii) is bound but not active. Thus it is a classical bound variable as in Predicate Logic. The reader is invited to draw her own conclusions on (piv).

A variable occurrence is bound in a formula if it is not 'visible' from the past, like x in $\exists x.A(x)$. Dually a variable is *trapped* if it is not visible from the future, travelling backwards in time. Examples here would be the occurrence of x in $\forall x(A(x))$ or the first ocurrence of x in $A(x).\exists x.B(x)$. A variable that is both bound and trapped is non-active and fully analogous to the bound variables of classical logic.

In the dynamic world there are really two candidates for correspondence to the classical notion of *sentence* in Predicate Logic (rather like the classical concept of mass divides into two in Relativity Theory): formulas in which all variables are bound and variable in which all variables are both bound and trapped. On our view sentences in the first sense are what describes states. Note that in contrast to sentences in Predicate Logic and to sentences in the second sense in DPL/DRT sentences in the first sense have more interesting meanings than just a truthvalue w.r.t. the given Model.   O

Our examples purported to illustrate the notion of state focussing on anaphoric phenomena. These are not the only relevant kind of phenomena. The processing of syntactic structure can be treated in an analogous way (at least for the admittedly modest fragments I have been considering). The simplest kind of model in this direction is what one gets when abstracting from what is between the brackets, i.o.w. when one just considers strings of brackets. Here a state is any string that has survived the bracket-test, i.e. any string where the bracket count, counting "(" as +1 and ")" as -1, has not sunk below 0. So these strings don't 'ask' for "("s at the beginning. This is saturation towards the past. We don't ask similar saturation

towards the future. The OTAT-principle simply tells us that when the bracket count has sunk below 0, nothing that comes after will set it right. The SSC's will be extensively discussed in section 6.4 of this paper.

1.4   The silly side of (our version of) Update Semantics: As we set it up in section 1.2 as soon as the output of a candidate update is not a state, the result is undefined. This seems to be definitely unrealistic. If I hear a fragment of conversation *He was smiling*, not knowing whom they are talking about, it would by simply ridiculous to 'become' undefined. For one thing it would provide people with overly simple ways of getting other people out of the way. The realistic way of handling the fragment is to set the problem of interpreting *he* aside as something to be dealt with later.

My hunch is that one should first get the silly model of updating straight before building more realistic models to describe how we actually handle semantically incomplete information. One hopeful sign is that the silly model more or less automatically leads to the notion of *partial state*. Perhaps *setting aside the problem of interpreting something* can be described as going into a partial state. (Still even partial states do not give us *error recovery*: the OTAT principle blocks this.)

1.5   Information Orderings: Till now we have just been thinking about the merging behaviour of meanings or information contents. But definitely the picture is incomplete if there are not ways to compare information contents. We will handle this problem by assuming that our meanings come with an ordering: the information ordering.

We write our information orderings in the Heyting Algebra style: so more informative is smaller. The top is the least informative item, the bottom the most informative one (in most situations the bottom is even over-informative.)

There are in fact two kinds of information ordering. The first one is the *synchronic* information ordering. For example I have two pieces of paper in my pocket. One states *Jan is wearing something new*, the other *Jan is wearing a new hat*. Evidently the first piece of paper is less informative than the second one. Whatever information state someone is in, being offered the second piece will leave her at least as informed as being offered the first. So we compare the effects of the pieces of paper when offered at the same time to the same person in different possible situations.

The second ordering is the *diachronic* ordering. Consider *Genever is a wonderful beverage. Not only the Dutch are fond of it*. Now the information content of both *Genever is a wonderful beverage* and of *Not only the Dutch are fond of it* are part of the information content of *Genever is a wonderful beverage. Not only the Dutch are fond of it*. But they are part by virtue of being brought into the whole via the process of consecutive presentation. Synchronic comparison of e.g. *Genever is a wonderful beverage. Not only the Dutch are fond of it* and *Not only the Dutch are fond of it* is a rather pointless exercise.

Both in the case of the synchronic ordering and of the diachronic ordering we may wish to distinguish ways in which one item is more informative than another one. This leads us to studying labeled orderings or categories, rather than ordinary orderings.

In this paper we will only study the synchronic unlabeled ordering. We will assume that the synchronic ordering together with the merger gives rise to the rich structure of a residuation lattice. This assumption is unfortunately not based an informally rigorous analysis, but just on the fact that some important examples satisfy it. So in a later stage of research we may have to retrace our steps.

Our basic structure is a *reduced residuation lattice* $\mathfrak{A} = \langle A, \vee, \wedge, \bullet, 1, \rightarrow, \leftarrow \rangle$. Define:

$a \leq b :\Leftrightarrow a \vee b = b$.

Let $\mathfrak{A}$ satisfy:

$\langle A, \vee, \wedge \rangle$ is a lattice, where we do not assume the top and the bottom;

$\langle A, \bullet, 1 \rangle$ is a monoid;

$a \bullet b \leq c \Leftrightarrow a \leq c \leftarrow b \Leftrightarrow b \leq a \rightarrow c$;

$\leftarrow$ is *left residuation* or *post-implication*. $\rightarrow$ is *right residuation* or *pre-implication*. We will consider the 'real order' of the arguments of post-implication to be opposite to the displayed one.

We left out the top and bottom just for temporary convenience: we could have left them, but that would make some formulations later on a bit heavy.

There are two intuitions about the synchronic ordering. The first takes presuppositions to be informative, the second takes presuppositions to be anti-informative. So

according to the second intuition the more information a content presupposes the less informative it is. As we will see later on these two intuitions correspond respectively to viewing undefined as *deadlock* or *error* and to viewing undefined as *not sufficient*.

In this paper the second intuition will be our choice. So according to us a presupposition will be on a negative place. (Clearly it could turn out that the right approach is to keep both options at the same time. This could lead to a treatment a bit like *bilattices*.)

Our choice leads immediately to a pleasant definition of the set of states in terms of the algebra. Remember that 1 is the tabula rasa mental state: it asks for nothing, it contains nothing. The items that are more informative than 1 are precisely the ones that presuppose less than 1 and contain at least as much (static) information as 1, i.e. precisely the ones that presuppose nothing, i.e. precisely the states. So we take: $S := \{a \in A | a \leq 1\}$.

$a, b, c, a', b', c', \ldots$ will range over A and $s, s', u, u', \ldots$ will range over S. Note that b has the OTAT-property precisely if for all a: $a \bullet b \leq 1 \Rightarrow a \leq 1$, i.o.w. precisely if $(1 \leftarrow b) \leq 1$. 0 (if present) has the OTAT-property just in case $1 = \top$.

Let's consider information orderings in terms of update functions. An update function is a partial function $F : S \rightarrow S$. Given an information ordering on S we can define two induced orderings, corresponding to the options we just discussed, on the update functions:

$$F \leq_1 G :\Leftrightarrow \forall s \ (sF\downarrow \Rightarrow sG\downarrow \text{ and } sF \leq sG),$$
$$F \leq_2 G :\Leftrightarrow \forall s \ (sG\downarrow \Rightarrow sF\downarrow \text{ and } sF \leq sG).$$

Clearly these are partial orderings.

Suppose for the moment that we would like to expand the ordering on the states with a new element 'undefined' or $\uparrow$ in such a way that $\leq_1$, respectively $\leq_2$, becomes the pointwise induced ordering. Consider the $\leq_1$-case. Noting e.g. that the nowhere defined function is the bottom, the unique way of achieving this is making $\uparrow$ a new bottom. Similarly in the $\leq_2$-case $\uparrow$ should be made a new top. In the $\leq_1$-case $\uparrow$ is even *more* informative than 'overdetermined' or 'false'. One way of understanding this is to view $\uparrow$ as an error state or a deadlock. In the $\leq_2$-case $\uparrow$ is even *less* informative than 'tabula rasa' or 'true'. One way of understanding this is

to view ↑ as the not-sufficient-state, something that strives to be a state, but needs something extra to be that. (It would be somewhat misleading to say that ↑ is open-ended, since the insufficiency is more naturally thought of as being 'on the side of the past'.)

In this paper we will study a specific set of update functions: the ones that update by merging with a fixed element a of our residuation lattice and whose domain is given as the set of all s below or equal to a fixed $a' \leq 1$. The idea is that a' represents a condition on the states: states below a' carry sufficient information to get access to the updating element a. (Note that the word *condition* is used here from the external pointnof view of the theoretician, not from the internal point of view of the framework.)

It may seem somewhat strange that no *intrinsic connection* is demanded between a' and a, but this can be understood by realizing that the update functions are supposed to be semantical objects. The coming together of presupposition state and update action is flows from the level of language use. If for example someone tells me *The present king of France is bald* the information contained in this sentence can only be processed by those having states providing a present king of France. The update simply has the form: *x is bald*. No intrisic connection is called for between kings of France and baldness.

We pick up the theme of updates again in section 3.

1.6 Excursion: Validity and implication: One of the major problems of the DPL/DRT approach is to gain an algebraic understanding of validity and implication. To give the reader some feeling for the problem let's briefly consider the problem in the case of Groenendijk & Stokhof's DPL.

DPL-meanings are relations between assignments. The merger simply becomes relation composition. We need also dynamic implication $\rightarrow$, where

$$f(R \rightarrow S)g :\Leftrightarrow f=g \text{ and } \forall h \ (fRh \Rightarrow \exists i \ hSi).$$

Given a classical model $\mathfrak{M}$ we may define:

$f\|\exists x\|g \quad :\Leftrightarrow$ for all variables y different from x yf=yg,

$f\|Px\|g \quad :\Leftrightarrow f=g$ and $xf\in\|P\|$,

$\|\perp\| \quad := \emptyset$

$$\|\phi . \psi\| \quad := \|\phi\| o \|\psi\|$$

$$\|(\phi \rightarrow \psi)\| := \|\phi\| \rightarrow \|\psi\|$$

$$\phi_0, ..., \phi_{n-1} \vDash_{\mathfrak{M}, f} \psi :\Leftrightarrow \forall g \ (f\|\phi_0\| o ... o \|\phi_{n-1}\| g \Rightarrow \exists h \ g\|\psi\| h)$$

$\forall x(\phi)$ can e.g. be considered as an abbreviation of: $(\exists x \rightarrow \phi)$.

At first sight some very basic progress is made here: we have before us a definition of (a form of) Predicate Logic that is a genuine special case of the corresponding version of propositional logic. The existential quantifier is just an atom, linked with the rest of the text by the propositional connectives. Granted, this is true. On the other hand, however, the propositional 'algebra' we have here is definitely unattractive. (i) As far as I know we have no axiomatization of the logic of o and $\rightarrow$ for the binary relations over an arbitrary domain. (ii) $\rightarrow$ not only handles 'negative place' but also throws away internal values assigned to variables. (iii) $\rightarrow$ is non-transitive. (As is illustrated by van Benthem's example: *Everyone who has a house, has a garden. Everyone who has a garden sprinkles it.* But not: *Everyone who has a house sprinkles it.*) (iv) Repetition of $\exists x$ with the same variable, is an obnoxious bug in the system, since it has the effect of throwing away all information about the values of the first occurrence of x.

Similar problems haunt also other related semantics like the one of DRT.

Johan van Benthem suggests to define $\rightarrow$ in terms of other more basic operations. This suggestion is surely on the right track, but just as surely not every definition can count as succes. E.g. in the residuation lattice of relations over a given domain extended with the converse-operation $^{\wedge}$ $\rightarrow$ can be defined as follows: $R \rightarrow S := 1 \wedge ((\top o S^{\wedge}) \leftarrow R)$.

Proof: $u(1 \wedge ((S^{\wedge} o \top) \leftarrow R))v \Leftrightarrow u = v$ and $\forall w \ (vRw \Rightarrow u(\top o S^{\wedge})w)$

$\qquad\qquad\qquad \Leftrightarrow u = v$ and $\forall w \ (uRw \Rightarrow \exists z \ u \top z S^{\wedge} w)$

$\qquad\qquad\qquad \Leftrightarrow u = v$ and $\forall w \ (uRw \Rightarrow \exists z \ z S^{\wedge} w)$

$\qquad\qquad\qquad \Leftrightarrow u = v$ and $\forall w \ (uRw \Rightarrow \exists z \ wSz)$ ❏

I submit, however, that this definition is too ad hoc to be enlightening. The problem is what Henk Barendregt calls, slightly adapting a Zen usage, a Koan. This means that what the problem is only becomes fully clear when we see the solution.

Open problem: Can $\rightarrow$ be defined in the residuation lattice of relations (without using $^{\wedge}$)?

(Lysbeth Zeinstra in her master's thesis defines (in a slightly different setting) $\rightarrow$ from a binary

connective "so". In this approach the 'trapping' of the variables in implications is effected by explicit 'downdates'. Still most of the problems of $\twoheadrightarrow$ also plague "so".)

In the present paper we will touch on the problems surrounding validity and implication only in passing.

2 Some elementary facts concerning residuation lattices

In this section some simple constructions in residuation lattices are described. More information on residuation algebras and action algebras can be found in Pratt[90] and Kozen[92].

Residuation lattices have obvious connections to category theory and linear logic (for the last see e.g. Abrusci[91]). Other close relatives are the bilattices due to Ginsberg (see e.g. Fitting[?]); in fact the construction described in that paper bears some similarity to our work in section 3).

Consider an a reduced residuation lattice $\mathfrak{A} = \langle A, \vee, \wedge, \bullet, 1, \rightarrow, \leftarrow \rangle$ (as introduced in 1.5). Define:

$a \leq b :\Leftrightarrow a \vee b = b;$

$b_1 \leftarrow a := (b \leftarrow a) \wedge 1;$

$S := \{a \in A | a \leq 1\}.$

a,b,c,a',b',c',... will range over A and s,s',u,u',... will range over S.

For completeness we state some principles valid in a (reduced) residuation lattice without proof. We only state principles for $\leftarrow$, but of course the corresponding ones for $\rightarrow$ also hold. The statements involving 0 and $\top$ only apply, when 0 is present.

$$x \bullet (y \vee z) = x \bullet y \vee x \bullet z,$$
$$(x \vee y) \bullet z = x \bullet z \vee y \bullet z,$$
$$0 \bullet x = x \bullet 0 = 0,$$
$$x \leftarrow (y \vee z) = (x \leftarrow y) \wedge (x \leftarrow z),$$
$$(x \wedge y) \leftarrow z = (x \leftarrow z) \wedge (y \leftarrow z),$$
$$x \leftarrow y \bullet z = (x \leftarrow z) \leftarrow y,$$
$$x \leftarrow 0 = \top \leftarrow x = \top, x \leftarrow 1 = x,$$
$$(x \leftarrow y) \bullet y \leq x,$$
$$(x \leftarrow y) \bullet (y \leftarrow z) \leq (x \leftarrow z).$$

We will amply use the following proof-generated property:

12

Ω $s'\bullet a \leq s'\bullet b$ and $s\leq s' \Rightarrow s\bullet a\leq s\bullet b$

Ω is equivalent to:

$$s\leq s' \Rightarrow (s'\rightarrow s'\bullet b) \leq (s\rightarrow s\bullet b).$$

So Ω says in a sense that in case of repetitions of states on the left before and after an inequality the occurrence on the negative place is the one that weights heavier.

We give an example of a residuation lattice not satisfying Ω. Let $A:=\{0,u,b,1\}$, where $0<u<b<1$, $u\bullet u=u\bullet b=b\bullet u=0$ and $b\bullet b=b$. It is easy to verify that this determines a residuation lattice (even an action lattice). Ω fails because $b\bullet 1\leq b\bullet b$, but $u\bullet 1\nleq u\bullet b$.

$\bullet$	0	u	b	1
0	0	0	0	0
u	0	0	0	u
b	0	0	b	b
1	0	u	b	1

$\rightarrow$	0	u	b	1
0	1	1	1	1
u	b	1	1	1
b	u	u	1	1
1	0	u	b	1

Truthtables for our example

There is a property that is somewhat more natural (but pria facie stronger) than Ω, a strengthening of Modus Ponens in case the antecedent is a state: SMP is the property: $(a_1\leftarrow s)\bullet s=s\wedge a$. SMP_1 is SMP for $a\leq 1$.

2.1   **Fact:** (i) Ω follows from SMP_1; (ii) SMP follows from SMP_1.

Proof: (i) Suppose SMP_1 and $s\bullet a\leq s\bullet b$ and $u\leq s$. Then $(u_1\leftarrow s)\bullet s\bullet a\leq(u_1\leftarrow s)\bullet s\bullet b$ and hence $u\bullet a\leq u\bullet b$. (ii) Suppose SMP_1. Then:

$$(a_1\leftarrow s)\bullet s = ((s\wedge a)_1\leftarrow s)\bullet s = s\wedge a. \qquad \square$$

Any set S' that (i) satisfies Ω (in the sense that we let the variables s and s' in the statement of Ω range over S'), that (ii) has a maximum m, (iii) is closed under $\bullet$, (iv) is downwards closed and (v) contains 1, is equal to S. In other words: m=1. Simply note that: $m\bullet m\leq m=m\bullet 1$. Apply Ω with $s:=m$ and $s':=1$, using $m\bullet m=m\bullet 1$ and $1\leq m$. We find: $m=1\bullet m\leq 1\bullet 1=1$.

2.2   The algebra $\mathfrak{A}_m$: Suppose $1\leq m$ and $m\bullet m\leq m$. $\mathfrak{A}_m$ is the (reduced) residuation lattice obtained by restricting the domain to $A_m:=\{a|a\leq m\}$ and by taking $\vee,\wedge,\bullet,1$ as before (it is easily seen that this can be done) and by taking as residuations $\rightarrow_m$ and $_m\leftarrow$, where:

$$a\rightarrow_m b := (a\rightarrow b)\wedge m, \text{ and } b_m\leftarrow a := (b\leftarrow a)\wedge m.$$

It is easily verified that the resulting algebra is as desired.

The specific example we will meet later is of course $\mathfrak{A}_1$. As is easily seen the construction under consideration also preserves Kleene's $*$, so if we start with a action lattice we get a new action lattice.

2.3   The relation $\leq_d$: We collect some facts about the ordering $\leq_d$, which will be useful later.

Define: $a\leq_d b :\Leftrightarrow d\bullet a\leq d\bullet b$ and $a=_d b :\Leftrightarrow d\bullet a=d\bullet b$. Clearly $\leq_d$ is a preordering with induced equivalence relation $=_d$. Below we will now and then conveniently confuse $\leq_d$ with its induced ordering on the $=_d$ equivalence classes. Note that $a\leq b \Rightarrow a\leq_d b$.

Define $N_d(b):=(d\rightarrow d\bullet b)$. As is well known N_d is a closure operation. We have:
$$a\leq_d b \quad \Rightarrow a\leq N_d(b)$$
$$\Rightarrow d\bullet a\leq d\bullet N_d(b)\leq d\bullet b$$
$$\Rightarrow a\leq_d b.$$
So $N_d(b)$ is the maximal element of te $=_d$-equivalence class of b.

$a\vee b$ is the $\leq_d$-suprememum of a and b, since:
$$a\vee b\leq_d c \Leftrightarrow d\bullet a\vee d\bullet b\leq d\bullet c$$
$$\Leftrightarrow a\leq_d c \text{ and } b\leq_d c.$$

$N_d(a)\wedge N_d(b)$ is the $\leq_d$-infimum of a and b, since:
$$c\leq_d N_d(a)\wedge N_d(b) \Rightarrow c\leq_d N_d(a) \text{ and } c\leq_d N_d(b)$$
$$\Rightarrow c\leq N_d(a) \text{ and } c\leq N_d(b)$$
$$\Rightarrow c\leq N_d(a)\wedge N_d(b)$$
$$\Rightarrow c\leq_d N_d(a)\wedge N_d(b).$$

Moreover $(d\rightarrow(d\bullet c\leftarrow b))$ is a kind of post-implication for $\leq_d$ and $\bullet$:

$$a \bullet b \leq_d c \quad \Rightarrow d \bullet a \leq d \bullet c \leftarrow b$$
$$\Rightarrow a \leq d \rightarrow (d \bullet c \leftarrow b)$$
$$\Rightarrow d \bullet a \leq d \bullet (d \rightarrow (d \bullet c \leftarrow b)) \leq d \bullet c \leftarrow b$$
$$\Rightarrow a \bullet b \leq_d c.$$

Note however that $\bullet$ is an operation modulo $=_d$ only in the first argument.

Finally in the presence of Ω we have yet another desirable property. Suppose Ω and $s \leq u \leq 1$, then:

$$c \leq_u N_s(a) \Rightarrow c \leq_s N_s(a)$$
$$\Rightarrow c \leq N_s(a)$$
$$\Rightarrow c \leq_u N_s(a).$$

Ergo: $c \leq N_s(a) \Leftrightarrow c \leq_u N_s(a) \Leftrightarrow c \leq N_u N_s(a)$, and so $N_u N_s(a) = N_s(a)$.

3   Update functions and partial actions

In this section we present the main construction of partial actions and prove its basic properties. The partial actions can be viewed (except for a few special elements) as update functions on states.

Fix a reduced residuation lattice $\mathfrak{A}$. Consider the update functions on $\mathfrak{A}$. We will consider these as ordered by $\leq_2$ of 1.5. We will designate $\leq_2$ simply by $\leq$. Remember that with every element a we associate its canonical update function Φ_a, given by: $s\Phi_a := s \bullet a$ if $s \bullet a \in S$, $s\Phi_a \uparrow$ otherwise. Note that:

$$s \bullet a \in S \Leftrightarrow s \bullet a \leq 1 \Leftrightarrow s \leq 1 \leftarrow a \Leftrightarrow s \leq 1_1 \leftarrow a.$$

(Reminder: $b_1 \leftarrow a =: (b \leftarrow a) \wedge 1$.) Thus $1_1 \leftarrow a$. is the canonical presupposition for updating with a. We define: $\text{pre}(a) := 1_1 \leftarrow a$. The class of canonical update functions is not always a good class: e.g. if OTAT fails for $\mathfrak{A}$ it may not be closed under composition.

We will study a somewhat larger class of update functions. These will be given by a presupposition state and an update action. Such updates can be considered as partial actions. (As we will see in section 3.5 this is slightly misleading, for even if there is a 'canonical embedding' of $\mathfrak{A}$ into the algebra given by these updates, $\leq$ and function composition, this embedding need not be a morphism of reduced residuation lattices.)

Consider a pair $\alpha := \langle a', a \rangle$ where $a' \in S$ and $a' \bullet a \in S$ (or: $a' \leq \text{pre}(a)$). Let $\Psi_\alpha : S \rightarrow S$ be given by: $s\Psi_\alpha := s \bullet a$ if $s \leq a'$, $:= \uparrow$ otherwise. Note that $s \bullet a \leq a' \bullet a \leq 1$, so $s \bullet a \in S$.

Note that $\Phi_a = \Psi_{\langle pre(a), a \rangle}$.

Let $U := \{\langle a', a \rangle \mid a \in S, a' \bullet a \in S\}$. $X := \{\Psi_\alpha : S \to S \mid \alpha \in U\}$. We show that X is closed under $\circ$.

3.1 Fact: X is closed under $\circ$.

Proof:

$$s\Psi_{\langle a', a \rangle} \circ \Psi_{\langle b', b \rangle} \downarrow \iff s \leq a' \text{ and } s \bullet a \leq b'$$
$$\iff s \leq a' \wedge (b' \leftarrow a).$$

Moreover if $s\Psi_{\langle a', a \rangle} \circ \Psi_{\langle b', b \rangle} \downarrow$, then $s\Psi_{\langle a', a \rangle} \circ \Psi_{\langle b', b \rangle} = s \bullet a \bullet b$. So:

$$\Psi_{\langle a', a \rangle} \circ \Psi_{\langle b', b \rangle} = \Psi_{\langle a' \wedge (b' \leftarrow a), a \bullet b \rangle} \qquad \square$$

It will be convenient to talk about the pairs $\langle a', a \rangle$ instead of about the corresponding update functions. To do this we need to know the induced merger and the induced preorder and corresponding equivalence relation on the pairs. Define:

$$\langle a', a \rangle \bullet \langle b', b \rangle := \langle a' \wedge (b' \leftarrow a), a \bullet b \rangle.$$

The proof of 3.1 gives us:

3.2 Fact: $\Psi_{\langle a', a \rangle} \circ \Psi_{\langle b', b \rangle} = \Psi_{\langle a', a \rangle \bullet \langle b', b \rangle}$.

To get a nice characterization of our preordering and equivalence relation we stipulate as our STANDING ASSUMPTION that $\mathfrak{A}$ satisfies Ω !

Define:

$$\langle a', a \rangle \leq \langle b', b \rangle : \iff b' \leq a' \text{ and } b' \bullet a \leq b' \bullet b,$$
$$\langle a', a \rangle \equiv \langle b', b \rangle : \iff a' = b' \text{ and } b' \bullet a = b' \bullet b,$$

We have:

3.3 Fact

$$\langle a', a \rangle \leq \langle b', b \rangle \iff \Psi_{\langle a', a \rangle} \leq \Psi_{\langle b', b \rangle},$$
$$\langle a', a \rangle \equiv \langle b', b \rangle \iff \Psi_{\langle a', a \rangle} = \Psi_{\langle b', b \rangle}.$$

Proof:

$$\Psi_{\langle a', a \rangle} \leq \Psi_{\langle b', b \rangle} \iff \forall s \leq b' \ (s \leq a' \text{ and } s \bullet a \leq s \bullet b)$$

$$\Leftrightarrow \text{b'}\leq\text{a'} \text{ and } \text{b'}\bullet\text{a}\leq\text{b'}\bullet\text{b}.$$

And:

$$\Psi_{\langle a',a\rangle}=\Psi_{\langle b',b\rangle} \Leftrightarrow \langle a',a\rangle\leq\langle b',b\rangle \text{ and } \langle b',b\rangle\leq\langle a',a\rangle$$

$$\Leftrightarrow \text{a'=b' and b'}\bullet\text{a=b'}\bullet\text{b}. \qquad \Box$$

We work with the representatives $\langle a',a\rangle$. But, of course, the real objects we are considering are the equivalence classes of $\equiv$ and via these the elements of X!

Let $Y:= X\cup\{0\}$ if $\mathfrak{A}$ has a bottom, $:= X\cup\{0,\top\}$ otherwise. We extend $\leq$ on Y by making 0, the bottom and (in the second case) $\top$ the top. Define $F\bullet G:=F\circ G$, $F\bullet 0=0\bullet F=0$, $F\bullet\top=\top\bullet G=\top$, $\top\bullet 0=0\bullet\top=0$. (Reader, please don't worry about the top and bottom at this point. Why they are added in this specific way will become clear in the proof of 3.4.)

Let $\mathfrak{U}_0(\mathfrak{A}):=\langle Y,\leq,\bullet\rangle$. The main result of this section is:

3.4   Theorem: Let $\mathfrak{A}$ be a reduced residuation lattice satisfying Ω, then:

i)   There is a unique residuation lattice $\mathfrak{U}(\mathfrak{A}) = \langle Y,\vee,\wedge,0,\bullet,1,\rightarrow,\leftarrow\rangle$, extending $\mathfrak{U}_0(\mathfrak{A})$, i.e. there is a unique residuation lattice $\mathfrak{U}(\mathfrak{A}) = \langle Y,\vee,\wedge,0,\bullet,1,\rightarrow,\leftarrow\rangle$, such that the order based on $\vee$ is the ordering $\leq$ of $\mathfrak{U}_0(\mathfrak{A})$ and $\bullet$ is the same in $\mathfrak{U}(\mathfrak{A})$ and $\mathfrak{U}_0(\mathfrak{A})$.

ii)   If $\mathfrak{A}$ is an action lattice then $\mathfrak{U}_0(\mathfrak{A})$ can be extended in a unique way to an action lattice $\mathfrak{U}^*(\mathfrak{A})$.

We will be forced to introduce 0 and $\top$ even if we were only aiming to find a reduced residuation lattice.

Proof: It is well known that if a structure of the form $\mathfrak{U}_0(\mathfrak{A})$ can be extended to a residuation (action) lattice (in the sense given above), then such an extension is unique. So the only thing we need to do to prove the theorem is to 'compute' the desired operations.

We first treat $\wedge$ and $\vee$ for the pairs (the treatment for 0 and $\top$ being selfevident). Define:

$$\langle a',a\rangle\vee\langle b',b\rangle := \langle a'\wedge b',a\vee b\rangle$$

$$\langle a',a\rangle\wedge\langle b',b\rangle := \langle a'\vee b',N_{a'}(a)\wedge N_{b'}(b)\rangle$$

It is easily seen that the values are in U.

17

$\vee$ is the supremum w.r.t. $\leq$ and $\wedge$ is the infimum: let $\alpha=\langle a',a\rangle$, $\beta=\langle b',b\rangle$, $\gamma=\langle c',c\rangle$, since:

$$\alpha\vee\beta\leq\gamma \quad \Leftrightarrow \quad c'\leq a'\wedge b' \text{ and } a\vee b\leq_{c'}c$$

$$\Leftrightarrow \quad c'\leq a', \ c'\leq b', \ a\leq_{c'}c, \ b\leq_{c'}c$$

$$\Leftrightarrow \quad \alpha\leq\gamma \text{ and } \beta\leq\gamma;$$

$$\gamma\leq\alpha\wedge\beta \quad \Leftrightarrow \quad a'\vee b'\leq c' \text{ and } c\leq_{a'\vee b'}N_{a'}(a)\wedge N_{b'}(b)$$

$$\Leftrightarrow \quad a'\leq c', \ b'\leq c', \ c\leq_{a'\vee b'}N_{a'\vee b'}N_{a'}(a)\wedge N_{a'\vee b'}N_{b'}(b)$$

$$\Leftrightarrow \quad a'\leq c', \ b'\leq c', \ c\leq N_{a'\vee b'}N_{a'}(a) \text{ and } c\leq N_{a'\vee b'}N_{b'}(b)$$

$$\Leftrightarrow \quad a'\leq c', \ b'\leq c', \ c\leq N_{a'}(a) \text{ and } c\leq N_{b'}(b)$$

$$\Leftrightarrow \quad \langle c',c\rangle\leq\langle a',a\rangle \text{ and } \langle c',c\rangle\leq\langle b',b\rangle.$$

(The reader is referred to 2.3 for the relevant facts on N which are used here.)

What about top and bottom in the new algebra? Since we always add a fresh bottom, the bottom can give no problems. If $\mathfrak{A}$ has no bottom, we add a new top. So again no problems. Suppose finally $\mathfrak{A}$ has a bottom 0. Clearly $0\in S$, and so $\langle 0,0\rangle\in U$. We get: $\langle a',a\rangle\leq\langle 0,0\rangle \Leftrightarrow 0\leq a'$ and $0\bullet a\leq 0\bullet 0$. So $\langle 0,0\rangle$ becomes the top of our new algebra. We will in the last case, perhaps confusingly, also designate $\langle 0,0\rangle$ by: $\top$.

3.4.1 Excursion: In case 0 is present in $\mathfrak{A}$, we also have:

$$\langle 1,0\rangle\leq\langle a',a\rangle \quad \Leftrightarrow \quad a'\leq 1 \text{ and } a'\bullet 0\leq a'\bullet a.$$

So $\langle 1,0\rangle$ the least of the pairs. We have:

$$\langle 1,0\rangle\bullet\langle b',b\rangle = \langle 1\wedge(b'\leftarrow 0),0\bullet b\rangle = \langle 1,0\rangle,$$

$$\langle a',a\rangle\bullet\langle 1,0\rangle = \langle a'\wedge(1\leftarrow a),a\bullet 0\rangle = \langle a',0\rangle \text{ (remember that } a'\bullet a\leq 1).$$

The last equation tells us that $\langle 1,0\rangle$ is not an annihilator for $\bullet$ and that thus $\langle 1,0\rangle$ can not play the role of the 0 of a residuation algebra. Hence adding the new bottom is really necessary. $\bigcirc$

We turn to the unit element of the merger. Let $1:=\langle 1,1\rangle$. We have

$$1\bullet\langle b',b\rangle = \langle 1\wedge(b'\leftarrow 1),1\bullet b\rangle=\langle 1\wedge b',b\rangle=\langle b',b\rangle$$

and

$$\langle a',a\rangle\bullet 1 = \langle a'\wedge(1\leftarrow a),a\bullet 1\rangle = \langle a',a\rangle,$$

since $a'\bullet a\leq 1$ and hence $a'\leq 1\leftarrow a$.

3.4.2 Excursion: In case our algebra has a bottom 0, $\langle 0,0\rangle$ has the role of top.

18

We have:
$$\top \cdot \langle b',b \rangle = \langle 0,0 \rangle \cdot \langle b',b \rangle = \langle 0 \wedge (b' \leftarrow 0), 0 \cdot b \rangle = \langle 0,0 \rangle = \top,$$
$$\langle a',a \rangle \cdot \top = \langle a',a \rangle \cdot \langle 0,0 \rangle = \langle a' \wedge (0 \leftarrow a), a \cdot 0 \rangle.$$
So the non-added top, doesn't quite behave like the added top!   ◯

We proceed with the computation of the residuations. First the "basic equation":
$$\langle a',a \rangle \cdot \langle b',b \rangle \leq \langle c',c \rangle \iff \langle a' \wedge (b' \leftarrow a), a \cdot b \rangle \leq \langle c',c \rangle$$
$$\iff c' \leq a' \wedge (b' \leftarrow a) \text{ and } c' \cdot a \cdot b \leq c' \cdot c .$$

We 'solve' $\langle a',a \rangle$ from the rhs.:
$$\langle a',a \rangle \cdot \langle b',b \rangle \leq \langle c',c \rangle \iff c' \leq a' \text{ and } c' \cdot a \leq b' \text{ and } c' \cdot a \cdot b \leq c' \cdot c$$
$$\iff c' \leq a' \text{ and } c' \cdot a \leq c' \cdot ((c' \rightarrow b') \wedge (c' \rightarrow (c' \cdot c \leftarrow b))).$$
We prove the non trivial part of the second equivalence: let P be:
$$c' \cdot a \leq c' \cdot ((c' \rightarrow b') \wedge (c' \rightarrow (c' \cdot c \leftarrow b))).$$
"⇒" Suppose $c' \cdot a \leq b'$ and $c' \cdot a \cdot b \leq c' \cdot c$, then $a \leq c' \rightarrow b'$ and $a \leq c' \rightarrow (c' \cdot c \leftarrow b)$. Ergo: $a \leq (c' \rightarrow b') \wedge (c' \rightarrow (c' \cdot c \leftarrow b))$ and hence P.
"⇐" Suppose P. Then $c' \cdot a \leq c' \cdot (c' \rightarrow b') \leq b'$ and
$$c' \cdot a \leq c' \cdot (c' \rightarrow (c' \cdot c \leftarrow b)) \leq c' \cdot c \leftarrow b,$$
hence $c' \cdot a \cdot b \leq c' \cdot c$.

Define:
$$\langle c',c \rangle \leftarrow \langle b',b \rangle := \langle c', (c' \rightarrow b') \wedge (c' \rightarrow (c' \cdot c \leftarrow b)) \rangle.$$
Clearly:
$$\langle a',a \rangle \leq \langle c',c \rangle \leftarrow \langle b',b \rangle \iff c' \leq a' \text{ and } c' \cdot a \leq c' \cdot ((c' \rightarrow b') \wedge (c' \rightarrow (c' \cdot c \leftarrow b))).$$

We 'solve' $\langle b',b \rangle$:
$$\langle a',a \rangle \cdot \langle b',b \rangle \leq \langle c',c \rangle \iff c' \leq a' \text{ and } c' \cdot a \leq b' \text{ and } c' \cdot a \cdot b \leq c' \cdot c$$
$$\iff c' \leq a' \text{ and } c' \cdot a \leq b' \text{ and } c' \cdot a \cdot b \leq c' \cdot a \cdot (c' \cdot a \rightarrow c' \cdot c)$$

Here we meet a problem: what to do with the clause $c' \leq a'$, which is independent of b' and b? The solution is use our new bottom 0. Set:
$$\langle a',a \rangle \rightarrow \langle c',c \rangle := \langle c' \cdot a, c' \cdot a \rightarrow c' \cdot c \rangle \text{ if } c' \leq a', := 0 \text{ otherwise.}$$
In case $c' \leq a$ we have:
$$\langle b',b \rangle \leq \langle a',a \rangle \rightarrow \langle c',c \rangle \iff c' \cdot a \leq b' \text{ and } c' \cdot a \cdot b \leq c' \cdot a \cdot (c' \cdot a \rightarrow c' \cdot c).$$
In case not $c' \leq a'$:

$\langle b',b\rangle \leq \langle a',a\rangle \rightarrow \langle c',c\rangle \iff \langle b,b\rangle \leq 0.$

Combining these we find the desired:

$\langle b',b\rangle \leq \langle a',a\rangle \rightarrow \langle c',c\rangle \iff c' \leq a'$ and $c' \cdot a \leq b'$ and $c' \cdot a \cdot b \leq c' \cdot a \cdot (c' \cdot a \rightarrow c' \cdot c).$

How is 0 going to behave w.r.t. $\leftarrow$? It is easy to see that we should have for $\alpha \in U \cup \{0, \top\}$ (where the top may be either added or constructed):

$$\alpha \leftarrow 0 := \top; \ 0 \leftarrow \alpha := 0 \ \text{for} \ \alpha \neq 0.$$

If we have a new top, we still must define its interaction with the residuations. A simple computation shows that we must set:

$$\alpha \leftarrow \top := 0 \ \text{for} \ \alpha \neq \top; \ \top \leftarrow \alpha := \top.$$

3.4.3 Summary: We restrict ourselves to the case where a new top is added. We give the connectives by specifying the corresponding operations on the representing pairs $\langle a',a\rangle$. The values of the operations on arguments involving $\{0,\top\}$ are specified in truthtables.

$$\begin{aligned}
\langle a',a\rangle \vee \langle b',b\rangle &:= \langle a' \wedge b', a \vee b\rangle, \\
\langle a',a\rangle \wedge \langle b',b\rangle &:= \langle a' \vee b', (a' \rightarrow a' \cdot a) \wedge (b' \rightarrow b' \cdot b)\rangle \\
&= \langle a' \vee b', N_{a'}(a) \wedge N_{b'}(b)\rangle, \\
\langle a',a\rangle \cdot \langle b',b\rangle &:= \langle a' \wedge (b' \leftarrow a), a \cdot b\rangle, \\
\langle b',b\rangle \leftarrow \langle a',a\rangle &:= \langle b', (b' \rightarrow a') \wedge (b' \rightarrow (b' \cdot b \leftarrow a))\rangle \\
&= \langle b', b' \rightarrow (a' \wedge (b' \cdot b \leftarrow a))\rangle, \\
\langle a',a\rangle \rightarrow \langle b',b\rangle &:= \langle b' \cdot a, b' \cdot a \rightarrow b' \cdot b\rangle \ \text{if} \ b' \leq a', := 0 \ \text{otherwise.}
\end{aligned}$$

We give the promised truthtables:

$\cdot$	0	1	α	$\top$
0	0	0	0	0
1	0	1	α	$\top$
α	0	α	-	$\top$
$\top$	0	$\top$	$\top$	$\top$

$\rightarrow$	0	1	α	$\top$
0	$\top$	$\top$	$\top$	$\top$
1	0	1	α	$\top$
α	0	-	-	$\top$
$\top$	0	0	0	$\top$

Truthtables for 0 and $\top$

The truthtables for $\rightarrow$ and $\leftarrow$ happen to be the same, so we only give $\rightarrow$. α

represents the 'generic' element. ○

3.4.4 Question: Does our new algebra still satify Ω?

Finally suppose $\mathfrak{A}$ is an action lattice. We give Kleene's * in $\mathfrak{U}(\mathfrak{A})$. Define:

$\langle a',a\rangle^* := \langle a'_1 \leftarrow a^*, a^*\rangle,$

$0^* := 1,$

$\top^* := \top$ (in case $\top$ is added).

Note that $(a'_1 \leftarrow a^*) \cdot a^* \leq a' \leq 1$. In case $\mathfrak{A}$ has bottom 0, we have:

$\top^* = \langle 0,0\rangle^* = \langle 0_1 \leftarrow 0^*, 0^*\rangle = \langle 0_1 \leftarrow 1,1\rangle = \langle 0,1\rangle \equiv \top.$

We check that * has the desired properties on the pairs, leaving the other elements to the industrious reader:

$\langle 1,1\rangle \leq \langle a',a\rangle^*$, since $a'_1 \leftarrow a^* \leq 1$ and $(a'_1 \leftarrow a^*) \cdot 1 \leq (a'_1 \leftarrow a^*)) \cdot a^*$,

since $1 \leq a^*$.

And:

$$\langle a',a\rangle^* \cdot \langle a',a\rangle^* = \langle a'_1 \leftarrow a^*, a^*\rangle \cdot \langle a'_1 \leftarrow a^*, a^*\rangle$$
$$= \langle (a'_1 \leftarrow a^*) \wedge ((a'_1 \leftarrow a^*) \leftarrow a^*), a^* \cdot a^*\rangle$$
$$= \langle 1 \wedge (a' \leftarrow a^*) \wedge (1 \leftarrow a^*) \wedge (a' \leftarrow a^* \cdot a^*), a^* \cdot a^*\rangle$$
$$= \langle a'_1 \leftarrow a^*, a^* \cdot a^*\rangle \qquad\qquad (a' \leq 1, \; a^* \cdot a^* \leq a^*)$$
$$\leq \langle a',a\rangle^*.$$

Finally:

$\langle a',a\rangle \leq \langle a',a\rangle^*$, since $a'_1 \leftarrow a^* \leq a'$ and $(a'_1 \leftarrow a^*) \cdot a \leq (a'_1 \leftarrow a^*) \cdot a^*$

since $1 \leq a^*$, and hence $(a' \leftarrow a^*) \leq (a' \leftarrow 1) = a'$, and $a \leq a^*$.

Consider any $\langle b',b\rangle$ and suppose:

$\langle 1,1\rangle \leq \langle b',b\rangle$, $\langle a',a\rangle \leq \langle b',b\rangle$ and $\langle b',b\rangle \cdot \langle b',b\rangle \leq \langle b',b\rangle.$

Then: $1 \leq N_{b'}(b)$, $a \leq N_{b'}(b)$ and $N_{b'}(b) \cdot N_{b'}(b) \leq N_{b'}(b)$, so $a^* \leq N_{b'}(b)$. Moreover $b' \leq b' \wedge (b' \leftarrow b)$ and hence $b' \cdot b \leq b'$. Since $a^* \leq N_{b'}(b)$, we have $b' \cdot a^* \leq b' \cdot b \leq b' \leq a'$ and so $b' \leq a'_1 \leftarrow a^*$. It follows that $\langle a',a\rangle^* \leq \langle b',b\rangle$. ❏

How does the old algebra fit into the new one? Consider any element a of $\mathfrak{A}$. Consider the update function Φ_a. Remember that $pre(a) := 1_1 \leftarrow a$, and that $s\Phi_a$ is defined iff $s \leq pre(a)$. Thus $\Phi_a = \Psi_{\langle pre(a),a\rangle}$. The natural embedding from $\mathfrak{A}$ into $\mathfrak{U}(\mathfrak{A})$ is given by: $emb(a) := \langle pre(a),a\rangle.$

3.5   Special elements in $\mathfrak{A}$: The mapping $a \mapsto \Phi_a$ induces an equivalence relation on $\mathfrak{A}$. Let $N(a):=N_{pre(a)}(a)$. We show that $N(a)$ is the maximal element equivalent to a. We have $a \leq N(a)$ and so $pre(N(a)) \leq pre(a)$. Note that $\langle pre(a),a \rangle \equiv \langle pre(a),N(a) \rangle$ and hence $pre(a) \cdot N(a) \leq 1$. Thus $pre(a) \leq pre(N(a))$. It follows that $pre(a)=pre(N(a))$. So $emb(N(a))=emb(a)$. On the other hand if $emb(b) = \langle pre(a),b \rangle \equiv \langle pre(a),a \rangle$, then $b \leq N(a)$.

3.6   Some properties of emb: In general emb doesn't behave like a morphism. We *do* have:
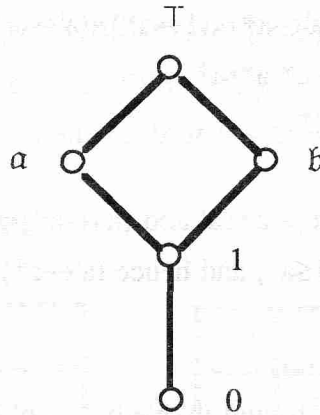
$$emb(a \vee b) = \langle 1_1 \leftarrow (a \vee b), a \vee b \rangle = \langle (1_1 \leftarrow a) \wedge (1_1 \leftarrow b), a \vee b \rangle = emb(a) \vee emb(b).$$

So, a fortiori, emb is order preserving.

Let $c := (1_1 \leftarrow a) \vee (1_1 \leftarrow b)$, then

$$emb(a \wedge b) = \langle 1_1 \leftarrow (a \wedge b), a \wedge b \rangle \leq \langle c, N_c(a) \wedge N_c(b) \rangle = emb(a) \wedge emb(b).$$

3.6.1 Example: To see that we cannot do better consider e.g. the algebra with domain $0, 1, a, b, \top$. The ordering relation is given by the following picture:

We stipulate that $0 \cdot x := x \cdot 0 := 0$, $1 \cdot x := x \cdot 1 := x$, for $x \neq 0$: $x \cdot \top := \top \cdot x := \top$, $a \cdot a := a$, $b \cdot b := b$, $a \cdot b := b \cdot a := \top$. It is easy to check that these stipulations determine a residuation lattice (even an action lattice) satisfying Ω. Note that $\cdot$ is commutative and idempotent.

We find that $emb(a \wedge b)=emb(1)=\langle 1,1 \rangle=1$, $emb(a)=\langle 0,a \rangle \equiv \top \equiv \langle 0,b \rangle=emb(b)$ and thus $emb(a) \wedge emb(b) \equiv \top$.

Below we give the truthtables of • and →.

•	0	1	α	b	T
0	0	0	0	0	0
1	0	1	α	b	T
α	0	α	α	T	T
b	0	b	T	b	T
T	0	T	T	T	T

→	0	1	α	b	T
0	T	T	T	T	T
1	0	1	α	b	T
α	0	0	α	0	T
b	0	0	0	b	T
T	0	0	0	0	T

Truthtables for • and →

○

We turn to the merger:

$$\mathrm{emb}(a \bullet b) = \langle 1_1 \leftarrow a \bullet b, a \bullet b \rangle \leq \langle (1_1 \leftarrow b)_1 \leftarrow a, a \bullet b \rangle =$$
$$\langle (1_1 \leftarrow a) \wedge ((1_1 \leftarrow b) \leftarrow a), a \bullet b \rangle = \mathrm{emb}(a) \bullet \mathrm{emb}(b).$$

If we have the OTAT property for b we find: $(1 \leftarrow b) \leq 1$ and hence $1_1 \leftarrow a \bullet b = (1_1 \leftarrow b) \leftarrow a$. So in this case: $\mathrm{emb}(a \bullet b) = \mathrm{emb}(a) \bullet \mathrm{emb}(b)$.

3.6.2 Example: Let $\mathfrak{B}$ be the structure $\langle \mathbb{Z}, \mathbf{min}, \mathbf{max}, +, \mathbf{0}, -, - \rangle$, (So $m \rightarrow n = n \leftarrow m = n-m$). Note that our ordering is the converse of the usual ordering on the integers. (We use boldface to emphasis that arithmetical objects and operations are intended and not those of the algebra. So e.g. $x+y=\min(x,y)$ and $x \bullet y = x+y$.) It is not difficult to see that $\mathfrak{B}$ is a reduced residuation lattice satisfying Ω. We have:

$$\mathrm{emb}(\mathbf{-1}) = \langle \mathbf{1}, \mathbf{-1} \rangle, \ \mathrm{emb}(\mathbf{0}) = \langle \mathbf{0}, \mathbf{0} \rangle,$$
$$\mathrm{emb}(\mathbf{-1+1}) = \langle \mathbf{0}, \mathbf{0} \rangle = 1,$$
$$\mathrm{emb}(\mathbf{-1}) \bullet \mathrm{emb}(\mathbf{1}) = \langle \mathbf{1}, \mathbf{-1} \rangle \bullet \langle \mathbf{0}, \mathbf{1} \rangle = \langle \mathbf{max}(\mathbf{1}, \mathbf{0--1}), \mathbf{0} \rangle = \langle \mathbf{1}, \mathbf{0} \rangle.$$

Note that $\langle \mathbf{0}, \mathbf{0} \rangle < \langle \mathbf{1}, \mathbf{0} \rangle$.

Clearly $\mathbf{-1+1=0}$, represents a flagrant violation of OTAT, since $\mathbf{-1}$ is not a state, but $\mathbf{0}$ is. More on $\mathfrak{B}$ in section 6.4.   ○

Finally suppose $\mathfrak{A}$ is a (reduced) action lattice. We have:

$$\mathrm{emb}(a)^* = \langle \mathrm{pre}(a), a^* \rangle = \langle (1_1 \leftarrow a)_1 \leftarrow a^*, a^* \rangle.$$

Now $(1_1 \leftarrow a)_1 \leftarrow a^* = (1_1 \leftarrow a^*) \wedge (1_1 \leftarrow a^* \bullet a)$. Moreover $a^* \bullet a \leq a^* \bullet a^* \leq a^*$, so:

23

$$(1_1 \leftarrow a)_1 \leftarrow a^* = 1_1 \leftarrow a^* = pre(a^*).$$

Ergo $emb(a^*) = emb(a)^*$.

3.7 **Emb on states:** Let s be a state of $\mathfrak{A}$. As is easily seen $emb(s) = \langle 1,s \rangle$. On the other hand if $\langle a',a \rangle \leq \langle 1,1 \rangle$, then $1 \leq a' \leq 1$ and so $a = a' \bullet a \leq 1$. So $\langle a',a \rangle$ is of the form $\langle 1,s \rangle$. So emb maps the states of $\mathfrak{A}$ surjectively on the states of $\mathfrak{U}(\mathfrak{A})$ minus 0. Obviously emb is injective modulo $\equiv$. We have:

$$emb(s \vee u) = emb(s) \vee emb(u),$$
$$emb(s \wedge u) = \langle 1, s \wedge u \rangle = \langle 1, N_1(s) \wedge N_1(u) \rangle = emb(s) \wedge emb(u),$$
$$emb(s \bullet u) = \langle 1, s \bullet u \rangle = emb(s) \bullet emb(u),$$
$$\langle 1,u \rangle \leftarrow \langle 1,s \rangle = \langle 1, (1 \rightarrow 1) \wedge (1 \rightarrow (1 \bullet u \leftarrow s)) \rangle = \langle 1, u_1 \leftarrow s \rangle = emb(u_1 \leftarrow s)$$
$$\langle 1,s \rangle \rightarrow \langle 1,u \rangle = \langle 1 \bullet s, 1 \bullet s \rightarrow 1 \bullet u \rangle = \langle s, s \rightarrow u \rangle.$$

Note that:

$$\langle 1,s \rangle \rightarrow_{\langle 1,1 \rangle} \langle 1,u \rangle = \langle s \vee 1, N_1(1) \wedge N_1(s \rightarrow u) \rangle = \langle 1, s \rightarrow_1 u \rangle = emb(s \rightarrow_1 u).$$

We may conclude that $\lambda a.(1 \wedge emb(a))$ is an isomorphism between $\mathfrak{A}_1$ and $\mathfrak{U}(\mathfrak{A})_1 / \{0\}$.

A trivial consequence of the results of 3.7 is:

3.8 **Fact:** SMP_1 (and hence SMP) is preserved under $\mathfrak{U}$.

What happens when we repeat $\mathfrak{U}$? Under the right circumstances we get nearly the same algebra. The precise identity is spoiled by the addition of new bottom elements in the construction.

To make our question sensible we must make sure that the circumstances that make $\mathfrak{U}$ meaningful and possible are preserved by $\mathfrak{U}$. Since we don't know whether Ω is preserved, the most reasonable option is to assume that the principle SMP_1 (which is preserved and implies Ω) holds in $\mathfrak{A}$. So assume $\mathfrak{A}$ satisfies SLP_1.

Consider the transitions $\mathfrak{A} \mapsto \mathfrak{U}(\mathfrak{A}) \mapsto \mathfrak{U}\mathfrak{U}(\mathfrak{A})$. Let's label the corresponding embeddings by superscript 1 respectively 2.

Let's compute $pre^2(\langle a',a \rangle)$. (Remember that $a' \bullet a \leq 1$, i.e. $a' \leq 1 \leftarrow a$.)

$$\langle 1,1 \rangle \leftarrow \langle a',a \rangle = \langle 1, (1 \rightarrow a') \wedge (1 \rightarrow (1 \bullet 1 \leftarrow a)) \rangle = \langle 1, a' \wedge (1 \leftarrow a) \rangle = \langle 1, a' \rangle.$$

Since $\langle 1,a'\rangle$ is already a state, we find: $\text{pre}^2(\langle a',a\rangle)=\langle 1,a'\rangle$

Let's assume $\mathfrak{U}(\mathfrak{A})$ has a new top. Now the elements of $\mathfrak{U}\mathfrak{U}(\mathfrak{A})$ have one of the forms:

$\langle 0,\top\rangle$ (note that $\text{pre}^2(\top)=0$),

$\langle\langle 1,s\rangle,\langle a',a\rangle\rangle$ with $\langle 1,s\rangle\leq\text{pre}^2(\langle a',a\rangle)$, i.e. $s\leq a'$,

or $\langle\langle 1,s\rangle,0\rangle$, or $\langle 0,\langle a',a\rangle\rangle$, or $\langle 0,0\rangle$ or 0.

Trivially for any α,β of the appropriate kind: $\langle 0,\alpha\rangle\equiv\langle 0,\beta\rangle$. So we may ignore the elements of the form $\langle 0,\langle a',a\rangle\rangle$ and $\langle 0,\top\rangle$ in favor of $\langle 0,0\rangle$. $\langle\langle 1,s\rangle,0\rangle\equiv\langle\langle 1,s\rangle,a\rangle$ precisely if $a=0$.

3.9   Fact: for $s\leq a'$: $\langle\langle 1,s\rangle,\langle a',a\rangle\rangle\equiv\langle\langle 1,s\rangle,\langle s,a\rangle\rangle$.

Proof: $\langle s,a\rangle$ is of the right form, since $s\bullet a\leq a'\bullet a\leq 1$. $\langle\langle 1,s\rangle,\langle s,a\rangle\rangle$ is of the right form since: $\langle 1,s\rangle\bullet\langle s,a\rangle=\langle(s_1\leftarrow s),s\bullet a\rangle=\langle 1,s\bullet a\rangle\leq\langle 1,1\rangle$. We check the equivalence: $\langle 1,s\rangle\bullet\langle a',a\rangle$ $=\langle(a'_1\leftarrow s),s\bullet a\rangle=\langle 1,s\bullet a\rangle=\langle 1,s\rangle\bullet\langle s,a\rangle$.   ❑

We find:

$\text{Emb}^2(\langle s,a\rangle)=\langle\text{pre}^2(\langle s,a\rangle),\langle s,a\rangle\rangle=\langle\langle 1,s\rangle,\langle s,a\rangle\rangle$.

Also:

$\text{Emb}^2(0)=\langle\text{pre}^2(0),0\rangle=\langle\langle 1,1\rangle,0\rangle$.

And:

$\text{Emb}^2(\top)=\langle\text{pre}^2(\top),\top\rangle=\langle 0,\top\rangle\equiv\langle 0,0\rangle$.

So the image of $\mathfrak{U}(\mathfrak{A})$ covers $\mathfrak{U}\mathfrak{U}(\mathfrak{A})$ except for the elements of the form $\langle\langle 1,s\rangle,0\rangle$ for $s\neq 1$.

If $\mathfrak{A}$ had a bottom 0, then the top of $\mathfrak{U}(\mathfrak{A})$ would be $\langle 0,0\rangle$ and $\text{Emb}^2(\langle 0,0\rangle)=\langle\langle 1,0\rangle,\langle 0,0\rangle\rangle$. So in this case $\langle 0,0\rangle$ of $\mathfrak{U}\mathfrak{U}(\mathfrak{A})$ would not be in the image of $\mathfrak{U}(\mathfrak{A})$, since it is above the image $\langle\langle 1,0\rangle,\langle 0,0\rangle\rangle$ of the top of $\mathfrak{U}(\mathfrak{A})$.

4   Further Properties

We study some specific properties further constraining residuation lattices.

4.1   S-injectivity: S-injectivity is the property: $s\bullet a\leq s\bullet b\Rightarrow a\leq b$. Note that this is equivalent to: $(s\to s\bullet b)=b$. Ω is a trivial consequence of S-injectivity. Note that for $\langle a',a\rangle$ and $\langle b',b\rangle$ in U: $\langle a',a\rangle\leq\langle b',b\rangle\Leftrightarrow b'\leq a'$ and $a\leq b$. Clearly S-injectivity is

only a reasonable property if $\mathfrak{A}$ has no bottom: the only residuation lattice with bottom satisfying S-injectivity is the degenerated one.

Our primary example of a reduced residuation is the algebra $\mathfrak{B}$ of example 3.6.2, see further e.g. section 6.4.

How does $\mathfrak{U}(\mathfrak{A})$ look if $\mathfrak{A}$ satisfies S-injectivity?

4.1.1 Lemma: $(s\rightarrow(s\bullet b\leftarrow a)) = b\leftarrow a.$

Proof:

$$\begin{aligned}
x\leq(s\rightarrow(s\bullet b\leftarrow a)) &\Leftrightarrow s\bullet x\leq s\bullet b\leftarrow a \\
&\Leftrightarrow s\bullet x\bullet a \leq s\bullet b \\
&\Leftrightarrow x\bullet a\leq b \\
&\Leftrightarrow x\leq b\leftarrow a \qquad\qquad \square
\end{aligned}$$

As is easily seen using 4.1.1 our results simplify to:

$$\begin{aligned}
\langle a',a\rangle\vee\langle b',b\rangle &:= \langle a'\wedge b',a\vee b\rangle \\
\langle a',a\rangle\wedge\langle b',b\rangle &:= \langle a'\vee b',a\wedge b\rangle \\
\langle a',a\rangle\bullet\langle b',b\rangle &:= \langle a'\wedge(b'\leftarrow a),a\bullet b\rangle \\
\langle b',b\rangle\leftarrow\langle a',a\rangle &:= \langle b',(b'\rightarrow a')\wedge(b\leftarrow a)\rangle \\
\langle a',a\rangle\rightarrow\langle b',b\rangle &:= \langle b'\bullet a,a\rightarrow b\rangle \text{ if } b'\leq a', := 0 \text{ otherwise.}
\end{aligned}$$

It is immediate that here emb commutes also with $\wedge$.

4.2   S-idempotency: S-idempotency is the property: $\forall s\in S \; s\bullet s=s$. A primary example of an S-idempotent residuation lattice is any Heyting Algebra $\mathfrak{H}$ with $\bullet:=\wedge$. Another example is the residuation lattice $\mathfrak{Rel}$ of binary relations on a given non-empty domain with order $\subseteq$ and as merger relation composition. Yet another example is the algebra of 3.6.1.

A noteworthy fact is that under certain circumstances $\bullet$ and $\wedge$ will coincide.

4.2.1 Fact: $s\bullet a\leq 1 \Rightarrow s\bullet a=s\wedge a$.

Proof: Clearly:

$$s \bullet a = (s \bullet s) \bullet a = s \bullet (s \bullet a) \leq s \bullet 1 = s \text{ and } s \bullet a \leq 1 \bullet a \leq a,$$

so $s \bullet u \leq s \wedge u$. Conversely:

$$s \wedge a \leq (s \wedge a) \bullet (s \wedge a) \leq s \bullet a. \qquad \Box$$

Note that 4.2.1 tells us that $\mathfrak{A}_1$ is in fact a Heyting Algebra! We show that S-idempotency implies SMP (and hence Ω).

4.2.2 Theorem: $(a_1 \leftarrow s) \bullet s = (a \leftarrow s) \wedge s = s \wedge a$ (and similarly for $\rightarrow$).

Proof: $(a_1 \leftarrow s) \bullet s \leq (a \leftarrow s) \bullet s \leq a$ and $(a_1 \leftarrow s) \bullet s \leq 1 \bullet s = s$, so $(a_1 \leftarrow s) \bullet s \leq s \wedge a$. Also by our previous lemma: $(a_1 \leftarrow s) \bullet s = (a_1 \leftarrow s) \wedge s = (a \leftarrow s) \wedge s \geq a \wedge s$. $\qquad \Box$

We show that under S-idempotency there are very pleasant designated representatives of our $\equiv$-equivalence classes:

4.2.3 Theorem: For any $\langle a',b \rangle \in U$, there is a unique $a \leq a'$, such that $\langle a',b \rangle \equiv \langle a',a \rangle$.

Proof: Consider any $\langle a',b \rangle \in U$. We have $a' \bullet b = a' \bullet a' \bullet b = a' \bullet (a' \wedge b)$, so $\langle a',b \rangle \equiv \langle a', a' \wedge b \rangle$. Take $a := a' \wedge b$. Clearly $\langle a',a \rangle \in U$. Finally suppose that for some $c \leq a'$ $a' \bullet b = a' \bullet c$. Then $a = (a' \wedge a) = a' \bullet a = a' \bullet c = (a' \wedge c) = c$. $\qquad \Box$

For representing the update functions in X, we can replace U by $U^i := \{\langle a',a \rangle | a \leq a' \leq 1\}$. We will do this in our representation of the operations in $\mathfrak{U}(\mathfrak{A})$.

A simple calculation shows that the new operations are:

$$\langle a',a \rangle \vee \langle b',b \rangle \quad := \langle a' \wedge b', (a \vee b) \wedge a' \wedge b' \rangle$$
$$\langle a',a \rangle \wedge \langle b',b \rangle \quad := \langle a' \vee b', (a' \vee b') \wedge (a' \rightarrow a) \wedge (b' \rightarrow b) \rangle$$
$$\langle a',a \rangle \bullet \langle b',b \rangle \quad := \langle a' \wedge (b' \leftarrow a), a \wedge b \rangle$$
$$\langle b',b \rangle \leftarrow \langle a',a \rangle \quad := \langle b', b' \wedge (b' \rightarrow a') \wedge (b' \rightarrow (b \leftarrow a)) \rangle = \langle b', b' \wedge a' \wedge (b \leftarrow a) \rangle$$
$$\langle a',a \rangle \rightarrow \langle b',b \rangle \quad := \langle b' \wedge a, b \wedge a \rangle \text{ if } b' \leq a', := 0 \text{ otherwise.}$$

Note that $\bullet$ becomes idempotent (and thus S-idempotency is preserved by $\mathfrak{U}$). Note also that $\mathfrak{U}(\mathfrak{A})$ is completely determined by $\mathfrak{A}_1$, i.o.w. $\mathfrak{U}(\mathfrak{A})$ is isomorphic to $\mathfrak{U}(\mathfrak{A}_1)$. As we noted $\mathfrak{A}_1$ is a Heyting Algebra.

6   Examples

In this section we introduce some motivating examples.

6.1   The simplest cases:

Consider the trivial one point Heyting Algebra $\mathfrak{Triv}$. It is easily seen that $\mathfrak{U}(\mathfrak{Triv})$ is precisely the two point Heyting Algebra $\mathfrak{Class}$ of Classical Logic, where $+=\wedge$. Let's look at $\mathfrak{U}(\mathfrak{Class})$. The elements are $\top:=\langle 0,0\rangle$, $1:=\langle 1,1\rangle$, $a:=\langle 1,0\rangle$ and 0, where $0<a<1<\top$. We give the truthtables for $\mathfrak{U}(\mathfrak{Class})$:

·	0	a	1	⊤
0	0	0	0	0
a	0	a	a	a
1	0	a	1	⊤
⊤	0	⊤	⊤	⊤

→	0	a	1	⊤
0	⊤	⊤	⊤	⊤
a	0	⊤	⊤	⊤
1	0	a	1	⊤
⊤	0	0	0	⊤

←	0	a	1	⊤
0	⊤	⊤	⊤	⊤
a	0	1	1	⊤
1	0	a	1	⊤
⊤	0	a	a	⊤

Truthtables for $\mathfrak{U}(\mathfrak{Class})$

Another salient four point algebra has the same elements and the same ordering as $\mathfrak{U}(\mathfrak{Class})$. Moreover the merger is also the same except that $a \cdot \top = \top$. These data determine the free commutative and idempotent residuation (action) algebra (lattice) on zero generators. (In this business even free algebras on zero generators tend to get fairly complicated. Our example here is the only simple one that I am aware of.)

6.2   A DRT-like Semantics for the positive part of Predicate Logic

Let Dref be a (finite or infinite) set of Discourse Referents (or: Variables). Let a non empty domain D be given. An assignment is a partial function from Dref to D. The set of assignments is Ass. We write $f \leq g$ for g extends f. A subset F of Ass is *persistent* if: $f \in F$ and $f \leq g \Rightarrow g \in F$.

Let $Ass_V := \{f \mid V \subseteq Dom(f)\}$. The objects of our algebra are pairs $\sigma = \langle V,F\rangle$, where V is a finite set of discourse referents and where F is a persistent set of partial assignments $f \in Ass_V$. We call these pairs M-states and we call the set of these pairs M. The idea is that V forms a context: it contains the mental objects that are present. The set F is a constraint on these objects, which codifies the actual information.

28

We do not demand: $f \in F \Rightarrow f{\restriction}V \in F$ (*Downwards Projection Property* or *DPP*). The reason is that a collection of M-states with this extra property would (with the obvious ordering; see below) not form a reduced residuation lattice. Absence of the DPP means that some variables are constrained even if they are not present in V. One could say that these variables are virtually present.

Define:
$$\langle V',F'\rangle \leq \langle V,F\rangle :\Leftrightarrow V \subseteq V' \text{ and } F' \subseteq F.$$
Define:
$$\langle V,F\rangle \wedge \langle V',F'\rangle = \langle V \cup V', F \cap F'\rangle,$$
$$\langle V,F\rangle \vee \langle V',F'\rangle = \langle V \cap V', F \cup F'\rangle.$$
It is easy to see that $\wedge$ and $\vee$ are indeed inf and sup for our ordering. (Note that $\wedge$ preserves the DPP, but $\vee$ does not.)

We take $\bullet := \wedge$. We see that we may take $1 := \langle \varnothing, \mathrm{Ass}\rangle$. Note that $1 = \top$ w.r.t our ordering, so M is really a set of states in our sense.

We show that the resulting structure is a reduced Heyting Algebra by giving $(\sigma \to \tau)$. Let F be any set of assignments, V any finite set of Discourse Referents. Define: $\mathrm{Int}_V(F) := \{f \in \mathrm{Ass}_V \mid \forall g \geq f\ g \in F\}$.

Suppose $\sigma = \langle V,F\rangle$ and $\tau = \langle W,G\rangle$. We show:
$$(\sigma \to \tau) = \langle W\backslash V, \mathrm{Int}_{W\backslash V}(F^c \cup G)\rangle$$

Consider $\rho = \langle X,H\rangle$. We find:
$$\rho \wedge \sigma \leq \tau \ \Leftrightarrow\ W \subseteq X \cup V \text{ and } H \cap F \subseteq G$$
$$\Leftrightarrow\ W\backslash V \subseteq X \text{ and } H \subseteq (F^c \cup G)$$

Note that if $W\backslash V \subseteq X$, then $H = \mathrm{Int}_X(H) = \mathrm{Int}_{W\backslash V}(H)$; also note that $\mathrm{Int}_{W\backslash V}$ is monotonic (w.r.t. $\subseteq$) and idempotent. We find:
$$\rho \wedge \sigma \leq \tau \ \Leftrightarrow\ W\backslash V \subseteq X \text{ and } H \subseteq \mathrm{Int}_{W\backslash V}(F^c \cup G)$$
$$\Leftrightarrow\ \rho \leq \langle W\backslash V, \mathrm{Int}_{W\backslash V}(F^c \cup G)\rangle.$$

Note that $\mathrm{Int}_{W\backslash V}(F^c \cup G) = \{h \in \mathrm{Ass}_{W\backslash V} \mid \forall f \geq h\ (f \in F \Rightarrow f \in G)\}$

Thus we consider the reduced residuation lattice:

$$\mathfrak{M}\text{-}\mathfrak{S} := \langle \text{M-states}, \vee, \wedge, \wedge, 1, \rightarrow, \rightarrow \rangle.$$

6.2.1 Special Elements: Given a set W of Discourse Referents, we define $\bot := \langle \emptyset, \emptyset \rangle$, Note that we use $\bot$ in a non-standard way, since it is not the bottom w.r.t. our ordering! $\top_\sigma := \bot \rightarrow \sigma$, $\bot_\sigma := \bot \wedge \sigma$.

It is easily seen that if $\sigma = \langle V, F \rangle$, then $\top_\sigma := \langle V, Ass_V \rangle$, $\bot_\sigma := \langle V, \emptyset \rangle$, $\bot := \bot_\bot$. Note that $\top = 1 = \top_1$. We have e.g.:

$$\bot_\sigma \leq \sigma \leq \top_\sigma,$$
$$\top_\sigma \wedge \top_\tau = \top_{\sigma \wedge \tau},$$
$$(\bot_\sigma \rightarrow \tau) = (\bot_\sigma \rightarrow \bot_\tau) = (\sigma \rightarrow \top_\tau) = (\top_\sigma \rightarrow \top_\tau) = \top_{\sigma \rightarrow \tau}. \qquad \bigcirc$$

We write $\sim\sigma$ for $(\sigma \rightarrow \bot)$.

6.2.2 Fact: σ has the DPP iff $\top_\sigma \leq (\sigma \vee \sim\sigma)$.

Proof: Left to the reader. $\qquad\qquad\qquad\qquad\qquad$ ❑

We turn to $\mathfrak{U}(\mathfrak{M}\text{-}\mathfrak{S})$.

6.2.3 Excursion on the connection with DRT: The elements of U are our semantic counterparts of DRS's. The relationship with semantic counterparts á la Zeevat is as follows. The objects we get as interpretations of predicate logical formulas (if an appropriate dynamical implication is added) have the form $\langle \top_\sigma, \tau \rangle$, i.o.w. $\langle \langle V, Ass_V \rangle, \langle W, G \rangle \rangle$, where $V \subseteq W$. These specific objects can also be written: $\langle V, G, W \rangle$. The DRS meanings in Zeevat's sense are in fact objects of the form $\langle W \backslash V, G \rangle$. So Zeevat just represents the discourse referents that are newly introduced ($\approx$ bound and active variables), but not the active variables that are imported ($\approx$ free variables). This difference leads to a slightly different logic.

6.2.4 Sample meanings: Let a suitable a first order model with domain D be given. We represent meanings as certain update functions on M-states, viz as elements of $\mathfrak{U}(\mathfrak{M}\text{-}\mathfrak{S})$. We exhibit some representations of meanings. We confuse a variable v with $\langle \{v\}, \emptyset \rangle$. So $v = \bot_v$.

$$\|\exists v\| := \langle \top, \top_v \rangle \quad (\text{"a" as in }),$$

$\|\iota v\| := \langle T_v, T_v \rangle$ ("he/she/it", and "the"),

$\|P(v,w)\| := \langle T_{v \wedge w}, \langle \{v,w\}, \{f \in Ass_{\{v,w\}} | \langle f(v),f(w) \rangle \in \|P\| \} \rangle$

(where $\|P\|$ is some binary relation on D associated to P in the given model).

$\|\phi.\psi\| := \|\phi\| \bullet \|\psi\|$.

We compute the meaning of $\exists v.P(v,w)$. Let $F := \{f \in Ass_{\{v,w\}} | \langle f(v),f(w) \rangle \in \|P\| \}$.

$$\|\exists v.P(v,w)\| = \langle T, T_v \rangle \bullet \langle T_{v \wedge w}, \langle \{v,w\},F \rangle \rangle$$
$$= \langle T \wedge (T_v \rightarrow T_{v \wedge w}), T_v \wedge \langle \{v,w\},F \rangle \rangle$$
$$= \langle T_w, \langle \{v,w\},F \rangle \rangle.$$

This means that the meaning of $\exists v.P(v,w)$ presupposes that w has a value and 'produces' a value for v.

6.2.5 Restriction:

Can we define $\Psi_\alpha \upharpoonright \text{dom}(\Psi_\beta)$ in our framework? Yes. Let $\alpha = \langle \sigma', \sigma \rangle$ and let $\beta = \langle \tau', \tau \rangle$. $\Psi_\alpha \upharpoonright \text{dom}(\Psi_\beta) = \Psi_{\langle \sigma' \wedge \tau', \sigma \wedge \tau' \rangle}$. Note that:

$$\langle \tau',\tau \rangle \leftarrow \langle \tau',\tau \rangle = \langle \tau', \tau' \wedge \tau' \wedge (\tau \leftarrow \tau) \rangle = \langle \tau', \tau' \rangle.$$

Also:

$$\langle \tau',\tau' \rangle \bullet \langle \sigma',\sigma \rangle = \langle \tau' \wedge (\sigma' \leftarrow \tau'), \tau' \wedge \sigma \rangle = \langle \sigma' \wedge \tau', \sigma \wedge \tau' \rangle.$$

So the pair representing $\Psi_\alpha \upharpoonright \text{dom}(\Psi_\beta)$ is $(\beta \leftarrow \beta) \bullet \alpha$. (Note that we only used S-idempotency.)

6.2.6 Conditions:

The function D with $\sigma D := \perp_\sigma$ can be viewed as producing the domain of σ. An update function F on M-states is a condition if it doesn't change the domains of its inputs, i.o.w. $F \circ D = D \upharpoonright \text{dom}(F)$. Clearly D itself is a condition, since it is idempotent and everywhere defined.

In $\mathfrak{U}(\mathfrak{M}\text{-}\mathfrak{S})$ D can be represented by $\delta := \langle T, \perp \rangle$. So α is a condition if:

$$\alpha \bullet \delta = (\alpha \leftarrow \alpha) \bullet \delta.$$

So $\langle \sigma', \sigma \rangle$ represents a condition if $\langle \sigma', \sigma \rangle \bullet \langle T, \perp \rangle = \langle \sigma', \sigma' \rangle \bullet \langle T, \perp \rangle$, i.e.

$$\langle \sigma' \wedge (\sigma \rightarrow T), \sigma \wedge \perp \rangle = \langle \sigma', \sigma \wedge \perp \rangle = \langle \sigma', \sigma' \wedge \perp \rangle = \langle \sigma' \wedge (\sigma' \rightarrow T), \sigma' \wedge \perp \rangle,$$

in other words $\sigma D = \sigma' D$.

Note that in our sample meanings above $\|\exists v\|$ is a state and $\|\iota v\|$ and $\|P(v,w)\|$ are conditions. $\bigcirc$

6.2.4 Discussion

31

i) I don't think the present approach is completely satisfactory. For one thing it has the consequence that if a variable v that is already active is existentially quantified, then the quantifier is ignored. E.g. ‖P(v).∃v.Q(v)‖=‖P(v).Q(v)‖. Thus the present treatment gives the existential quantifier the meaning: *introduce as new if not already present, otherwise ignore*. A second objection is the presence of the ill understood virtual referents due to the absence in general of the DPP. Still I think it is worthwile to pursue the present approach a bit, since it is the simplest approach known to existential quantification, that gets the presuppositional aspect right. For another aproach see e.g.Vermeulen[91b] or my forthcoming *Meanings in Time*.

ii) We do not go into the treatement of validity and dynamic implication here (see also section 1.6). As far as I can see to do this reasonably one should extend the algebraic framework. E.g. van Benthem suggests to define validity using the precondition operator ◊. $◊⟨σ',σ⟩$ gives the weakest precondition for getting truth (rather than definedness) after applying $\Psi_{⟨σ',σ⟩}$. We have $σ≤◊⟨σ',σ⟩≤σ'$. We can proceed to define validity in the style of Groenendijk & Stokhof as:

$$α\vDash_σ β \;:⟺\; σ\Psi_α≤◊\Psi_β.$$

6.3 The algebra of relations $\mathfrak{Rel}$

Let $\mathfrak{Rel}_D := ⟨\mathfrak{P}(D×D),∪,∩,\circ,ID,→,←⟩$, where D is some non-empty set. $\mathfrak{P}(D×D)$ is the set of binary relations on D and ID is the identity relation or i.o.w. the diagonal. The residuations are given by:

$$u(R→S)v \;:⟺\; ∀w\,(wRu ⇒ wSv)$$
$$u(S←R)v \;:⟺\; ∀w\,(vRw ⇒ uSw).$$

In this structure the states are precisely the tests: subsets of the diagonal. Relation Composition is idempotent on the states. Thus $\mathfrak{U}(\mathfrak{Rel}_D)=\mathfrak{U}(\mathfrak{Set}_D)$, where $\mathfrak{Set}_D$ is the residuation lattice $⟨\mathfrak{P}(D),∪,∩,∩,D,→,→⟩$, and $X→Y:=X^c∪Y$.

Since the states are tests, I think the ordering on $\mathfrak{Rel}_D$ is wrong for doing Dynamic Semantics. It is an open problem to find an ordering on (a suitable subset of) $\mathfrak{P}(D×D)$ that does what we want.

6.4 Simple Stacking Cells: The monoid of Simple Stacking Cells (SSC's)
is simply the free monoid on two generators (and), satisfying the equation ()=1. (In this monoid we follow the usual convention of notationally suppressing •.) SSC's are in a sense the integers of the well known bracket test.

An application of SSC's is to be found in my forthcoming paper *Meanings in Time*.

We will show that SSC's can be viewed as integers under a presupposition. Before proceeding to the formal development, let me first sketch the basic intuition.

Consider a clerk in charge of a store of items. At certain times either a number of items is demanded or a number of items is delivered. Whenever more items are asked for than are in store, the clerk's firm will go bankrupt. We assume that delivery may not be postponed. Now consider a sequence of demands and deliveries. The effect of such a sequence can be well described by two numbers: first the minimal number that has to be in store for the factory not to go bankrupt in the process. Secondly the sum of the deliveries minus the sum of the demands. The two numbers, say m and n, together give us an update function for numbers s of items in store: if m≤s then the result of updating is s+n, the result is undefined otherwise. Note that we use 'undefined' to model bankruptcy.

If we equate *more in store* with *more informed*, then it becomes clear that the order on the integers needed to model the clerk example is the converse of the usual ordering. This leads us to consider the reduced residuation lattice $\beta = \langle \mathbb{Z}, \textbf{min}, \textbf{max}, +, \textbf{0}, \textbf{-}, \textbf{-} \rangle$ as our initial structure. The states of β are the s≤1, i.e. the s≥0, i.o.w. precisely the non-negative integers. $\mathfrak{U}(\beta)$ is precisely the appropriate structure to model the process of deliveries an demands.

Note that if we would allow the clerk to postpone giving out items that are asked for untill sufficiently many items are in store the appropriate objects to describe the process would be simply the integers.

To return to our original way of describing the SSC's every string of brackets can be rewritten using the conversion [() $\mapsto$ empty string] to a string)...)(...(of first m right brackets and then k left brackets. The corresponding representative in $\mathfrak{U}(\beta)$ will be $\langle n, k-n \rangle$.

Since β satisfies S-injectivity, the operations of $\mathfrak{U}(\beta)$ take the particularly simple form given in section 4.1.

6.5   A clerk with several kinds of items: What happens if the clerk of

example 6.4 watches over a store of different kinds of items, say apples, pears, bananas, etc.? The obvious idea is to model the store as a *multiset* of kinds. A multiset can be represented as a function from the possible kinds to the natural numbers ℕ. We could allow that there is an infinity of possible kinds. Since we are only interested in finite multisets, we stipulate that the representing functions are almost everywhere **0**. We are thus lead to the following idea.

Let K be the (possibly) infinite set of the possible kinds that may be in store. For any reduced residuation lattice we define $\mathfrak{Finsup}(K,\mathfrak{A})$ to be the reduced residuation lattice of the functions of finite support from K to $\mathfrak{A}$. A function has finite support if its value is 1 almost everywhere. The operations of $\mathfrak{Finsup}(K,\mathfrak{A})$ are the pointwise induced ones. Note that if K is infinite and if $\mathfrak{A}$ is non-trivial, then $\mathfrak{Finsup}(K,\mathfrak{A})$ has no bottom. It is easily seen that this construction preserves properties like S-injectivity, SMP, Ω, etc.

We now model our clerk's work by $\mathfrak{B}:=\mathfrak{U}(\mathfrak{Finsup}(K,\mathfrak{B}))$. As soon as of any kind more is asked than is in store the firm will go bankrupt. An alternative modelling is: $\mathfrak{C}:=\mathfrak{Finsup}(K,\mathfrak{U}(\mathfrak{B}))=\mathfrak{Finsup}(K,\mathfrak{SSC})$. We leave it to the reader to trace the similarities and differences between these solutions.

As a specific un-clerklike example of our clerk, consider the strictly positive rational numbers. The inverse divisibility ordering on the strictly positive rationals $\mathbb{Q}^+$ is defined as follows:

$$m/n \le i/j :\Leftrightarrow i \times n \text{ divides } m \times j \qquad (m,n,i,j \in \mathbb{N}/\{0\}).$$

Note that if we assume that m and n, resp. i and j have no common divisors this simplifies to:

$$m/n \le i/j :\Leftrightarrow i \text{ divides } m \text{ and } n \text{ divides } j.$$

Let P be the set of primes. Consider $\mathfrak{Finsup}(P,\mathfrak{B})$; this is, say, $\langle Q,\vee,\wedge,\bullet,1,\rightarrow,\leftarrow\rangle$. It is easy to see that $\langle\mathbb{Q}^+,\le,\times\rangle$ is isomorphic to $\langle Q,\le,\bullet\rangle$ (where the $\le$ of the last structure is derived from $\vee$). So $\langle\mathbb{Q}^+,\le,\times\rangle$ can be extended (uniquely) to a reduced residuation lattice. Implication is division here. $\mathfrak{U}(\mathfrak{Finsup}(P,\mathfrak{B}))$ becomes a kind of presuppositional version of the non-negative rationals. Note that the 0 of this last structure is like the usual 0 of the rationals.

6.5.1 Question: Can one make a presuppositional version of all the rationals with both operations + and ×, which has a 'reasonable' structure?

7   An alternative representation of the update functions

In this section we try to increase the analogy of our update functions to the integers.

Again fix a reduced residuation algebra $\mathfrak{A}$ satisfying Ω.

It could be felt as a defect that we didn't really give a construction of actions from states, but of actions from actions. Thus it fails the analogy with the construction of the integers from the natural numbers. An alternative representation of the update functions goes some way to remedy this defect, but not in all cases the full way .

Consider $\Psi_{\langle a',A \rangle}$ for $\langle a',A \rangle \in U$. Note that (in the presence of Ω) $\Psi_{\langle a',A \rangle}$ is fully detemined by a' and a'•A. Thus we can represent our update functions also by pairs of states [a',a], where a'$\leq$1 and for some A a=a'•A$\leq$1. Let V be the set of these pairs. Below we will describe $\mathfrak{U}(\mathfrak{A})$ in terms of these alternative pairs on the assumption of SMP.

We transform our earlier results to the new format:

$$[a',a] \leq [b',b] \quad \Leftrightarrow b' \leq a' \text{ and } b' \bullet A \leq b$$
$$\Leftrightarrow b' \leq a' \text{ and } (b'_1 \leftarrow a') \bullet a \leq b$$
$$[a',a] \vee [b',b] = [a' \wedge b', (a' \wedge b') \bullet (A \vee B)]$$
$$= [a' \wedge b', ((a' \wedge b')_1 \leftarrow a') \bullet a \vee ((a' \wedge b')_1 \leftarrow b') \bullet b)]$$
$$= [a' \wedge b', (b'_1 \leftarrow a') \bullet a \vee (a'_1 \leftarrow b') \bullet b)]$$
$$[a',a] \wedge [b',b] = [a' \vee b', (a' \vee b') \bullet ((a' \rightarrow a) \wedge (b' \rightarrow b))]$$
$$[a',a] \bullet [b',b] = [a' \wedge (b' \leftarrow A), (a' \wedge (b' \leftarrow A)) \bullet A \bullet B]$$
$$= [((b' \leftarrow A)_1 \leftarrow a') \bullet a', ((b' \leftarrow A)_1 \leftarrow a') \bullet a' \bullet A \bullet B]$$
$$= [(b'_1 \leftarrow a) \bullet a', (b'_1 \leftarrow a) \bullet a \bullet B]$$
$$= [(b'_1 \leftarrow a) \bullet a', (a_1 \leftarrow b') \bullet b' \bullet B]$$
$$= [(b'_1 \leftarrow a) \bullet a', (a_1 \leftarrow b') \bullet b]$$
$$[b',b] \leftarrow [a',a] = [b', b' \bullet ((b' \rightarrow a') \wedge (b' \rightarrow (b \leftarrow A)))]$$
$$= [b', b' \bullet (b' \rightarrow (a' \wedge (b \leftarrow A)))]$$
$$= [b', b' \bullet (b' \rightarrow ((b \leftarrow A)_1 \leftarrow a') \bullet a')]$$
$$= [b', b' \bullet (b' \rightarrow ((b_1 \leftarrow a) \bullet a'))]$$
$$[a',a] \rightarrow [b',b] = [b' \bullet A, b' \bullet A \bullet (b' \bullet A \rightarrow b)]$$
$$= [(b'_1 \leftarrow a') \bullet a, (b'_1 \leftarrow a') \bullet a \bullet ((b'_1 \leftarrow a') \bullet a \rightarrow b)] \text{ if } b' \leq a',$$
$$= 0 \text{ otherwise.}$$

So we find:

$$[a',a] \leq [b',b] \quad :\Leftrightarrow \quad b' \leq a' \text{ and } (b'_1 \leftarrow a') \bullet a \leq b$$

$$[a',a] \vee [b',b] \quad := \quad [a' \wedge b', (b'_1 \leftarrow a') \bullet a \vee (a'_1 \leftarrow b') \bullet b]$$

$$[a',a] \wedge [b',b] \quad := \quad [a' \vee b', (a' \vee b') \bullet ((a' \rightarrow a) \wedge (b' \rightarrow b))]$$

$$[a',a] \bullet [b',b] \quad := \quad [(b'_1 \leftarrow a) \bullet a', (a_1 \leftarrow b') \bullet b]$$

$$[b',b] \leftarrow [a',a] \quad := \quad [b', b' \bullet (b' \rightarrow (b_1 \leftarrow a) \bullet a')]$$

$$[a',a] \rightarrow [b',b] \quad := \quad [(b'_1 \leftarrow a') \bullet a, (b'_1 \leftarrow a') \bullet a \bullet ((b'_1 \leftarrow a') \bullet a \rightarrow b)] \text{ if } b' \leq a',$$
$$\quad := \quad 0 \text{ otherwise.}$$

Note that in case $\mathfrak{A}$ satisfies S-idempotency, our new representation collapses to our earlier one since $a = a' \bullet A = a' \wedge A$.

The new way of representing is not really a construction from actions from states, since (i) V is still generally dependent on all of $\mathfrak{A}$ and (ii) in our formulations unrelativized pre-implications still occur in an essential way. In case $\mathfrak{A}_1$ is idempotent a simple inspection shows that the dependency disappears. This leads us to the following list of problems.

7.1   Open problems

i)   Under what conditions can the dependence of $\mathfrak{U}(\mathfrak{A})$ on the full pre-implications of $\mathfrak{A}$ be eliminated?

Consider a reduced residuation algebra $\mathfrak{S}$ with the property that $1 = \top$. Let's say that $\mathfrak{S} \preccurlyeq \mathfrak{B}$ if $\mathfrak{S}$ is isomorphic to $\mathfrak{B}_1$.

ii)   What $V_{\mathfrak{B}}$ are possible? Are there nice properties characterizing possible $V_{\mathfrak{B}}$?

iii)   What is the structure of the $V_{\mathfrak{B}}$ with $\subseteq$?

It is easy to see that $V_{\mathfrak{B}}$ is isomorphic to $V_{\mathfrak{U}(\mathfrak{B})}$.

7.2   The alternative representation of $\mathfrak{S}\mathfrak{S}\mathfrak{C}$:

It is easy to see that $V_{\mathfrak{Z}}$ is $\{[a',a] \mid a',a \in \mathbb{N}\}$. A simple computation gives simplified relations and operations on $\mathfrak{S}\mathfrak{S}\mathfrak{C}(=\mathfrak{U}(\mathfrak{Z}))$:

$$[a',a] \leq [b',b] \quad \Leftrightarrow \quad b' \leq a' \text{ and } (b'_1 \leftarrow a') \bullet a \leq b$$

$$[a',a] \vee [b',b] \quad = \quad [a' \wedge b', (b'_1 \leftarrow a') \bullet a \vee (a'_1 \leftarrow b') \bullet b)]$$

$$[a',a] \wedge [b',b] \quad = \quad [a' \vee b', (a' \vee b') \bullet ((a' \rightarrow a) \wedge (b' \rightarrow b))]$$

$$[a',a] \bullet [b',b] \quad = \quad [(b'_1 \leftarrow a) \bullet a', (a_1 \leftarrow b') \bullet b]$$

$$[b',b] \leftarrow [a',a] \quad = \quad [b', (b_1 \leftarrow a) \bullet a']$$

$[a',a] \to [b',b] = [(b'_1 \leftarrow a') \cdot a, b]$ if $b' \leq a'$, $= 0$ otherwise.

These operations on the representing pairs are fully in terms of $\mathfrak{B}_1$, i.e. the reduced residuation algebra $\mathfrak{R} := \langle \mathbb{N}, \mathbf{min}, \mathbf{max}, +, \mathbf{0}, \dot{-}, \dot{-} \rangle$ (where $\dot{-}$ is cut-off substraction). So $\mathbb{SSC}$ satisfies the ideal of being constructible in a way analoguous to the construction of $\mathbb{Z}$. More information on $\mathbb{SSC}$ in the appendix 8.

7.3   Negative Strings?: A rather obvious idea is to go and use our framework to create *negative strings*. However it turns out that the relevant analogue of $\mathfrak{R}$ is not a reduced residuation algebra, but just a model of process algebra, since pre-implication is lacking. Thus this problem escapes our present framework. I hope to report on this puzzle in a later publication.

8   Appendix: subalgebras of $\mathbb{SSC}$

In this appendix we collect some data on the SSC's.

It is instructive to write out the representation of section 7 of SSC's in purely numerical terms. Thus an SSC α is either 0 or $\top$ or a pair $[pop_\alpha, push_\alpha]$. We specify the relations and operations on the pairs.

$\alpha \leq \beta \quad \Leftrightarrow pop_\alpha \leq pop_\beta$ and $push_\beta \leq push_\alpha + (pop_\beta \dot{-} pop_\alpha)$

$\alpha \vee \beta \quad = [\mathbf{max}(pop_\alpha, pop_\beta), \mathbf{min}(push_\alpha + (pop_\beta \dot{-} pop_\alpha), push_\beta + (pop_\alpha \dot{-} pop_\beta))]$

$\alpha \wedge \beta \quad = [\mathbf{min}(pop_\alpha, pop_\beta), \mathbf{min}(pop_\alpha, pop_\beta) + \mathbf{max}(push_\alpha - pop_\alpha, push_\beta - pop_\beta)]$

$\alpha \cdot \beta \quad = [pop_\alpha + (pop_\beta \dot{-} push_\alpha), push_\beta + (push_\alpha \dot{-} pop_\beta)]$

$\beta \leftarrow \alpha \quad = [pop_\beta, pop_\alpha + (push_\beta \dot{-} push_\alpha)]$

$\alpha \to \beta \quad = [push_\alpha + (pop_\beta \dot{-} pop_\alpha), push_\beta]$ if $pop_\alpha \leq pop_\beta$, $= 0$ otherwise.

Some well known algebras are subalgebras of $\mathbb{SSC}$ for all operations, with the exception of one of the implications.

The algebra $\mathfrak{Pop}$ is given by $0, \top$ and the pairs of the form $[pop, \mathbf{0}]$. A simple computation shows:

$\quad \alpha \leq \beta \quad \Leftrightarrow pop_\alpha \leq pop_\beta$

$\quad \alpha \vee \beta \quad = [\mathbf{max}(pop_\alpha, pop_\beta), \mathbf{0}]$

$\quad \alpha \wedge \beta \quad = [\mathbf{min}(pop_\alpha, pop_\beta), \mathbf{0}]$

$\quad \alpha \cdot \beta \quad = [pop_\alpha + pop_\beta, \mathbf{0}]$

$\quad \beta \leftarrow \alpha \quad = [pop_\beta, pop_\alpha]$

$\quad \alpha \to \beta \quad = [pop_\beta - pop_\alpha, \mathbf{0}]$ if $pop_\alpha \leq pop_\beta$, $= 0$ otherwise.

The true internal post-implication of $\mathfrak{Pop}$ is the maximum in $\mathfrak{Pop}$ below $\beta\leftarrow\alpha$ (= [pop_β, pop_α]). We have:

$$[p,\mathbf{0}] \leq [pop_\beta, pop_\alpha] \Leftrightarrow p+pop_\alpha \leq pop_\beta.$$

So $\alpha\rightarrow\beta$ is the maximum of the elements of $\mathfrak{Pop}$ below $\beta\leftarrow\alpha$. (The treatment of 0 and $\top$ is as is to be expected.) The resulting algebra is the +,max residuation lattice. If we rename $0=:\perp$ and identify [n,0] with n, the ordering of this algebra looks as follows: $\perp,0,1,2,....\top$. $\cdot$ is addition, where $\perp+\alpha=\alpha+\perp=\perp$ and for $\alpha\neq\perp$: $\top+\alpha=\alpha+\top=\top$. Finally both residuations are given by: $\top\dot{-}\alpha=\top$, $\alpha\dot{-}\perp=\top$, $\alpha\dot{-}\beta=\perp$ if $\alpha<\beta$, and m$\dot{-}$n=m-n if n$\leq$m. Good alternative notations for $\perp$ and $\top$ here would have been -∞ and ∞.

Next consider the algebra $\mathfrak{Push}$ consisting of the elements 0 and [0,p]. We have:

$$\alpha\leq\beta \quad \Leftrightarrow push_\beta \leq push_\alpha$$
$$\alpha\vee\beta = [\mathbf{0},\mathbf{min}(push_\alpha, push_\beta)]$$
$$\alpha\wedge\beta = [\mathbf{0},\mathbf{max}(push_\alpha, push_\beta)]$$
$$\alpha\cdot\beta = [\mathbf{0}, push_\beta + push_\alpha]$$
$$\beta\leftarrow\alpha = [\mathbf{0}, push_\beta \dot{-} push_\alpha]$$
$$\alpha\rightarrow\beta = [push_\alpha, push_\beta].$$

Clearly $\mathfrak{Push}$ is just $\mathfrak{SSC}_1$ (With the proper internal pre-implication, viz. $\rightarrow_1$ and the proper treatement of 0.) the resulting algebra is the +,min residuation lattice or the tropical residuation lattice. If we identify [0,n] with n, this looks as follows. The ordering is $\perp....2,1,0$. $\cdot$ is addition. The residuations are both cut off substraction $\dot{-}$ on the natural numbers and $\alpha\dot{-}\perp=0$, $\perp\dot{-}n=\perp$. (An alternative notation for $\perp$ could have been ∞.)

Finally consider $\mathfrak{Zero}$ the subalgebra given by 0, $\top$ and the pairs [z,z] (or $\langle z,\mathbf{0}\rangle$). We have:

$$\alpha\leq\beta \quad \Leftrightarrow z_\alpha \leq z_\beta$$
$$\alpha\vee\beta = [\mathbf{max}(z_\alpha,z_\beta), \mathbf{max}(z_\alpha,z_\beta)] = \langle\mathbf{max}(z_\alpha,z_\beta),\mathbf{0}\rangle$$
$$\alpha\wedge\beta = [\mathbf{min}(z_\alpha,z_\beta), \mathbf{min}(z_\alpha,z_\beta)] = \langle\mathbf{min}(z_\alpha,z_\beta),\mathbf{0}\rangle$$
$$\alpha\cdot\beta = [\mathbf{max}(z_\alpha,z_\beta), \mathbf{max}(z_\alpha,z_\beta)] = \langle\mathbf{max}(z_\alpha,z_\beta),\mathbf{0}\rangle$$
$$\beta\leftarrow\alpha = [z_\beta, \mathbf{max}(z_\alpha,z_\beta)] = \langle z_\beta, z_\alpha\dot{-}z_\beta\rangle$$
$$\alpha\rightarrow\beta = [z_\beta, z_\beta] = \langle z_\beta,\mathbf{0}\rangle \text{ if } z_\alpha\leq z_\beta, = 0 \text{ otherwise.}$$

Note that $[z,z] \leq [u,v] \Leftrightarrow z \leq u$ and $v \leq u$. So $\alpha \to \beta$ is the maximal element in $\mathfrak{Z}ero$ below $\beta \leftarrow \alpha$. So identifying $[z,z]$ with z the ordering of our algebra is: $\bot, \mathbf{0}, \mathbf{1}, ..., \top$. • is **max** on $\mathbb{N} \cup \{\top\}$, but $\bot \cdot \alpha = \alpha \cdot \bot = \bot$. Finally both residuations are equal and are as described by $\chi(\alpha \leq \beta) \cdot \beta$, where $\chi(\alpha \leq \beta) := \mathbf{0}$ if $\alpha \leq \beta$, $:= \bot$ otherwise.

References

Abrusci, V.M., 1991, *Phase Semantics and Sequent Calculus for Pure Noncommutative Classical Linear Propositional Logic,* JSL 56, 1403-1451.

Benthem, J.F.A.K. van, 1991 , *Language in action*, North Holland, Amsterdam.

Bergstra, J.A. & Tucker, J.V., 1990, *The inescapable stack, an exercise in algebraic specification with total functions*, Report P8804b, University of Amsterdam.

Conway, J.H., 1971, *Regular Algebra and Finite Machines*, Chapman and Hall, London.

Eijck, J. van, editor, 1990, *Proceedings on Logics in AI, JELIA '90,* Springer Lecture Notes in Computer Science 478.

Eijck, J. van & Vries, F.J. de, 1991, *Dynamic Interpretation and Hoare Deduction,* CWI Technical Report CS-R9115. (To appear in JoLLI.)

Fernando, T., 1991, *Transition systems and dynamic semantics,* forthcoming.

Frege, G., 1975, *Funktion, Begriff, Bedeutung,* edited by G. Patzig, Vandenhoeck & Ruprecht, Göttingen.

Fitting, M., 199?, *Kleene's Logic, Generalized,* JLC @@@@

Groenendijk, J. & Stokhof, M., 1991, *Dynamic Predicate Logic,* Linguistics and Philosophy 14, 39-100.

Kozen, D., 1981, *A Completeness Theorem for Kleene Algebras and the Algebra of Regular Events,* Proceedings of the 6th IEEE Symposium on Logic in Computer Science, 214-225.

Kozen, D., 1992, *On Action Algebras,* these proceedings.

Leeuwen, J. van, editor, 1990, *Formal Models and Semantics, Handbook of Theoretical Computer Science, volume B*, Elsevier, Amsterdam.

Perrin, D., 1990, *Finite Automata,* in Leeuwen[90], 1-57.

Pratt, V., 1990, *Action Logic and Pure Induction,* in Eijck[90], 97-120.

Vermeulen, C., 1991a, *Sequence Semantics for Dynamic Predicate Logic,* Logic Group Preprint Series 60, Department of Philosophy, University of Utrecht.

Vermeulen, C., 1991b, *Merging without Mystery,* Logic Group Preprint Series 70, Department of Philosophy, University of Utrecht.

Visser, A., 199?, *Meanings in Time*, forthcoming.

Zeevat, H.W., 1991, *A compositional version of Discourse Representation Theory*, Linguistics and Philosophy 12, 95-131.

Zeinstra, L., 1990, *Reasoning as Discourse*, Master's Thesis, Department of Philosophy, University of Utrecht.

Logic Group Preprint Series
Department of Philosophy, University of Utrecht
Heidelberglaan 8, 3584 CS Utrecht
The Netherlands

1 C.P.J. Koymans, J.L.M. Vrancken, *Extending Process Algebra with the empty process*, September 1985

2 J.A. Bergstra, *A process creation mechanism in Process Algebra*, September 1985

3 J.A. Bergstra, *Put and get, primitives for synchronous unreliable message passing*, October 1985

4 A. Visser, *Evaluation, provably deductive equivalence in Heyting's arithmetic of substitution instances of propositional formulas*, November 1985

5 G.R. Renardel de Lavalette, *Interpolation in a fragment of intuitionistic propositional logic*, January 1986

6 C.P.J. Koymans, J.C. Mulder, *A modular approach to protocol verification using Process Algebra*, April 1986

7 D. van Dalen, F.J. de Vries, *Intuitionistic free abelian groups*, April 1986

8 F. Voorbraak, *A simplification of the completeness proofs for Guaspari and Solovay's R*, May 1986

9 H.B.M. Jonkers, C.P.J. Koymans & G.R. Renardel de Lavalette, *A semantic framework for the COLD-family of languages*, May 1986

10 G.R. Renardel de Lavalette, *Strictheidsanalyse*, May 1986

11 A. Visser, *Kunnen wij elke machine verslaan? Beschouwingen rondom Lucas' argument*, July 1986

12 E.C.W. Krabbe, *Naess's dichotomy of tenability and relevance*, June 1986

13 H. van Ditmarsch, *Abstractie in wiskunde, expertsystemen en argumentatie*, Augustus 1986

14 A. Visser, *Peano's Smart Children, a provability logical study of systems with built-in consistency*, October 1986

15 G.R. Renardel de Lavalette, *Interpolation in natural fragments of intuitionistic propositional logic*, October 1986

16 J.A. Bergstra, *Module Algebra for relational specifications*, November 1986

17 F.P.J.M. Voorbraak, *Tensed Intuitionistic Logic*, January 1987

18 J.A. Bergstra, J. Tiuryn, *Process Algebra semantics for queues*, January 1987

19 F.J. de Vries, *A functional program for the fast Fourier transform*, March 1987

20 A. Visser, *A course in bimodal provability logic*, May 1987

21 F.P.J.M. Voorbraak, *The logic of actual obligation, an alternative approach to deontic logic*, May 1987

22 E.C.W. Krabbe, *Creative reasoning in formal discussion*, June 1987

23 F.J. de Vries, *A functional program for Gaussian elimination*, September 1987

24 G.R. Renardel de Lavalette, *Interpolation in fragments of intuitionistic propositional logic*, October 1987 (revised version of no. 15)

25 F.J. de Vries, *Applications of constructive logic to sheaf constructions in toposes*, October 1987

26 F.P.J.M. Voorbraak, *Redeneren met onzekerheid in expertsystemen*, November 1987

27 P.H. Rodenburg, D.J. Hoekzema, *Specification of the fast Fourier transform algorithm as a term rewriting system*, December 1987

28 D. van Dalen, *The war of the frogs and the mice, or the crisis of the Mathematische Annalen*, December 1987

29 A. Visser, *Preliminary Notes on Interpretability Logic*, January 1988

30 D.J. Hoekzema, P.H. Rodenburg, *Gauß elimination as a term rewriting system*, January 1988

31 C. Smoryński, *Hilbert's Programme*, January 1988

32 G.R. Renardel de Lavalette, *Modularisation, Parameterisation, Interpolation*, January 1988

33 G.R. Renardel de Lavalette, *Strictness analysis for POLYREC, a language with polymorphic and recursive types*, March 1988

34 A. Visser, *A Descending Hierarchy of Reflection Principles*, April 1988

35 F.P.J.M. Voorbraak, *A computationally efficient approximation of Dempster-Shafer theory*, April 1988

36 C. Smoryński, *Arithmetic Analogues of McAloon's Unique Rosser Sentences*, April 1988

37   P.H. Rodenburg, F.J. van der Linden, *Manufacturing a cartesian closed category with exactly two objects*, May 1988

38   P.H. Rodenburg, J.L.M.Vrancken, *Parallel object-oriented term rewriting : The Booleans*, July 1988

39   D. de Jongh, L. Hendriks, G.R. Renardel de Lavalette, *Computations in fragments of intuitionistic propositional logic*, July 1988

40   A. Visser, *Interpretability Logic*, September 1988

41   M. Doorman, *The existence property in the presence of function symbols*, October 1988

42   F. Voorbraak, *On the justification of Dempster's rule of combination*, December 1988

43   A. Visser, *An inside view of EXP, or: The closed fragment of the provability logic of $I\Delta_0+\Omega_1$*, February 1989

44   D.H.J. de Jongh & A. Visser, *Explicit Fixed Points in Interpretability Logic*, March 1989

45   S. van Denneheuvel & G.R. Renardel de Lavalette, *Normalisation of database expressions involving calculations*, March 1989

46   M.F.J. Drossaers, *A Perceptron Network Theorem Prover for the Propositional Calculus*, July 1989

47   A. Visser, *The Formalization of Interpretability*, August 1989

48   J.L.M. Vrancken, *Parallel Object Oriented Term Rewriting : a first implementation in Pool2*, September 1989

49   G.R. Renardel de Lavalette, *Choice in applicative theories*, September 1989

50   C.P.J. Koymans & G.R. Renardel de Lavalette, *Inductive definitions in COLD-K*, September 1989

51   F. Voorbraak, *Conditionals, probability, and belief revision (preliminary version)*, October 1989

52   A. Visser, *On the Σ_1^0-Conservativity of Σ_1^0-Completeness*, October 1989

53   G.R. Renardel de Lavalette, *Counterexamples in applicative theories with choice*, January 1990

54   D. van Dalen, *L.E.J. Brouwer. Wiskundige en Mysticus*, June 1990

55   F. Voorbraak, *The logic of objective knowledge and rational belief*, September 1990

56   J.L.M. Vrancken, *Reflections on Parallel and Functional Languages*, September 1990

57   A. Visser, *An inside view of EXP, or: The closed fragment of the provability logic of $I\Delta_0+\Omega_1$*, revised version with new appendices, October 1990

58   S. van Denneheuvel, K. Kwast, G.R. Renardel de Lavalette, E. Spaan, *Query optimization using rewrite rules*, October 1990

59   G.R. Renardel de Lavalette, *Strictness analysis via abstract interpretation for recursively defined types*, October 1990

60   C.F.M. Vermeulen, *Sequence Semantics for Dynamic Predicate Logic*, January 1991

61   M.B. Kalsbeek, *Towards the Interpretability Logic of $I\Delta_0+EXP$*, January 1991.

62   D. van Dalen, *I<R, Some Intuitionistic Elementary Equivalences*, February 1991.

63   M. Bezem, *Strong termination of Logic Programs*, March 1991.

64   A. Visser, *The Unprovability of Small Inconsistency*, March 1991.

65   C.M. Jonker, *On the Semantics of Conflict Resolution in Truth Maintenance Systems*, March 1991.

66   F. Voorbraak, *A Preferential Model Semantics for Default Logic*, July 1991.

67   M. Kracht, *Splittings and the finite model property*, October 1991.

68   M. Bezem, *Impredicative recursion: terms depending on order types (extended abstract)*, November 1991.

69   E. Barendsen, M. Bezem, *Bar recursion versus polymorphism (extended abstract)*, December 1991.

70   C.F.M. Vermeulen, *Merging without Mystery*, December 1991.

71   F. Voorbraak, *Generalized Kripke models for epistemic logic*, December 1991.

72   W. Veldman, M. Bezem, *Ramsey's theorem and the pigeonhole principle in intuitionistic mathematics*, January 1992.

73   M. Kracht, *Prefinitely axiomatizable modal and intermediate logics*, January 1992.

74   C. M. Jonker, G. R. Renardel de Lavalette, *A tractable algorithm for the wellfounded model*, February 1992.

75   M. Kracht, *The Theory of Syntactic Domains*, February 1992.

76   A. Visser, *Actions under Presuppositions*, February 1992.