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RAMSEY'S THEOREM AND THE PIGEONHOLE PRINCIPLE IN INTUITIONISTIC MATHEMATICS

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Abstract

At first sight, the argument which F.P. Ramsey gave for (the infinite case of) his famous theorem from 1927, is hopelessly unconstructive. If suitably reformulated, the theorem is true intuitionistically as well as classically: we offer a proof which should convince both the classical and the intuitionistic reader.

1. Introduction

1.1 In 1927, F.P. Ramsey proved the following theorem (cf. *Ramsey 1928*):

For every binary relation R on $\mathbb{N}$ there exists an infinite subset A of $\mathbb{N}$ which is R-*homogeneous*, i.e. such that
either $\forall m \in A \forall n \in A[m < n \rightarrow R(m,n)]$ *or* $\forall m \in A \forall n \in A[m < n \rightarrow \neg R(m,n)]$.

(Note that we write "$R(m,n)$" for "$(m,n) \in R$".)

What is the constructive content of this theorem?
This question may be treated in different ways.
Some kind of an answer has been given by E.R. Specker (cf. *Specker 1971*) and also by C.G. Jockusch (cf. *Jockusch 1972*). They exhibited *recursive* binary relations R on $\mathbb{N}$, such that there is no *recursive* infinite R-homogeneous subset of $\mathbb{N}$, and thereby showed Ramsey's theorem to be false in that particular branch of constructive mathematics which one calls *recursive* mathematics (cf. *Bridges and Richman 1987*, or *Troelstra and van Dalen 1988*, for a survey of possible positions within constructive mathematics).
One may be surprised that classical logic is not generally avoided in recursion theory. It does not seem proper to use nonconstructive arguments when treating algorithmic objects.
We study Ramsey's theorem from an *intuitionistic* point of view, and accordingly take seriously the "either... or..." which occurs in its above formulation. The theorem does not stand this reading. The following one-dimensional case and easy consequence of the theorem, which sometimes goes by the name of *pigeonhole principle*, already fails to be true.

For every subset R of $\mathbb{N}$ there exists an infinite subset A of $\mathbb{N}$ such that
either $\forall m \in A[m \in R]$ *or* $\forall m \in A[m \notin R]$.

A suitable *Brouwerian counterexample* refutes this principle:

Let $p : \mathbb{N} \rightarrow \{0, 1 \ldots, 9\}$ be the decimal development of the real number π. Consider $R := \{n \in \mathbb{N} \mid \exists \ell \leq n \forall i < 99[p(\ell+i) = 9]\}$. Whoever claims that this set R is (intuitionistically) infinite, implies, probably recklessly, that there exists an unbroken sequence of 99 9's in the decimal development of π; whoever claims that $\mathbb{N} \backslash R$ is infinite, implies, probably recklessly, that there is no such sequence.

Observe that the set R constructed in this example is recursive, and that from a classical point of view, either R itself or its complement is recursive and infinite. There do exist classical recursion-theoretic counterexamples to the pigeonhole principle, i.e. subsets R of $\mathbb{N}$ such that neither R nor its complement contains an infinite recursive subset. Such sets must be nonrecursive and are called *bi-immune* subsets of $\mathbb{N}$. It is interesting

to note that C.G. Jockusch first studied bi-immune subsets of $\mathbb{N}$ and then found his recursive counterexample to Ramsey's theorem.
We will see in Section 3 of this paper, that the pigeonhole principle plays a key role in some classical proofs of Ramsey's theorem. We will see in Section 8 of this paper that the construction of a recursive counterexample to Ramsey's theorem is related to the construction of a bi-immune subset of $\mathbb{N}$.

1.2 Whenever a theorem from classical mathematics proves to be false intuitionistically, one may try, pondering the classical arguments and, while preserving its classical meaning, reformulating the theorem in many different ways, to find intuitionistically valid versions of it.
We did so for Ramsey's theorem. The Intuitionistic Ramsey Theorem that we present in Section 6 of this paper is a negationless statement of intuitionistic analysis, classically equivalent to Ramsey's theorem itself. In the proof of this theorem we obey the laws of intuitionistic logic, but we use no more than one intuitionistic axiom, viz. the principle of induction on monotone bars. As this principle is not, like the famous continuity principles, contrary to classical assumptions, but admits of a not too difficult classical justification, the proof is acceptable classically as well as intuitionistically.
Before proving the Intuitionistic Ramsey Theorem, we first establish, in Section 5 of this paper, a similar Intuitionistic Pigeonhole Principle.

1.3 The paper is organized as follows. In Section 2 we recapitulate the principles of intuitionistic analysis, and we arrange some notations. In Section 3 we formulate a classical proof of Ramsey's theorem. In Section 4 we develop an intuitionistic argument from this classical proof. In Section 5 we study the (infinite) pigeonhole principle. In Section 6 we treat the intuitionistic Ramsey Theorem. In Section 7 we generalize our theorem from binary to ternary, and further to n-ary relations on $\mathbb{N}$. In Section 8 we discuss the recursion-theoretic counterexamples mentioned in 1.1 from an intuitionistic point of view. In Section 9 we show how a recent result of Thierry Coquand's easily follows from our main theorem. Section 10 contains some concluding remarks.

2. Intuitionistic analysis

2.1 Our discussion concerns the set $\mathbb{N}$ of natural numbers and the set $\mathcal{N}$ of functions from $\mathbb{N}$ to $\mathbb{N}$. We use $m, n, p, q, \ldots$ as variables over the set $\mathbb{N}$ and $\alpha, \beta, \ldots$ as variables over the set $\mathcal{N}$.
We use intuitionistic logic, as we interpret connectives and quantifiers constructively. This means, in particular, that a proof of a disjunction $A \vee B$ should contain either a

proof of A or a proof of B (and we are able to decide which one), and that a proof of an existential statement $\exists x[A(x)]$ should contain an effective method to find an object x with the property $A(x)$.

2.2 We introduce an axiom of countable choice:

$\mathbf{AC_{0,0}}$: For every binary relation C on $\mathbb{N}$:
If $\forall m \exists n[C(m,n)]$, then $\exists \alpha \forall m[C(m,\alpha(m))]$.

We accept this axiom, as, in our view, an element α of $\mathcal{N}$ may be constructed step-by-step, first $\alpha(0)$, then $\alpha(1)$, and so on. We feel no obligation to describe α in finitely many words. Observe that C need not be a decidable relation on $\mathbb{N}$, therefore it is not possible, in general, to define α by:
for all $m \in \mathbb{N}$: $\alpha(m)$ is the least $n \in \mathbb{N}$ such that $C(m,n)$.
We will not use in our main argument the stronger intuitionistic axiom of countable choice $\mathbf{AC_{0,1}}$, or any continuity principle.

2.3 $\mathbb{N}^* = \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$ is the set of all finite sequences of natural numbers. A finite sequence $a = \langle a(0), a(1), \ldots, a(n-1) \rangle$ may be thought of as a function from the set $\{0, 1, \ldots, n-1\}$ to $\mathbb{N}$. $\langle\,\rangle$ is the empty sequence, the only sequence of length 0.
$* : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^*$ is the concatenation function, i.e.: for all $a, b \in \mathbb{N}^*$, $a * b$ denotes the finite sequence which results from putting b behind a.

2.4 We introduce the principle of induction on monotone bars:

$\mathbf{BI_M}$: Let P, Q be subsets of $\mathbb{N}^*$ such that
(i) $\forall a \in \mathbb{N}^* \forall m[P(a) \to P(a * \langle m \rangle)]$ (P is *monotone*)
(ii) $\forall \alpha \exists n[P(\langle \alpha(0), \alpha(1), \ldots, \alpha(n-1) \rangle)]$ (P is a *bar* in $\mathcal{N}$)
(iii) $\forall a \in \mathbb{N}^*[P(a) \to Q(a)]$ ($P \subseteq Q$) and
(iv) $\forall a \in \mathbb{N}^*[\forall n[Q(a * \langle n \rangle)] \to Q(a)]$ (Q is *inductive*).
Then: $Q(\langle\,\rangle)$.

(We write "$P(a)$" for "$a \in P$" and similarly in similar cases.)
This principle, which seems to be implicit in Brouwer's argument for his *bar theorem* was made an axiom of intuitionistic analysis by S.C. Kleene.
On this much-discussed axiom, the reader may consult *Kleene and Vesley 1965* or *Troelstra and van Dalen 1988.*

2.5 $S := \{a \in \mathbb{N}^* \mid \forall n[n+1 < \mathrm{length}(a) \to a(n) < a(n+1)]\}$ is the set of all finite *strictly increasing* sequences of natural numbers.

$\mathcal{S} := \{\alpha \in \mathcal{N} \mid \forall n[\alpha(n) < \alpha(n{+}1)]\}$ is the set of all *strictly increasing* functions from $\mathbb{N}$ to $\mathbb{N}$.

In elementary intuitionistic analysis, the following principle is an easy consequence of $\mathbf{BI_M}$ itself:

$\mathbf{BI'_M}$: Let P, Q be subsets of S such that

(i) $\forall a \in S[P(a) \to \forall m[S(a * \langle m \rangle) \to P(a * \langle m \rangle)]]$

(ii) $\forall \alpha \in \mathcal{S} \exists n[P(\langle \alpha(0), \alpha(1), \ldots, \alpha(n-1) \rangle)]$

(iii) $\forall a \in S[P(a) \to Q(a)]$ and

(iv) $\forall a \in S[\forall n[S(a * \langle n \rangle) \to Q(a * \langle n \rangle)] \to Q(a)]$.

Then: $Q(\langle\,\rangle)$.

3. A classical proof of Ramsey's theorem

Let R be a binary relation on $\mathbb{N}$. We sketch a classical argument, by which one obtains an infinite R-homogeneous subset of $\mathbb{N}$.

3.1 One defines, for each finite sequence $a = \langle a(0), a(1), \ldots, a(n-1) \rangle$ of natural numbers:
a is *R-homeogeneous* iff a is strictly increasing and
for all $m, p, q < n$ such that $m < p$ and $m < q$: $R\big(a(m), a(p)\big) \leftrightarrow R\big(a(m), a(q)\big)$.
Observe that every finite sequence of length ≤ 2 is R-homeogeneous.

3.2 One uses the following fact:
For all $a \in \mathbb{N}^*$, and $i, j, k \in \mathbb{N}$ such that $i < j < k$:
If $a * \langle i \rangle$, $a * \langle j \rangle$ and $a * \langle k \rangle$ are all three R-homeogeneous, then at least one of the three sequences $a * \langle i, j \rangle$, $a * \langle i, k \rangle$ and $a * \langle j, k \rangle$ is R-homeogeneous.

(*Proof of this fact*: let $a = \langle a(0), a(1), \ldots a(n-1) \rangle$ and suppose: $a * \langle i \rangle$, $a * \langle j \rangle$ and $a * \langle k \rangle$ are R-homeogeneous, and $a * \langle i, j \rangle$ is not R-homeogeneous.
Then: $\neg\big(R(a(n-1), a(i)\big) \leftrightarrow R\big(a(n-1), a(j)\big)$.
Therefore either $R\big(a(n-1), a(i)\big) \leftrightarrow R\big(a(n-1, a(k)\big)$ or
$R\big(a(n-1), a(j)\big) \leftrightarrow R\big(a(n-1), a(k)\big)$,
i.e. either $a * \langle i, k \rangle$ is R-homeogeneous, or $a * \langle j, k \rangle$ is R-homeogeneous.)

3.3 One defines, for each finite sequence $a = \langle a(0), \ldots a(n-1) \rangle$ of natural numbers:
a is *safe for R* $:= \forall p \exists q[p < q \;\wedge\; a * \langle q \rangle$ is R-homeogeneous$]$, i.e.: there are infinitely many immediate extensions of a which are R-homeogeneous.
Hence each finite sequence $a \in \mathbb{N}^*$, which is safe for R, is also R-homeogeneous.

3.4 One makes the following important remark:
For each $a \in S$: If a is safe for R, then $\exists q[a * \langle q \rangle$ is safe for $R]$

(Proof: Let $a \in S$ be safe for R. Then $A := \{q \in \mathbb{N} \mid a * \langle q \rangle$ is R-homeogeneous$\}$ is an infinite set of natural numbers. We distinguish two cases.
Case (i): $\forall p \in A \forall q \in A[p < q \rightarrow a * \langle p, q \rangle$ is R-homeogeneous$]$;
then $\forall q \in A[a * \langle q \rangle$ is safe for $R]$.
Case (ii): $\exists p \in A \exists q \in A[p < q \wedge a * \langle p, q \rangle$ is not R-homeogeneous$]$. Let p_0, q_0 be members of A such that $p_0 < q_0$ and $a * \langle p_0, q_0 \rangle$ is not R-homeogeneous. Then (cf. 3.2), for each $k \in A$ such that $q_0 < k$: either $a * \langle p_0, k \rangle$ is R-homeogeneous, or $a * \langle q_0, k \rangle$ is R-homeogeneous; therefore: either the set $\{k \in A \mid a * \langle p_0, k \rangle$ is R-homeogeneous$\}$ is infinite, or the set $\{k \in A \mid a * \langle q_0, k \rangle$ is R-homeogeneous$\}$ is infinite, i.e.: either $a * \langle p_0 \rangle$ is safe for R or $a * \langle q_0 \rangle$ is safe for R.)

3.5 The empty sequence $\langle\ \rangle$ is of course safe for R.
Using remark 3.4, one builds an infinite strictly increasing sequence $\alpha \in \mathcal{S}$ such that $\forall n[\langle \alpha(0), \alpha(1), \ldots, \alpha(n-1) \rangle$ is safe for $R]$.
One then observes: $\forall n[\langle \alpha(0), \alpha(1), \ldots, \alpha(n-1) \rangle$ is R-homeogeneous$]$.
Finally, one considers the two sets $B_0 := \{\alpha(n) | n \in \mathbb{N} | R(\alpha(n), \alpha(n+1))\}$ and $B_1 := \{\alpha(n) | n \in \mathbb{N} | \neg R(\alpha(n), \alpha(n+1))\}$ and one concludes: these two sets are R-homogeneous, and at least one of them is infinite.

4. A first attempt to make sense of the classical proof

4.1 From now on, we reason intuitionistically. First, we translate the classical argument from section 3 into intuitionistic language, using double negations, as in the Gödel-Gentzen translation of classical arithmetic into intuitionistic arithmetic. This approach to Ramsey's theorem is due to the second author. In later sections, we will see other ways of understanding the classical argument.
In sections 3.4 and 3.5, one applied the following classical truth:

> For all subsets A, B of $\mathbb{N}$:
> If A is infinite and $B \subseteq A$, then either B is infinite or $A \backslash B$ is infinite.

We saw in 1.1 that this so-called pigeonhole principle is intuitionistically false.
There is, however, a weak version of the principle which is intuitionistically provable.

4.2 Reasoning with negative statements is an art with its own peculiarities. We mention two of them.

Firstly, it occurs, when we are busy proving a negative statement, that we *reinforce our assumptions*, leaving out some double negation and replacing an assumption "$\neg\neg P$" by "P". We may do so, as, in intuitionistic logic, $\neg\neg P \to \neg Q$ follows from $P \to \neg Q$. In the sequel, we will indicate such reinforcements by the words "*Suppose even more*". Secondly, we may, again when we are aiming for a negative conclusion, use some statements of the form "$P \vee \neg P$" as an extra assumption. (This is a special case of the first procedure, as $\neg\neg(P \vee \neg P)$ is an intuitionistic truth). We then may continue the proof by distinction of cases: "Case (i): Suppose $P \ldots$, Case (ii): Suppose $\neg P \ldots$". In the sequel, whenever we apply this device, we use the words: "*Distinguish two cases*".

4.3 We define, for each subset A of $\mathbb{N}$:

$$A \text{ is } \textit{weakly infinite} \quad := \quad \forall p \neg\neg \exists q[p < q \ \wedge \ q \in A]$$

4.4 Lemma: For all subsets A, B of $\mathbb{N}$:
If A is weakly infinite and $B \subseteq A$,
then $\neg\neg$ (B is weakly infinite $\vee$ $A \backslash B$ is weakly infinite).

Proof: Suppose $B \subseteq A \subseteq \mathbb{N}$ and: A is weakly infinite, and B is not weakly infinite.
Then: $\neg\neg\exists p \forall q[p < q \ \to \ q \notin B]$
Suppose even more: $\exists p \forall q[p < q \ \to \ q \notin B]$. Calculate $p_0 \in \mathbb{N}$ such that $\forall q[p_0 < q \ \to \ q \notin B]$.
A is weakly infinite, therefore $\forall p \neg\neg\exists q[q > p \ \wedge \ q > p_0 \ \wedge \ q \in A]$ and:
$\forall p \neg\neg\exists q[q > p \ \wedge \ q \in A \ \wedge \ q \notin B]$, i.e. $A \backslash B$ is weakly infinite.
$\diamond$

4.5 We define, for each binary relation R and $\mathbb{N}$ and each finite sequence $a = \langle a(0), a(1), \ldots, a(n-1) \rangle$ of natural numbers:

a is *weakly safe* for R $:=$ $\forall p \neg\neg\exists q[p < q \wedge a * \langle q \rangle$ is R-homeogeneous$]$.

4.6 Lemma: For all binary relations R on $\mathbb{N}$, for all $a \in \mathbb{N}^*$:
If a is weakly safe for R, then $\neg\neg\exists q[a * \langle q \rangle$ is weakly safe for $R]$.

Proof: Suppose a is weakly safe for R.
Define $A := \{q \in \mathbb{N} \mid a * \langle q \rangle$ is R-homeogeneous$\}$.
Then A is weakly infinite.
Observe that our goal is a negative conclusion, and *distinguish two cases*:
Case (i): $\forall p \in A \forall q \in A[p < q \ \to \ \neg\neg(a * \langle p, q \rangle$ is R-homeogeneous$)]$.
Then: $\forall p \in A[a * \langle p \rangle$ is weakly safe for $R]$.
Case (ii): $\neg\forall p \in A \forall q \in A[p < q \ \to \ \neg\neg(a * \langle p, q \rangle$ is R-homeogeneous$)]$,

i.e. $\neg\neg\exists p \in A \exists q \in A[p < q \wedge \neg(a * \langle p, q\rangle$ is R-homeogeneous$)]$. *Suppose even more*, and let p_0, q_0 be such that $p_0 < q_0$ and $a * \langle p_0\rangle$, $a * \langle q_0\rangle$ are R-homeogeneous, and $a * \langle p_0, q_0\rangle$ is *not* R-homeogeneous. Then, for each $k \in A$ such that $q_0 < k$:
$\neg\neg(a * \langle p_0, k\rangle$ is R-homeogeneous $\vee$ $a * \langle q_0, k\rangle$ is R-homeogeneous). (This follows by the argument given in 3.2). Using Lemma 4.4, we conclude:
$\neg\neg(\{k \in A \mid a * \langle p_0, k\rangle$ is R-homeogeneous$\}$ is weakly infinite $\vee$
$\{k \in A \mid a * \langle q_0, k\rangle$ is R-homeogeneous$\}$ is weakly infinite), i.e.:
$\neg\neg(a * \langle p_0\rangle$ is weakly safe for R $\vee$ $a * \langle q_0\rangle$ is weakly safe for $R)$
therefore: $\neg\neg\exists p[a * \langle p\rangle$ is weakly safe for $R]$
◇

In view of a later application, viz. Lemma 4.9, we rephrase Lemma 4.6.

4.7 Corollary: For all binary relations R on $\mathbb{N}$, for all $a \in \mathbb{N}^*$:
If $\forall q[a * \langle q\rangle$ is not weakly safe for $R]$, then a is not weakly safe for R.

Proof. Obvious. ◇

4.8 The difficulty now is that we do not see how to iterate Lemma 4.6 countably many times. (This is done in the classical proof, cf. 3.5). The following conclusion, let us call it $(*)$, is surely out of reach:

$(*)$ For all binary relations R on $\mathbb{N}$:
$\exists \alpha \in \mathcal{S} \forall n[\langle \alpha(0), \alpha(1), \ldots, \alpha(n-1)\rangle$ is R-homeogeneous$]$.

Assume $(*)$. Given a subset A of $\mathbb{N}$ we may apply $(*)$ to the binary relation R on $\mathbb{N}$ which is defined by: $\forall m \forall n[R(m,n) \leftrightarrow A(n)]$.
Thus, $(*)$ is seen to imply:

$(**)$ For every subset A of $\mathbb{N}$: $\exists \alpha \in \mathcal{S} \forall n[A(\alpha(0)) \leftrightarrow A(\alpha(n))]$.

$(**)$ is refuted by the Brouwerian example mentioned in the introduction, 1.1. One is tempted to try the following weakening of $(*)$:

For all binary relations R on $\mathbb{N}$:
$\neg\neg\exists \alpha \in \mathcal{S} \forall n[\langle \alpha(0), \alpha(1), \ldots, \alpha(n-1)\rangle$ is R-homeogeneous$]$.

or its corollary:

For every subset A of $\mathbb{N}$: $\neg\neg\exists \alpha \in \mathcal{S} \forall n[A(\alpha(0)) \leftrightarrow A(\alpha(n))]$.

In Section 8.6.2, however, we will see, by a metamathematical argument, that these conclusions cannot be obtained from the usual axioms of intuitionistic analysis.
In view of these impossibilities, we are happy with the following lemma, first established by the second author:

4.9 Lemma: For all binary relations R on $\mathbb{N}$:

$$\neg\forall\alpha \in \mathcal{S}\exists n[\langle\alpha(0), \alpha(1), \ldots, \alpha(n-1)\rangle \text{ is not } R\text{-homeogeneous}].$$

Proof. Let R be a binary relation on $\mathbb{N}$ such that:
$\forall\alpha \in \mathcal{S}\exists n[\langle\alpha(0), \alpha(1), \ldots, \alpha(n-1)\rangle$ is not R-homeogeneous]
We will use the principle of induction on monotone bars, $\mathbf{BI}'_{\mathbf{M}}$. (Cf. 2.5).
We define subsets P and Q of S as follows. For each $a \in S$:
$P(a) :=$ a is not R-homeogeneous, and $Q(a) :=$ a is not weakly safe for R.
Observe that P is a monotone bar in $\mathcal{S}$, that $P \subseteq Q$, and that, according to Corollary 4.7, Q is inductive.
Using $\mathbf{BI}'_{\mathbf{M}}$, we conclude: $Q(\langle\,\rangle)$, i.e.: the empty sequence $\langle\,\rangle$ is not weakly safe for R.
But: $\langle\,\rangle$ is weakly safe for R, as $\forall q[\langle q\rangle$ is R-homeogeneous]. Contradiction.
◇

4.10 Lemma 4.9 says that, for every binary relation R on $\mathbb{N}$, the assumption that the range of every strictly increasing sequence of natural numbers is, in a strong sense, not R-homeogeneous, leads to a contradiction. In this very weak sense, the classical theorem that there exists an infinite R-homeogeneous subset of $\mathbb{N}$, is true.
In 6.2, we will establish a similar weak version of Ramsey's theorem itself, i.e. the theorem that, for every binary relation R on $\mathbb{N}$, there exists an infinite R-*homogeneous* subset of $\mathbb{N}$.
To obtain this result, we need one further step which we discuss in the next section.

5. Almost full subsets of $\mathbb{N}$

5.1 Containing as many double negations as it does, Lemma 4.4 is not a very welcome substitute for the classical pigeonhole principle. In this section, we want to approximate the principle by a lemma and a theorem, viz. 5.2 and 5.4, which, having a more positive appearance, are less like ghosts from the lost classical paradise.

5.2 Lemma: For all subsets A of $\mathbb{N}$:

$$\text{If } \forall\gamma \in \mathcal{S}\exists n[A(\gamma(n))], \text{ then } \exists\gamma \in \mathcal{S}\forall n[A(\gamma(n))].$$

Proof: We use $\mathbf{AC_{0,0}}$, the axiom of countable choice introduced in 2.2.
Let A be a subset of $\mathbb{N}$ such that $\forall\gamma \in \mathcal{S}\exists n[A(\gamma(n))]$.

Then $\forall m \exists n[n > m \wedge A(n)]$. (In order to see this, one considers, for each $m \in \mathbb{N}$, the function $\gamma \in \mathcal{S}$ defined by $\forall n[\gamma(n) = m+n+1]$).
Applying $\mathbf{AC_{0,0}}$ we find $\alpha \in \mathcal{N}$ such that $\forall m[\alpha(m) > m \wedge A(\alpha(m))]$.
Define $\gamma \in \mathcal{N}$ by: $\gamma(0) = \alpha(0)$ and $\forall n[\gamma(n+1) = \alpha(\gamma(n))]$.
Then $\gamma \in \mathcal{S}$ and $\forall n[A(\gamma(n))]$.
◇

5.3 Considering Lemma 5.2, one might cherish some hopes for the following version of Ramsey's Theorem:

For all binary relations R on $\mathbb{N}$:
If $\forall \gamma \in \mathcal{S} \exists m \exists n[m < n \wedge R(\gamma(m), \gamma(n))]$,
then $\exists \gamma \in \mathcal{S} \forall m \forall n[m < n \rightarrow R(\gamma(m), \gamma(n))]$.

Such hopes are idle, as we show be an example, due to Mervyn Jansen.
Let $p : \mathbb{N} \rightarrow \{0, 1, \ldots, 9\}$ be the decimal development of the number π,
let $A := \{n \in \mathbb{N} \mid \exists \ell \leq n \forall i < 99[p(\ell+i) = 9]\}$ and
let $R := \{(m, n) \in \mathbb{N}^2 \mid A(m) \leftrightarrow A(n)\}$.
We claim that $\forall \gamma \in \mathcal{S} \exists m \exists n[m < n \wedge R(\gamma(m), \gamma(n))]$.
(Let $\gamma \in S$ and consider $\gamma(1)$. Either: $\neg A(\gamma(1))$, and therefore: $R(\gamma(0), \gamma(1))$, or: $A(\gamma(1))$, and therefore: $R(\gamma(1), \gamma(2))$).
Suppose now: $\gamma \in \mathcal{S}$ and $\forall m \forall n[m < n \rightarrow R(\gamma(m), \gamma(n))]$. Either $A(\gamma(0))$ and $\forall n[A(\gamma(n))]$, a reckless conclusion, or $\neg A(\gamma(0))$ and $\forall n[\neg A(\gamma(n))]$, again a reckless conclusion.

5.4 Theorem: *(Intuitionistic Pigeonhole Principle):*
For all subsets A, B of $\mathbb{N}$:
If $\forall \gamma \in \mathcal{S} \exists n[A(\gamma(n))]$ and $\forall \gamma \in \mathcal{S} \exists n[B(\gamma(n))]$,
then $\forall \gamma \in \mathcal{S} \exists n[A(\gamma(n)) \wedge B(\gamma(n))]$.

First proof: (Using $\mathbf{AC_{0,0}}$).
Let A, B be subsets of $\mathbb{N}$ which fulfil the requirements of the theorem.
Let $\gamma \in \mathcal{S}$. Observe: $\forall \delta \in \mathcal{S} \exists n[A(\gamma \circ \delta(n))]$.
Apply lemma 5.2 and construct $\delta \in \mathcal{S}$ such that $\forall n[A(\gamma \circ \delta(n))]$.
Calculate n_0 such that $B(\gamma \circ \delta(n_0))$, let $p := \delta(n_0)$ and observe: $A(\gamma(p)) \wedge B(\gamma(p))$.
◇

Second proof: (Using $\mathbf{BI'_M}$).
Let A, B be subsets of $\mathbb{N}$ which fulfil the requirements of the theorem.
Define a subset P of the set S of strictly increasing finite sequences by:
For all $a = \langle a(0), a(1), \ldots a(n-1) \rangle \in S$:

$P(a) := \exists i < n \exists j < n[i \leq j \wedge A(a(i)) \wedge B(a(j))]$.
We claim that P is a monotone bar in $\mathcal{S}$.
(*Proof of this claim:* Let $\gamma \in \mathcal{S}$. Calculate $i_0 \in \mathbb{N}$ such that $A(\gamma(i_0))$. Define $\delta \in \mathcal{S}$ by: $\forall n[\delta(n) = \gamma(i_0+n)]$. Calculate k_0 such that $B(\delta(k_0))$. Let $j_0 := i_0+k_0$ and observe: $i_0 \leq j_0$ and $A(\gamma(i_0)) \wedge B(\gamma(j_0))$. *End of proof of claim*).
Define a subset Q of the set S of strictly increasing sequences of natural numbers by: For all $a \in S$: $Q(a) := \forall\gamma \in \mathcal{S}\exists n[P(a * \langle\gamma(n)\rangle)]$.
Observe that $P \subseteq Q$.
We claim that Q is inductive, i.e.: $\forall a \in S[\forall q[S(a * \langle q\rangle) \rightarrow Q(a * \langle q\rangle)] \rightarrow Q(a)]$.
(*Proof of this claim:* let $a \in S$ be such that $\forall q[S(a * \langle q\rangle) \rightarrow Q(a * \langle q\rangle)]$. Let $\gamma \in \mathcal{S}$. Calculate $n_0 := \mu p[S(a * \langle\gamma(p)\rangle)]$.
Remark: $Q(a * \langle\gamma(n_0)\rangle)$, and calculate n_1 such that $P(a * \langle\gamma(n_0), \gamma(n_1)\rangle)$, observe: either $P(a * \langle\gamma(n_0)\rangle)$ or $P(a * \langle\gamma(n_1)\rangle)$ or $A(\gamma(n_0)) \wedge B(\gamma(n_1))$.
Remark: $Q(a * \langle\gamma(n_1)\rangle)$, and calculate n_2 such that $P(a * \langle\gamma(n_1), \gamma(n_2)\rangle)$, observe: either $P(a * \langle\gamma(n_1)\rangle)$ or $P(a * \langle\gamma(n_2)\rangle)$ or $A(\gamma(n_1)) \wedge B(\gamma(n_2))$.
We conclude: $P(a*\langle\gamma(n_0)\rangle) \vee P(a*\langle\gamma(n_1)\rangle) \vee P(a*\langle\gamma(n_2)\rangle)$, and: $\exists n \in \mathbb{N}[P(a*\langle\gamma(n)\rangle)]$.
Therefore: $\forall\gamma \in \mathcal{S}\exists n[P(a * \langle\gamma(n)\rangle)]$, i.e.: $Q(a)$. *End of proof of claim.*)
Using $\mathbf{BI}'_{\mathbf{M}}$ (cf. 2.5), we conclude $Q(\langle\,\rangle)$, i.e.: $\forall\gamma \in \mathcal{S}\exists n[P(\langle\gamma(n)\rangle)]$,
therefore $\forall\gamma \in \mathcal{S}\exists n[A(\gamma(n)) \wedge B(\gamma(n))]$.
◇

5.5 We define, for each subset A of $\mathbb{N}$:

$$A \text{ is } \textit{almost full} := \forall\gamma \in \mathcal{S}\exists n[A(\gamma(n))]$$

Observe that, for each subset A of $\mathbb{N}$: if $\exists n\forall m[m > n \rightarrow A(m)]$, then $\forall\gamma \in \mathcal{S}\exists n[A(\gamma(n))]$. Intuitionistically, the converse is not true: let $p : \mathbb{N} \rightarrow \{0, 1, \ldots, 9\}$ be the decimal development of π and consider

$$A := \{n \in \mathbb{N} \mid \neg\forall i < 99[p(n+i) = 9] \ \vee\ \exists \ell < n\forall i < 99[p(\ell+i) = 9]\}$$

(Observe that A is a decidable subset of $\mathbb{N}$, i.e.: $\forall n[n \in A \ \vee\ \neg(n \in A)]$ and $\forall m\forall n[(m \notin A \ \wedge\ n \notin A) \ \rightarrow\ m = n]$, i.e.: $\mathbb{N}\backslash A$ has at most one element.
Therefore: $\forall\gamma \in \mathcal{S}[A(\gamma(0)) \vee A(\gamma(1))]$.
Whoever claims: $\exists n\forall m[m > n \ \rightarrow\ A(m)]$, declares himself able to decide: $\exists m[\neg A(m)] \vee \forall m[A(m)]$, and is probably reckless.
(This example is related to the one given in section 1 of *Veldman 1982*).)
It is evident from theorem 5.4 that the class of all almost full subsets of $\mathbb{N}$ is a filter: $\mathbb{N}$ itself is of course almost full; for all subsets A, B of $\mathbb{N}$: if A is almost full and $A \subseteq B$, then B is almost full, and:

if both A and B are almost full, then $A \cap B$ is almost full.

Observe that Theorem 5.4 is classically equivalent to the pigeonhole principle which says that for every subset A of $\mathbb{N}$, either A or its complement is infinite. Theorem 5.4 *implies this principle*, as, for any subset A of $\mathbb{N}$, if neither A nor its complement have an infinite subset, both $\mathbb{N}\backslash A$ and A are almost full, which is contradictory, according to Theorem 5.4. Theorem 5.4 *is implied by this principle*, as, for all subsets A, B of $\mathbb{N}$, if both A and B are almost full, every infinite subset of $\mathbb{N}$ contains an infinite subset of A and therefore a member of $A \cap B$.

6. The intuitionistic Ramsey theorem

We start with an application of the Intuitionistic Pigeonhole Principle.

6.1 Lemma: For all binary relations R, T on $\mathbb{N}$:
If $\forall \alpha \in \mathcal{S} \exists m \exists n[m < n \ \wedge \ R(\alpha(m), \alpha(n))]$ and $\forall \alpha \in \mathcal{S} \exists m \exists n[m < n \ \wedge \ T(\alpha(m), \alpha(n))]$, then $\forall \alpha \in \mathcal{S} \exists m \exists n \exists p[m < n \ \wedge \ m < p \ \wedge \ R(\alpha(m), \alpha(n)) \ \wedge \ T(\alpha(m), \alpha(p))]$

Proof: Let R, T be binary relations on $\mathbb{N}$ which fulfil the requirements of the theorem.
Let $\alpha \in S$. Define subsets A_α, B_α of $\mathbb{N}$ by:
$A_\alpha := \{m \in \mathbb{N} \mid \exists n[m < n \ \wedge \ R(\alpha(m), \alpha(n))]\}$ and
$B_\alpha := \{m \in \mathbb{N} \mid \exists n[m < n \ \wedge \ T(\alpha(m), \alpha(n))]\}$.
Observe that both A_α and B_α are almost full subsets of $\mathbb{N}$.
($\forall \beta \in \mathcal{S} \exists m \exists n[m < n \ \wedge \ R(\alpha \circ \beta(m), \alpha \circ \beta(n))]$, therefore: $\forall \beta \in \mathcal{S} \exists m[A_\alpha(\beta(m))]$.)
Using theorem 5.4, we conclude: $A_\alpha \cap B_\alpha$ is almost full, in particular:
$\exists m[A_\alpha(m) \wedge B_\alpha(m)]$, i.e.:
$\exists m \exists n \exists p[m < n \ \wedge \ m < p \ \wedge \ R(\alpha(m), \alpha(n)) \ \wedge \ T(\alpha(m), \alpha(p))]$.
◇

6.2 Corollary: For all binary relations R on $\mathbb{N}$:

$$\neg \forall \alpha \in \mathcal{S} \exists m \exists n \exists p \exists q[m < n \ \wedge \ R(\alpha(m), \alpha(n)) \ \wedge \ p < q \ \wedge \ \neg R(\alpha(p), \alpha(q))].$$

Proof: Let R be a binary relation on $\mathbb{N}$ such that
$\forall \alpha \in \mathcal{S} \exists m \exists n \exists p \exists q[m < n \ \wedge \ R(\alpha(m), \alpha(n)) \ \wedge \ p < q \ \wedge \ \neg R(\alpha(p), \alpha(q))]$.
Applying lemma 6.1, we find:
$\forall \alpha \in \mathcal{S} \exists m \exists n \exists p[m < n \ \wedge \ m < p \ \wedge \ R(\alpha(m), \alpha(n)) \ \wedge \ \neg R(\alpha(m), \alpha(p))]$.
Therefore $\forall \alpha \in \mathcal{S} \exists n[\langle \alpha(0), \alpha(1), \ldots, \alpha(n-1) \rangle$ is not R-homeogeneous].
This leads to a contradiction, according to Lemma 4.9.

◇

Corollary 6.2 is the result we announced in 4.10. The second author, when he first proved Corollary 6.2 for *decidable* binary relations, called it the intuitionistic Ramsey theorem, as it is equivalent to Ramsey's theorem when one reads it classically. We now prefer to reserve this name for the stronger theorem 6.5.
In the proof of this stronger theorem, we apply the so-called finite Ramsey theorem, which we state here as another lemma. In most classical proofs of the infinite Ramsey theorem, one does not use the finite Ramsey theorem. One even goes the other way around, proving the finite Ramsey theorem from the infinite one by a so-called *compactness* argument, (cf. *Graham, Rothschild and Spencer, 1980, esp. pp. 13-17)*. (In Section 9 we will reconstruct this compactness argument intuitionistically, using the fan theorem.) Ramsey himself however, gave the proof of the infinite theorem only to prepare the reader for the finite theorem, which was the one he needed.

6.3 Lemma *(Finite Ramsey theorem)*:
For all natural numbers n, k there exists a natural number N such that for every function g from $\{0,1,\ldots,N-1\}\times\{0,1,\ldots,N-1\}$ to $\{0,1,\ldots,k-1\}$ there exists a subset A of $\{0,1,\ldots,N-1\}$ of at least n elements which is g-homogeneous, i.e.: such that for all $p,q,r,s \in A$ such that $p < q$ and $r < s$: $g(p,q) = g(r,s)$.

We do not give the proof. Ramsey himself, indicating how, given the numbers n, k, the number N may be calculated, already argued constructively.
In the sequel, we frequently follow the set theoretic convention of identifying a natural number n with the set of its predecessors $\{0,1,\ldots,n-1\}$.

6.4 Lemma: For all binary relations R, T on $\mathbb{N}$:
If $\forall\alpha \in \mathcal{S}\exists m\exists n[m < n \wedge R(\alpha(m),\alpha(n))]$
and $\forall\alpha \in \mathcal{S}\exists m\exists n[m < n \wedge T(\alpha(m),\alpha(n))]$,
then $\exists m\exists n[m < n \wedge R(m,n) \wedge T(m,n)]$.

Proof: Let R, T be binary relations on $\mathbb{N}$ which fulfil the requirements of the lemma. We define a subset P of the set S of strictly increasing finite sequences of natural numbers: for each $a = \langle a(0), a(1),\ldots,a(n-1)\rangle \in S$:

$$P(a) := \exists i < n\exists j < n\exists k < n[i < j \wedge i < k \wedge R(a(i),a(j)) \wedge T(a(i),a(k))]$$

Observe that, in consequence of Lemma 6.1, P is a monotone bar in S. We define a proposition QED, as follows: $QED := \exists m\exists n[m < n \wedge R(m,n) \wedge T(m,n)]$

We also define a subset Q of the set S of strictly increasing finite sequences of natural numbers: for each $a \in S$: $Q(a) := \forall \gamma \in \mathcal{S} \exists n[P(a * \langle \gamma(n) \rangle) \vee QED]$
Observe that $P \subseteq Q$.
We claim that Q is inductive, i.e.: $\forall a \in S[\forall q[S(a * \langle q \rangle) \rightarrow Q(a * \langle q \rangle)] \rightarrow Q(a)]$

Proof of this claim.
Let $a = \langle a(0), a(1), \ldots, a(n-1) \rangle$ be such that $\forall q[S(a * \langle q \rangle) \rightarrow Q(a * \langle q \rangle)]$. Using lemma 6.3, the finite Ramsey-theorem, we calculate $N \in \mathbb{N}$ such that, for each function g from $N \times N$ to n there exists $i, j, k < N$ such that $i < j < k$ and $g(i,j) = g(i,k) = g(j,k)$. Now let $\gamma \in \mathcal{S}$. Calculate $n_0 := \mu p[S(a * \langle \gamma(p) \rangle)]$. As $Q(a * \langle \gamma(n_0) \rangle)$, we calculate $n_1 \in \mathbb{N}$ such that $P(a * \langle \gamma(n_0), \gamma(n_1) \rangle) \vee QED$. As $Q(a * \langle \gamma(n_0) \rangle)$ and $Q(a * \langle \gamma(n_1) \rangle)$, we calculate, using theorem 5.4, the Intuitionistic Pigeonhole Principle, $n_2 \in \mathbb{N}$, such that both $P(a * \langle \gamma(n_0), \gamma(n_2) \rangle) \vee QED$ and $P(a * \langle \gamma(n_1), \gamma(n_2) \rangle) \vee QED$. As $\forall i < 3[Q(a * \langle \gamma(n_i) \rangle)]$, we calculate, using theorem 5.4, $n_3 \in \mathbb{N}$ such that
$\forall i < 3[P(a * \langle \gamma(n_i), \gamma(n_3) \rangle)) \vee QED]$. We continue this procedure for N steps. In the end, we have defined natural numbers $n_0, n_1, \ldots, n_{N-1}$ such that:
$\forall i < N \forall j < N[i < j \rightarrow (P(a * \langle \gamma(n_i), \gamma(n_j) \rangle) \vee QED)]$.
We distinguish two cases:
Case (i): QED
Case (ii): $\forall i < N \forall j < N[i < j \rightarrow P(a * \langle \gamma(n_i), \gamma(n_j) \rangle)]$.
We define, for each $i, j < N$ such that $i < j$: $a_{i,j} := a * \langle \gamma(n_i), \gamma(n_j) \rangle$
We build, in finitely many steps, three functions, f, g, h, from N^2 to $n+2$ such that, for all $i, j < N$ such that $i < j$: $f(i,j) < g(i,j)$ and $f(i,j) < h(i,j)$ and $R\Big(a_{i,j}(f(i,j)), a_{i,j}(g(i,j))\Big)$ and $T\Big(a_{i,j}(f(i,j)), a_{i,j}(h(i,j))\Big)$. We distinguish three subcases:
Case (ii)a: $\exists i < N \exists j < N[i < j \ \wedge \ g(i,j) = h(i,j)]$
Let $i, j < N$ be such numbers. Let $s := a_{i,j}(f(i,j))$ and $t := a_{i,j}(g(i,j))$.
Observe: $R(\langle s, t \rangle) \wedge T(\langle s, t \rangle)$, therefore: QED.
Case (ii)b: $\exists i < N \exists j < N[i < j \ \wedge \ (g(i,j) < n \ \vee \ h(i,j) < n)]$. Let $i, j < N$ be such numbers.
We distinguish two cases:

- $g(i,j) = n+1$ or $h(i,j) = n+1$. Then: $P(a * \langle \gamma(n_j) \rangle)$
- $g(i,j) \leq n$ and $h(i,j) \leq n$. Then: $P(a * \langle \gamma(n_i) \rangle)$.

In both cases: $\exists p[P(a * \langle \gamma(p) \rangle)]$.
Case (ii)c: $\forall i < N \forall j < N[i < j \rightarrow \{g(i,j), h(i,j)\} = \{n, n+1\}]$.
Then $\forall i < N \forall j < N[i < j \rightarrow f(i,j) < n]$.
In view of our choice of N, let $i, j, k < N$ be such that $i < j < k$ and $f(i,j) = f(i,k) = f(j,k)$.

Now, let $s := a(f(i,j))$, $t := \gamma(n_i)$, $u := \gamma(n_j)$ and $v := \gamma(n_k)$.
Observe: $(R(s,t) \vee R(s,u)) \wedge (R(s,t) \vee R(s,v)) \wedge (R(s,u) \vee R(s,v))$.
Spelling this out, we find two sequences from $\{\langle s,t\rangle, \langle s,u\rangle, \langle s,v\rangle\}$ that belong to R.
In a similar way, we find two sequences from $\{\langle s,t\rangle, \langle s,u\rangle, \langle s,v\rangle\}$ that belong to T.
Combining, we find a sequence from $\{\langle s,t\rangle, \langle s,u\rangle, \langle s,v\rangle\}$ that belongs to $R \cap T$.
We conclude: $\exists p[P(a * \langle p\rangle)]$ and QED.
In all cases, we have: $\exists p[P(a * \langle\gamma(p)\rangle) \vee QED]$.
This ends the proof of our claim, that Q is inductive.

Using the principle of induction on monotone bars, $\mathbf{BI}'_{\mathbf{M}}$ (cf. 2.5). we conclude: $Q(\langle\ \rangle)$,
i.e.: $\forall\gamma \in \mathcal{S}\exists n[P(\langle\gamma(n)\rangle) \vee QED]$, therefore QED,
i.e.: $\exists m\exists n[m < n \wedge R(m,n) \wedge T(m,n)]$.
◇

6.5 Theorem: *(Intuitionistic Ramsey Theorem):*
For all binary relations R, T on $\mathbb{N}$:
If $\forall\alpha \in \mathcal{S}\exists m\exists n[m < n \wedge R(\alpha(m), \alpha(n))]$
and $\forall\alpha \in \mathcal{S}\exists m\exists n[m < n \wedge T(\alpha(m), \alpha(n))]$,
then $\forall\alpha \in \mathcal{S}\exists m\exists n[m < n \wedge R(\alpha(m), \alpha(n)) \wedge T(\alpha(m), \alpha(n))]$.

Proof: Let R, T be binary relations on $\mathbb{N}$ which fulfil the requirements of the theorem.
Let $\alpha \in \mathcal{S}$. Define binary relations R_α, T_α on $\mathbb{N}$ by:
for all $m, n \in \mathbb{N}$: $R_\alpha(m,n) := R(\alpha(m), \alpha(n))$ and $T_\alpha(m,n) = T(\alpha(m), \alpha(n))$.
Observe that $\forall\beta \in \mathcal{S}\exists m\exists n[m < n \wedge R_\alpha(\beta(m), \beta(n))]$ and
$\forall\beta \in \mathcal{S}\exists m\exists n[m < n \wedge T_\alpha(\beta(m), \beta(n))]$. Use lemma 6.4 and conclude:
$\exists m\exists n[m < n \wedge R(\alpha(m), \alpha(n)) \wedge T(\alpha(m), \alpha(n))]$.
◇

6.6 Let R be a binary relation on $\mathbb{N}$. We define

$$R \text{ is } \textit{almost full} := \forall\alpha \in \mathcal{S}\exists m\exists n[m < n \wedge R(\alpha(m), \alpha(n))].$$

Theorem 6.5 says that the class of all almost full binary relations on $\mathbb{N}$ is a filter. Observe that Theorem 6.5 is classically equivalent to Ramsey's theorem. *It implies Ramsey's theorem*, as, for any binary relation R on $\mathbb{N}$, if there is no R-homogeneous subset of $\mathbb{N}$, both R and its complement are almost full, which is contradictory, according to theorem 6.5. *It is implied by Ramsey's theorem* as, for all binary relations R, T on $\mathbb{N}$, if both R and T are almost full, every infinite subset A of $\mathbb{N}$ contains an infinite subset B such that $\forall m \in B \forall n \in B[m < n \rightarrow R(m,n)]$, and there are $m,n \in B$ such that $m < n$ and $T(m,n)$.

7. Generalization of the Intuitionistic Ramsey Theorem

7.1 We consider finite sequences of natural numbers as functions a whose domain, Dom (a), is a natural number $n = \{0, 1, \ldots, n-1\}$.
For each $b = \langle b(0), b(1), \ldots b(m-1)\rangle \in \mathbb{N}^*$ and each $a = \langle a(0), a(1), \ldots, a(n-1)\rangle \in \mathbb{N}^*$ we define $b \circ a \in \mathbb{N}^*$ by: Dom $(b \circ a) = \{k \in \mathbb{N} \mid k < n \ \wedge \ \forall i \leq k[a(i) < m]\}$ and $\forall k \in$ Dom $(b \circ a)[b \circ a(k) = b(a(k))]$.
$b \circ a$ is called the *composition* of b and a.
For each $\alpha \in \mathcal{N}$ and each $a = \langle a(0), \ldots a(n-1)\rangle \in \mathbb{N}^*$ we may form the finite sequence $\alpha \circ a := \langle \alpha(a(0)), \alpha(a(1)), \ldots, \alpha(a(n-1))\rangle$. The finite sequence $\alpha \circ a$ is called the composition of α and a.
For each $k \in \mathbb{N}$, S_k will denote the set of strictly increasing finite sequences of natural numbers of length k (equivalently, with k as domain).

7.2 Let $k \in \mathbb{N}$ and let R be a k-ary relation on $\mathbb{N}$.
We define

$$R \text{ is } \textit{almost full} := \forall \alpha \in \mathcal{S} \exists a \in S_k[R(\alpha \circ a)]$$

7.3 Theorem *(Generalized Ramsey Theorem)*:

For each $k \in \mathbb{N}$, $k > 0$: If R and T are almost full k-ary relations on $\mathbb{N}$, then $R \cap T$ is an almost full k-ary relation on $\mathbb{N}$.

Proof: The proof is by induction.
The cases $k = 1$, $k = 2$ have been treated in Theorems 5.4 and 6.5 respectively.
Assume that $k \in \mathbb{N}$ and that the theorem has been proved for the cases $1, \ldots, k$.
We sketch the proof for the case $k+1$. The proof is in three steps. Let R, T be almost full $k+1$-ary relations on $\mathbb{N}$. Without loss of generality, we may assume that both R and T are subsets of S_{k+1}.

Step one (cf. Lemma 6.1):
Define k-ary relations R^- and T^- on $\mathbb{N}$ by:
for all $a \in \mathbb{N}^k$: $R^-(a) := \exists n[R(a * \langle n\rangle)]$ and: $T^-(a) := \exists n[T(a * \langle n\rangle)]$
Observe that R^- and T^- are almost full.
Using the induction hypothesis we conclude that $R^- \cap T^-$ is almost full, in particular: $\exists a \in S_k[R^-(a) \ \wedge \ T^-(a)]$.

We generalize this result as follows: let $\gamma \in \mathcal{S}$ and define k-ary relations R_γ^- and T_γ^- on $\mathbb{N}$ by: for all $a \in \mathbb{N}^k$: $R_\gamma^-(a) := \exists n[R(\gamma \circ (a * \langle n \rangle))]$ and:
$T_\gamma^-(a) := \exists n[T(\gamma \circ (a * \langle n \rangle))]$
Again, R_γ^- and T_γ^-, and therefore: $R_\gamma^- \cap T_\gamma^-$ are almost full,
in particular: $\exists a \in S_k[R_\gamma^-(a) \wedge T_\gamma^-(a)]$.

Step two (cf. Lemma 6.4):
Define a subset P of $\mathbb{N}^*$ by:
for all $a \in \mathbb{N}^*$: $P(a) := \exists b \in S_k \exists m \exists n[R(a \circ (b * \langle m \rangle)) \wedge T(a \circ (b * \langle n \rangle))]$
In **Step one**, we have seen: P is a monotone bar in $\mathcal{S}$.
We define a proposition QED by: $QED := \exists a \in S_{k+1}[R(a) \wedge T(a)]$
We define a subset Q of $\mathbb{N}^*$ by:
For all $a \in \mathbb{N}^* : Q(a) := \forall \gamma \in \mathcal{S} \exists n[P(a * \langle \gamma(n) \rangle) \vee QED]$
Observe that $P \subseteq Q$.
Arguing very much like we did in the proof of Lemma 6.4, we may establish, applying once more the finite Ramsey theorem (Lemma 6.3), that Q is inductive.
Using $\mathbf{BI}'_\mathbf{M}$, the principle of induction on monotone bars (cf. 2.5), we conclude: $Q(\langle\,\rangle)$, and therefore: QED, i.e.: $\exists a \in S_{k+1}[R(a) \wedge T(a)]$.

Step three (cf. Theorem 6.5):
Let $\gamma \in \mathcal{S}$ and consider the subsets R_γ and T_γ of S_{k+1} defined by:
for all $a \in S_{k+1} : R_\gamma(a) := R(\gamma \circ a)$ and $T_\gamma(a) := T(\gamma \circ a)$
Observe that R_γ and T_γ are almost full $k{+}1$-ary relations on $\mathbb{N}$. Using the result of step two, we conclude: $\exists a \in S_{k+1}[R_\gamma(a) \wedge T_\gamma(a)]$, i.e : $\exists a \in S_{k+1}[R(\gamma \circ a) \wedge T(\gamma \circ a)]$.
Therefore: $\forall \gamma \in \mathcal{S} \exists a \in S_{k+1}[R(\gamma \circ a) \wedge T(\gamma \circ a)]$, i.e.: $R \cap T$ is almost full.
◇

8. Some remarks on the recursion-theoretic counterexamples

In this section, we examine the construction, due to C.G. Jockusch (cf. *Jockusch* 1972), of a recursive binary relation R on $\mathbb{N}$ such that every infinite recursive subset W of $\mathbb{N}$ contains numbers m, n such that $m < n$ and $R(m,n)$ *and* numbers p, q such that $p < q$ and $\neg R(p,q)$. (The earlier counterexample to Ramsey's theorem found by E.R. Specker (cf. *Specker* 1971) is a bit more involved, as it is based on the existence of two recursively enumerable sets of incomparable degrees of unsolvability.)
We will see that this construction is closely related to the construction of a Δ^0_2-subset A of $\mathbb{N}$ such that every infinite recursive subset W of $\mathbb{N}$ is not a subset of A and not

a subset of $\mathbb{N}\backslash A$. (A subset A of $\mathbb{N}$ with the latter property is sometimes called a *bi-immune* set.)
We want to explore the intuitionistic meaning of these constructions. Moreover, we will draw some information from them concerning the formal strength of our axioms.
In doing so, we reinforce a remark by the second author, that the double negation of the classical Ramsey theorem, i.e. the statement that, for all binary relations R on $\mathbb{N}$
$\neg\neg\exists\alpha \in S[\forall m\forall n[m < n \to R(\alpha(m), \alpha(n))] \vee \forall m\forall n[m < n \to \neg R(\alpha(m), \alpha(n))]]$
is not derivable in any of the usual formalizations of intuitionistic analysis.

8.1 The question as to the existence and complexity of bi-immune subsets of $\mathbb{N}$ is connected with the following more general problem:

> Given some sequence $V_0, V_1, V_2, \ldots$ of subsets of $\mathbb{N}$, find subsets A, B of $\mathbb{N}$ such that $A \cap B = \emptyset$ and: $\forall n[V_n$ is infinite $\to (V_n \cap A \neq \emptyset \wedge V_n \cap B \neq \emptyset)]$

Classically, one easily forms such sets A, B by going through the sequence $V_0, V_1, V_2, \ldots$, disregarding the finite sets among them, and choosing from each infinite set two elements which are both different from every element chosen from one of the earlier mentioned sets. One lets A consist of the first members of all pairs thus found, and B of the second members. Intuitionistically, the main difficulty with this procedure is that we cannot decide in general if a given set is infinite or not. In some special cases, however, this difficulty is absent.

8.2 Recall from 7.1, that, for each $n \in \mathbb{N}$, S_n is the set of all strictly increasing finite sequences of natural numbers of length n. We call a subset X of $\mathbb{N}$ *finite* if and only if there exist $n \in \mathbb{N}$ and $a \in S_n$ such that $X = \{a(0), a(1), \ldots, a(n-1)\}$.
We also define, for each subset X of $\mathbb{N}$ and each $n \in \mathbb{N}$:

$$\#X \geq n \ (X \text{ has at least } n \text{ elements }) := \exists a \in S_n \forall j < n[a(j) \in X].$$

Observe that, for each finite subset X of $\mathbb{N}$ and each $n \in \mathbb{N}$, we may decide:
$\#X \geq n \ \vee \ \neg(\#X \geq n)$.

8.3 Lemma: There exists a functional F which assigns to each function f from $\mathbb{N}$ to the set of finite subsets of $\mathbb{N}$ a one-to-one sequence $\alpha = F(f) \in \mathcal{N}$ such that: $\forall n[\#f(n) \geq 2n+2 \to (\alpha(2n) \in f(n) \wedge \alpha(2n+1) \in f(n))]$.

Proof: Let $f = f(0), f(1), f(2), \ldots$ be a sequence of finite subsets of $\mathbb{N}$. We define $\alpha = F(f)$ by recursion:
for each $n \in \mathbb{N}$:

if $\#f(n) \geq 2n+2$, then $\alpha(2n) := \mu p \in f(n)[\forall j < 2n[p \neq \alpha(j)]])$ and
$\alpha(2n+1) := \mu p \in f(n)[\forall j < 2n+1[p \neq \alpha(j)]]$,
and, if $\neg(\#f(n) \geq 2n+2)$, then $\alpha(2n) := \mu p \in \mathbb{N}[\forall j < 2n[p \neq \alpha(j)]]$ and
$\alpha(2n+1) := \mu p \in \mathbb{N}[\forall j < 2n+1[p \neq \alpha(j)]]$.
Note that, if $\neg(\#f(0) \geq 2)$, then $\alpha(0) = 0$ and $\alpha(1) = 1$.

◇

8.4 We call a subset V of $\mathbb{N}$ *enumerable* if and only if there exists a sequence $V_0, V_1, V_2, \ldots$ of finite subsets of $\mathbb{N}$ such that $V = \bigcup_{m \in \mathbb{N}} V_m$. In the following, we restrict attention to *recursively enumerable* subsets of $\mathbb{N}$.
We know from recursion theory that there exists a (recursive) *universal double-sequence* $(W_{n,m})_{n,m \in \mathbb{N}}$ of finite subsets of $\mathbb{N}$, such that $\forall n \forall m[W_{n,m} \subseteq W_{n,m+1}]$ and such that, if we define, for each $n \in \mathbb{N}$, $W_n := \bigcup_{m \in \mathbb{N}} W_{n,m}$, then $W_0, W_1, W_2, \ldots$ is a complete list of the recursively enumerable subsets of $\mathbb{N}$.
We use such a universal double-sequence in the statement and the proof of the next theorem.

8.5 Theorem:
(i) There exist recursive binary relations R, T on $\mathbb{N}$ such that:
(i)a $R \cap T = \emptyset$
(i)b $\forall n[\#W_n \geq 2n+2 \rightarrow \exists m \forall q \geq m \exists r \in W_n \exists s \in W_n[R(r,q) \wedge T(s,q)]]$
(i)c $\forall n[\#W_n \geq 2n+2 \rightarrow \neg\neg\exists r \in W_n \exists s \in W_n \exists m \forall q \geq m[R(r,q) \wedge T(s,q)]]$
(i)d $\forall p \neg\neg\exists m[\forall q \geq m[R(p,q)] \vee \forall q \geq m[T(p,q)]]$
(ii) There exist $\sum_2^0$-subsets A, B of $\mathbb{N}$ such that:
(ii)a $A \cap B = \emptyset$
(ii)b $\forall n[\#W_n \geq 2n+2 \rightarrow \neg\neg\exists r \in W_n \exists s \in W_n[r \in A \wedge s \in B]]$
(ii)c $\forall p[\neg\neg(p \in A \vee p \in B)]$

Proof:
Let $(W_{n,m})_{n,m \in \mathbb{N}}$ be the universal double sequence that we mentioned in 8.4.
Define, for each $m \in \mathbb{N}$, a sequence f_m of finite subsets of $\mathbb{N}$ by:
for all $n \in \mathbb{N}$: $\quad f_m(n) := W_{n,m}$.
Let F be the functional defined in the proof of Lemma 8.3. Define, for each $m \in \mathbb{N}$: $\alpha_m := F(f_m)$.
Define binary relations R, T on $\mathbb{N}$ by:
$R := \{\langle \alpha_m(2n), m\rangle \mid m, n \in \mathbb{N}\}$ and $T := \{\langle \alpha_m(2n+1), m\rangle \mid m, n \in \mathbb{N}\}$
We claim that R, T fulfil the requirements.

Proof of this claim:

(i)a According to Lemma 8.3, each α_m is injective, therefore: $R \cap T = \emptyset$.

(i)b Assume $n \in \mathbb{N}$ and $\#W_n \geq 2n{+}2$. Calculate $m \in \mathbb{N}$ such that $\#W_{n,m} \geq 2n{+}2$. Observe that, for each $q \geq m$, $\#W_{n,q} \geq 2n{+}2$, and therefore: $\alpha_q(2n) \in W_n$ and $\alpha_q(2n{+}1) \in W_n$ and $R(\alpha_q(2n), q)$ and $T(\alpha_q(2n+1), q)$.

(i)c Assume $n \in \mathbb{N}$ and $\#W_n \geq 2n{+}2$.

We have to prove a *negative* conclusion, viz.:

$\neg\neg\exists r \in W_n \exists s \in W_n \exists m \forall q \geq m[R(r,q) \wedge T(s,q)]$

Therefore, as we saw in Section 4.2, we assume *without risk*:

$\forall j \leq n[\#W_j \geq 2j{+}2 \vee \neg(\#W_j \geq 2j{+}2).]$

Calculate $p \in \mathbb{N}$ such that $\forall j \leq n[\#W_j \geq 2j{+}2 \leftrightarrow \#W_{j,p} \geq 2j{+}2]$.

Calculate $t \in \mathbb{N}$ such that $\forall j \leq n \forall x \in W_{j,p}[x \leq t]$.

We make another *harmless* assumption: $\forall x \leq t \forall j \leq n[x \in W_j \vee \neg(x \in W_j)]$.

Calculate $m \in \mathbb{N}$ such that: $\forall x \leq t \forall j \leq n[x \in W_j \rightarrow x \in W_{j,m}]$. Observe that, for each $j \leq n$, if W_j has at least $2j{+}2$ elements, then the first $2j{+}2$ elements of W_j belong to $W_{j,m}$ already. From the proof of Lemma 8.3 and the fact that $\forall j \leq 2n+2 \forall q \geq m[W_{j,m} \subseteq W_{j,q}]$ we conclude:

$\forall j \leq 2n{+}2 \forall q \geq m[\alpha_q(j) = \alpha_m(j)]$.

We define $r := \alpha_m(2n)$ and $s := \alpha_m(2n{+}1)$. Observe: $\forall q \geq m[R(r,q) \wedge T(s,q)]$.

(i)d Let $p \in \mathbb{N}$. The set $\{x \in \mathbb{N} \mid x \geq p\}$ is recursively enumerable and, being so, it occurs in the sequence $W_0, W_1, W_2, \ldots$. Calculate $n \in \mathbb{N}$ such that $W_n = \{x \in \mathbb{N} \mid x \geq p\}$. As in (i)c, we are striving for a negative conclusion. Strengthening our assumptions and reasoning as in the proof of (i)c, we find $m \in \mathbb{N}$ such that, for all $j \leq n$, if W_j has at least $2j{+}2$ elements, then the first $2j{+}2$ elements of W_j belong to $W_{j,m}$ already.

Observe that p is the first element of the infinite set W_n and also the first element of $W_{n,m}$. Therefore, we may calculate $i < 2n{+}2$ such that $p = \alpha_m(i)$.

Observe: $\forall q \geq m[\alpha_q(i) = \alpha_m(i)]$

Either i is even, and $\forall q \geq m[R(p,q)]$, or i is odd and $\forall q \geq m[T(p,q)]$.

End of proof of claim.

Define subsets A, B of $\mathbb{N}$ by:

$A := \{p \in \mathbb{N} \mid \exists m \forall q \geq m[R(p,q)]\}$ and $B := \{p \in \mathbb{N} \mid \exists m \forall q \geq m[T(p,q)]\}$

(ii)a,b,c follow straightforwardly from (i)a,c,d.

One easily sees that R, T are recursive, and A, B are $\sum_2^0$.

◇

8.6 We may draw some metamathematical conclusions from Theorem 8.5. Let **EL** be the formal system for intuitionistic analysis which is explained in *Troelstra and van Dalen 1988.*

Let **CT** be Church's thesis in the following form:

$$\forall\alpha\exists e\forall n\exists z[T(e,n,z)\wedge U(z)=\alpha(n)].$$

(T is the recursive subset of $\mathbb{N}^3$, introduced by S.C. Kleene, U is a total recursive function from $\mathbb{N}$ to $\mathbb{N}$, the so-called result-extracting function.)
Let $\mathbf{CT_0}$ be Church's thesis in the following schematic form:

For each *arithmetically definable* binary relation R on $\mathbb{N}$:
If $\forall m\exists n[R(m,n)]$, then $\exists e\forall m\exists z[T(e,m,z)\wedge R(m,U(z))]$.

8.6.1 Observe that in **EL+CT**, Theorem 8.5(i) plainly contradicts the Intuitionistic Ramsey Theorem 6.5. (Observe that, in **EL+CT**, Theorem 8.5(i) says that there exist almost full binary relations R,T on $\mathbb{N}$ such that $R\cap T=\emptyset$.) Therefore: $\mathbf{EL+CT+BI_M}$ is inconsistent.
This is a wellknown fact, first shown by S.C. Kleene, who in *Kleene 1952* gave an example of a recursive subtree T of the binary tree $\{0,1\}^*$, which has arbitrarily long finite branches, but is such that we may calculate, for each recursive infinite branch, an initial part that does not belong to T.
Remark that, as **EL+CT** is consistent, it is impossible to prove the Intuitionistic Ramsey Theorem 6.5 in **EL**.

8.6.2 Let A,B be subsets of $\mathbb{N}$ which have the properties mentioned in theorem 8.5(ii). We claim that, in $\mathbf{EL+CT_0}$, the assumption $\exists\alpha\in\mathcal{S}\forall n[A(\alpha(n))\leftrightarrow A(\alpha(0))]$ leads to a contradiction.
(Proof of this claim: Assume: $\alpha\in\mathcal{S}$ and $\forall n[A(\alpha(n))\leftrightarrow A(\alpha(0))]$. With a view to our negative goal we *distinguish two cases.*
Case (i): $A(\alpha(0))$. Then $\forall n[A(\alpha(n))]$, therefore $\forall m\exists n>m[A(n)]$.
Applying $\mathbf{CT_0}$, we find a total recursive function f from $\mathbb{N}$ to $\mathbb{N}$ such that $\forall m[f(m)>m\ \wedge\ A(f(m))]$.
Determine $e\in\mathbb{N}$ such that $W_e=\{f(m)\mid m\in\mathbb{N}\}$.
Observe: $\#W_e\geq 2e{+}2$ and $\forall n\in W_e[A(n)]$, in contradiction with 8.5(ii)a,b.
Case (ii): $\neg A(\alpha(0))$. Reasoning as in case (i), we find $e\in\mathbb{N}$ such that $\#W_e\geq 2e{+}2$ and $\forall n\in W_e[\neg A(n)]$, again in contradiction with 8.5(ii)b.
End of proof of claim.)
It follows from *Kreisel and Troelstra 1970* that the system $\mathbf{EL+BI_M+CT_0}$ is consistent. Therefore, we are unable to prove, in this system:

$$\neg\neg\exists\alpha\in\mathcal{S}\forall n[A(\alpha(0))\leftrightarrow A(\alpha(n))].$$

$\mathbf{CT_0}$ may be consistently added to much stronger systems for intuitionistic analysis, for instance the system **FIM** of *Kleene and Vesley 1965.* In no such system the above-mentioned statement is provable.

8.6.3 Let us consider the following form of Markov's principle.

> **MP**: For every ***arithmetically definable*** subset A of $\mathbb{N}$:
> If $\forall n[A(n) \vee \neg A(n)]$ and $\neg\neg\exists n[A(n)]$, then $\exists n[A(n)]$.

We claim that, in $\mathbf{EL}+\mathbf{CT_0}+\mathbf{MP}$ one may prove:

> For every ***arithmetically definable*** subset A of $\mathbb{N}$:
> If $\forall n[A(n) \ \vee \ \neg A(n)]$, then $\neg\neg\exists\alpha \in \mathcal{S}\forall n[A(\alpha(0)) \leftrightarrow A(\alpha(n))]$.

(Proof of this claim: Assume: $\forall n[A(n) \ \vee \ \neg A(n)]$.
According to Lemma 4.4: $\neg\neg$ (A is weakly infinite $\vee$ $\mathbb{N}\backslash A$ is weakly infinite). As we have to prove a negative conclusion, we ***distinguish two cases***:
Case (i): A is weakly infinite, i.e.: $\forall p\neg\neg\exists q[p < q \ \wedge \ A(q)]$.
Using **MP**, we conclude: $\forall p\exists q[p < q \ \wedge \ A(q)]$.
Using $\mathbf{CT_0}$, we find a recursive function f from $\mathbb{N}$ to $\mathbb{N}$ such that $\forall p \in \mathbb{N}[p < f(p) \ \wedge \ A(f(p))]$.
Define $\alpha \in \mathcal{S}$ by: $\alpha(0) = f(0)$ and for all $n \in \mathbb{N}$: $\alpha(n+1) = f(\alpha(n))$.
Then: $\forall n[A(\alpha(0)) \leftrightarrow A(\alpha(n))]$
Case (ii): $\mathbb{N}\backslash A$ is weakly infinite, is treated in a similar way.
End of proof of claim.)
As $\mathbf{EL}+\mathbf{CT_0}+\mathbf{MP}$ is consistent, cf. *Luckhardt 1977*, we conclude: We cannot prove in **EL** that there exist ***decidable*** subsets A, B of $\mathbb{N}$ with the properties mentioned in theorem 8.5(ii). Neither can we do so from any set of axioms to which $\mathbf{CT_0}+\mathbf{MP}$ may be added consistently, as, for instance, the system **FIM** of *Kleene and Vesley 1965.*

8.7 We conclude this section with a remark on the intuitionistic continuity principle. The following strong version of the pigeonhole principle and, a fortiori, the corresponding strong version of Ramsey's theorem, are intuitionistically unprovable:

$$(*) \qquad \forall\beta \in 2^{\mathbb{N}}\exists\alpha \in \mathcal{S}\forall n[\beta(\alpha(0)) = \beta(\alpha(n))]$$

This is clear from the example given in the introduction. It is possible to derive a contradiction from $(*)$ by the following weak principle of continuity:

CP: For all subsets A of $\mathcal{N} \times \mathbb{N}$:

If $\forall\beta\exists n[A(\beta,n)]$, then $\forall\beta\exists m\exists n\forall\gamma[\overline{\gamma}m = \overline{\beta}m \to A(\gamma,n)]$

($\overline{\gamma}m$ denotes the finite sequence $\langle\gamma(0),\gamma(1),\dots,\gamma(m-1)\rangle$).

Our claim is that **EL**+(∗)+**CP** is inconsistent.

(Sketch of the proof: Suppose (∗) and **CP**.
Observe: $\forall\beta \in 2^{\mathbf{N}}\exists n\exists\alpha \in \mathcal{S}[\alpha(0) = n \wedge \forall m[\beta(\alpha(m)) = \beta(n)]]$. Let $\underline{0} \in 2^{\mathbf{N}}$ be such that $\forall p[\underline{0}(p) = 0]$. Calculate $p, n \in \mathbf{N}$ such that
$\forall\gamma \in 2^{\mathbf{N}}[\overline{\gamma}p = \overline{\underline{0}}p \to \exists\alpha \in \mathcal{S}[\alpha(0) = n \wedge \forall m[\gamma(\alpha(m)) = \gamma(n)]]$.
Let N be the maximum of the numbers p, $n+1$ and consider $\beta = \overline{\underline{0}}N * \underline{1}$ ($\underline{1} \in 2^{\mathbf{N}}$ is such that $\forall p \in \mathbf{N}[\underline{1}(p) = 1]$)).
This application of **CP** exemplifies a well-known technique in intuitionistic analysis: weak counterexamples may be used, together with **CP**, to obtain proofs of inconsistency.

9. An application

9.1 We will show how a recent result of Thierry Coquand's (cf. *Coquand 1991*) may be derived from the Intuitionistic Ramsey Theorem. We became aware of this result after the other sections of this paper had been completed. We first extend definition 7.2.

9.2. Let R be a subset of the set S of finite strictly increasing sequences of natural numbers. We define

$$R \text{ is } \textit{almost full} := \forall\alpha \in \mathcal{S}\exists a \in S[R(\alpha \circ a)].$$

(We drop the restriction that all sequences in R are of the same length.)
If we consider R in the obvious way as a set of finite sets of natural numbers, then R is almost full if and only if every infinite set of natural numbers has a subset that belongs to R.
An important example of an almost full subset of S is the following:

$$R_{PH} := \{a \in S \mid a \neq \langle\,\rangle \text{ and length } (a) > a(0)\}.$$

The letters PH refer to *Paris and Harrington 1977.*

9.3. Let R be a subset of S.
Let $k \in \mathbf{N}$ and let δ be a function from S_k to $\{0,1\}$.
We define a subset R^δ of R:

$$R^\delta := \{a \in R \mid \forall b \in S_k\,[b(k-1) < \text{length}(a) \to \delta(a \circ b) = \delta(\langle a(0),\dots,a(k-1)\rangle)]\}.$$

If we consider δ as a colouring of the k-element-subsets of $\mathbf{N}$, then R^δ is the set of δ-monochromatic members of R.

9.4. Theorem (Th. Coquand):

For every subset R of S, for every $k \in \mathbb{N}$, for every $\delta : S_k \to \{0,1\}$:
If R is almost full, then R^δ is almost full.

Proof: Let R be an almost full subset of S, and let $\delta : S_k \to \{0,1\}$.
Let $\alpha \in S$.
Let QED be the proposition: $\exists a \in S[R^\delta(\alpha \circ a)]$.
Let $\gamma \in S$. Determine $a \in S$ such that $R(\alpha \circ \gamma \circ a)$.
We may decide: *either* $R^\delta(\alpha \circ \gamma \circ a)$ and therefore: QED, *or* $\exists b \in S_k[\delta(\alpha \circ \gamma \circ a \circ b) = 0]$.
Therefore: $\forall \gamma \in S \exists a \in S_k[\delta(\alpha \circ \gamma \circ a) = 0 \vee QED\,]$ and, similarly:
$\forall \gamma \in S \exists a \in S_k[\delta(\alpha \circ \gamma \circ a) = 1 \vee QED\,]$.
Apply Theorem 7.3 and conclude QED.
◇

9.5 In order to derive a useful corollary from Theorem 9.4, we need a well-known principle of intuitionistic analysis, the fan theorem. The version of this theorem that we use (*27.9 in *Kleene and Vesley 1965*, p.76, or FAN(T) in *Troelstra and van Dalen 1988*, p.218) is a corollary of the principle induction on monotone bars, $\mathbf{BI_M}$ (cf. 2.4). Let C be intuitionistic Cantor space, i.e. the binary fan, the set of all functions from $\mathbb{N}$ to $\{0,1\}$.

9.5.1 FT (fan theorem):

Let Q be a subset of the set of binary finite sequences such that
$\forall \alpha \in C \exists n[Q(\langle \alpha(0), \ldots, \alpha(n-1) \rangle)]$.
Then $\exists N \forall \alpha \in C \exists n \leq N[Q(\langle \alpha(0), \ldots, \alpha(n-1) \rangle)]$.

9.5.2 Corollary:

Let R be an almost full subset of S, and let $k \in \mathbb{N}$.
Then $\forall \alpha \in S \exists N \forall \delta : S_k \to \{0,1\} \exists a \in S[R^\delta(\langle \alpha(0), \ldots, \alpha(N-1) \rangle \circ a)]$.

Proof: By enumerating S_k, we may identify C with the set of all functions from S_k to $\{0,1\}$.
Now apply 9.4 and 9.5.1.
◇

Observe that Corollary 9.5.2 is an intuitionistic version of the compactness argument, by which in *Paris and Harrington 1977* a sharpened version of the finitary Ramsey theorem is proved. (In order to obtain the desired conclusion, specialize α, in the second line of 9.5.2, to sequences of the form $\lambda m.(m + p)$).

9.6 Thierry Coquand observed that from Theorem 9.4, an important case of Theorem 7.3 may be derived, as follows:
Let $k \in \mathbb{N}$ and suppose that R, T are *decidable* almost full subsets of S_k.
Let $Q := \{a \in S \mid \exists b \in S[R(a \circ b)] \wedge \exists b \in S[T(a \circ b)]\}$. Let $\delta : S_k \to \{0,1\}$ be the characteristic function of T.
Observe that Q is almost full, and that for all $a \in S$, if $Q^\delta(a)$, then $\exists b \in S[R(a \circ b) \wedge T(a \circ b)]$.]
As, by Theorem 9.4, Q^δ is almost full, also $R \cap T$ is almost full.
In the above argument, it suffices to assume that T is a decidable subset of S.
In *Coquand 1991* another constructive version of Ramsey's theorem is stated and proved that avoids the application of the intuitionistic principle of bar induction.

10. Concluding remarks

10.1 As far as we know, the first one who studied Ramsey's theorem from an intuitionistic point of view, was Mervyn Jansen who in Nijmegen, in 1974, wrote a Master's Thesis on the subject, under the guidance of Johan J. de Iongh.
He found the example mentioned in 5.3 and, formulating an admittedly unpleasant ad hoc condition, he proved, using the principle of bar induction, that every binary relation R on $\mathbb{N}$ which satisfies this condition, has the property mentioned in the conclusion of 6.2, i.e.: it is impossible that both R and its complement are almost full.
The second author, who did not know about the earlier attempt by Mervyn Jansen, obtained a stronger result, which he announced in the first "Stelling", added to his dissertation *Bezem 1986*: he proved for every *decidable* binary relation R on $\mathbb{N}$ that not both R and $(\mathbb{N} \times \mathbb{N}) \setminus R$ are almost full. This result sparked off the research which led to the present paper. The first author studied the second author's proof and found Theorems 6.5, 7.3 and 5.4. The negationless wording of these theorems owes something to a question posed by John Burgess: in a letter from 1983, he had asked for a constructive argument establishing $\forall\alpha\forall\beta\exists m\exists n[m < n \wedge \alpha(m) \leq \alpha(n) \wedge \beta(m) \leq \beta(n)]$. (We challenge the reader to find the elementary proof of this special case of theorem 6.5).
In Section 4 of this paper, one finds a paraphrase of the original argument of the second author: Lemma 4.6, Corollary 4.7 and a slight weakening of both Lemma 6.1 and Corollary 6.2 are due to him.
Section 8 of this paper elaborates another observation of the second author, to the effect that from consistency results about Church's Thesis $\mathbf{CT_0}$, one may obtain unprovability results concerning versions of Ramsey's theorem.
The application of the intuitionistic Ramsey theorem in Section 9 is due to the first

author, who also wrote the final version of the paper.

10.2 Further generalizations of the intuitionistic Ramsey Theorem, such as an analogue to the classical *clopen* Ramsey Theorem (cf. *Fraïssé 1986*), are possible. We hope to treat them in a sequel to the present paper.

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