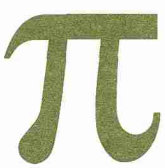
Department of Philosophy - Utrecht University

Bar Recursion versus Polymorphism

Extended abstract



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Extended abstract

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1. Introduction

By constructing a *counter model* we show that a certain appealing equation E has no solution in Girard's [1972] second order lambda calculus (= **λ2T**, or the *polymorphic* lambda calculus). The counter model is a new PER model based on an applicative structure consisting of untyped lambda calculus extended with a specific higher order oracle in the style of Kleene [1962]. A novelty of this model is the presence of discontinuous functionals, which seem to be absent in all known models of **λ2T**. The model construction is interesting in itself and can be applied more widely to obtain independence and consistency results.

The equation E = E(Φ) (with Φ a type **3** variable) is a simple functional equation in the language of Gödel's [1958] system of higher order primitive recursive functionals (= **λT**). The computational interpretation of E(Φ) is that Φ performs a certain minimization.

In contrast to the negative result concerning **λ2T** we show that E has a solution in Spector's [1962] system of *bar recursive* functionals (= **λTB**, the extension of **λT** with bar recursion). This indicates that **λTB** is in some sense stronger than **λ2T**.

Surprisingly, E has a solution in **λT**, but only provably so in **λTB** (thus showing that **λTB** is not conservative over **λT**).

We stress the fact that that our results show that **λ2T** and **λTB** have different classes of definable *functionals* (at least of type **3**), whereas the meta-mathematical results from Spector [1962] and Girard [1972] show that they have the same class of definable *functions*, namely the provably total functions of analysis.

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2. Syntax

2.1. INTRODUCTION. We present the syntax of the following equational systems based on simply typed lambda calculus: $\boldsymbol{\lambda}^\tau, \boldsymbol{\lambda T}, \boldsymbol{\lambda TB}$ and $\boldsymbol{\lambda 2}, \boldsymbol{\lambda 2T}$. To the systems $\boldsymbol{\lambda T}, \boldsymbol{\lambda TB}, \boldsymbol{\lambda 2T}$ we can add quantifier-free arithmetic, which will be expressed by the denotation $\boldsymbol{\lambda\Box_A}$. With $\longrightarrow$ expressing the subsystem relation we can depict the situation as follows.

$$\begin{array}{ccccc} \boldsymbol{\lambda}^\tau & \longrightarrow & \boldsymbol{\lambda T}_{(\mathbf{A})} & \longrightarrow & \boldsymbol{\lambda TB}_{(\mathbf{A})} \quad \text{first order systems} \\ \downarrow & & \downarrow & & \\ \boldsymbol{\lambda 2} & \longrightarrow & \boldsymbol{\lambda 2T}_{(\mathbf{A})} & & \quad \text{second order systems} \end{array}$$

2.2. TYPES. The first and second order systems are distinguished by their respective sets of *types*, $\mathbb{T}_1, \mathbb{T}_2$, given by the following abstract syntax.

$$\begin{aligned} \mathbb{T}_1 &= \mathbf{0} \mid \mathbb{T}_1 \to \mathbb{T}_1, \\ \mathbb{T}_2 &= \mathbf{0} \mid \mathbb{V} \mid \mathbb{T}_2 \to \mathbb{T}_2 \mid \forall \mathbb{V}.\mathbb{T}_2. \end{aligned}$$

Here $\mathbb{V} = \{\alpha, \beta, \alpha', \ldots\}$ is an infinite set of *type variables*. Note that $\mathbb{T}_1 \subset \mathbb{T}_2$. In the sequel, $\sigma, \tau, \sigma', \ldots$ range over $\mathbb{T}_2$. It is convenient to single out some special $\mathbb{T}_1$ types: $\mathbf{1} \equiv \mathbf{0}\to\mathbf{0}$, $\mathbf{2} \equiv \mathbf{1}\to\mathbf{0}$, and so on.

2.3. TERMS. (i) The systems $\boldsymbol{\lambda}^\tau$ and $\boldsymbol{\lambda 2}$ are the well-known simply typed lambda calculus and its second order (polymorphic) extension. Typical examples of $\boldsymbol{\lambda}^\tau$-terms are: $x^{\mathbf{0}}$, $f^{\mathbf{1}}$, $(\lambda f^{\mathbf{1}} x^{\mathbf{0}}.f(fx))(\lambda y^{\mathbf{0}}.y)$. Typical examples of $\boldsymbol{\lambda 2}$-terms are: all typical examples of $\boldsymbol{\lambda}^\tau$-terms, $\Lambda\alpha\lambda x^\alpha.x$ (the so-called polymorphic identity), $(\Lambda\alpha\lambda f^{\alpha\to\alpha} x^\alpha.f(fx))(\forall\alpha.\sigma)$.

(ii) The systems $\boldsymbol{\lambda T}_{(\mathbf{A})}$, $\boldsymbol{\lambda TB}_{(\mathbf{A})}$ (resp. $\boldsymbol{\lambda 2T}_{(\mathbf{A})}$) are based on $\boldsymbol{\lambda}^\tau$ (resp. $\boldsymbol{\lambda 2}$), but allow the occurrence of certain typed *constants* in the terms. These constants are

$$\begin{aligned} \boldsymbol{0}, \boldsymbol{S}, \boldsymbol{R}_\sigma \quad &\text{for the systems} \quad \boldsymbol{\lambda T}_{(\mathbf{A})}, \boldsymbol{\lambda TB}_{(\mathbf{A})}, \boldsymbol{\lambda 2T}_{(\mathbf{A})}, \\ \boldsymbol{B} \quad &\text{for the systems} \quad \boldsymbol{\lambda TB}_{(\mathbf{A})}. \end{aligned}$$

Here $\boldsymbol{0}$ is of type $\mathbf{0}$ and $\boldsymbol{S}$ is of type $\mathbf{1}$, with intended interpretation *zero* respectively the *successor* function (the intended interpretation of $\mathbf{0}$ is the set of natural numbers). We use the abbreviations $\boldsymbol{1} \equiv \boldsymbol{S0}$, $\boldsymbol{2} \equiv \boldsymbol{S}(\boldsymbol{S0})$, and so on. The constants $\boldsymbol{R}_\sigma$ of type $\sigma\to(\mathbf{0}\to\sigma\to\sigma)\to\mathbf{0}\to\sigma$ are added for all types σ of the system in question; their intended interpretation is that of *primitive recursor*. The type of the constant $\boldsymbol{B}$ is $\mathbf{2}\to\mathbf{1}\to(\mathbf{1}\to\mathbf{0}\to\mathbf{0})\to\mathbf{0}\to\mathbf{0}$; intended interpretation of $\boldsymbol{B}$ is *bar recursor*. A typical example of a $\boldsymbol{\lambda T}$-term is $\mathbf{Cond} \equiv \lambda x^{\mathbf{0}} y^{\mathbf{0}} z^{\mathbf{0}}.\boldsymbol{R_0} z(\lambda n^{\mathbf{0}} m^{\mathbf{0}}.y)x$. In the sequel, $\mathrm{Term}_\sigma(\boldsymbol{\lambda\Box})$ stands for the set of $\boldsymbol{\lambda\Box}$-terms of type σ. $F, M, N, M', \ldots$ range over terms. We shall tacitly assume that terms are well-typed and shall reduce type superscripts to a minimum that is needed to reconstruct the types of the constituents of a well-typed term.

2.4. EQUATIONS. The formulas of $\boldsymbol{\lambda}\Box$ are *equations* $M =_\sigma N$ with σ a type of $\boldsymbol{\lambda}\Box$ and M and N terms of $\boldsymbol{\lambda}\Box$ of type σ. Typical examples of such equations can be found in the inference rules that define the theories $\boldsymbol{\lambda}\Box$ below. In the sequel, $E, E', E(x,y), \ldots$ range over equations. We shall omit the type subscript in equations when confusion is not likely.

2.5. THEORY. The theories $\boldsymbol{\lambda}\Box$ are built up from axioms and rules that naturally divide into three groups.

1. *Lambda calculus* axioms and rules. We distinguish the following subgroups.

   (L1) Basic axioms and rules for simply typed lambda calculus (first order).

   (β_1) $(\lambda x^\sigma.M)N =_\tau M[x := N]$,

   (η_1) $\lambda x^\sigma.Mx =_{\sigma\to\tau} M$ provided $x \notin \mathrm{FV}(M)$.

   Equality and compatibility axioms/rules (cf. Barendregt [1984]).

   (L2) Axioms and rules for polymorphism (second order).

   (β_2) $(\Lambda\alpha.M)\tau =_{\sigma[\alpha:=\tau]} M[\alpha := \tau]$,

   (η_2) $\Lambda\alpha.M\alpha =_{\forall\alpha.\sigma} M$ (optional, not used in this paper).

   Equality and compatibility axioms/rules extended to the second order case.

2. Defining equations for *constants*, divided into the following subgroups.

   (CR) $\boldsymbol{R}_\sigma MN\boldsymbol{0} =_\sigma M$, $\quad \boldsymbol{R}_\sigma MN(\boldsymbol{S}P) =_\sigma NP(\boldsymbol{R}_\sigma MNP)$.

   (CB) Rules for the bar recursor, to be explained below.

$$\frac{Y[s] < \mathrm{lh}(s)}{\boldsymbol{B}YGHs = Gs} \qquad \frac{Y[s] \geq \mathrm{lh}(s)}{\boldsymbol{B}YGHs = H(\lambda x^{\boldsymbol{0}}.\boldsymbol{B}YGH(s * \langle x\rangle))s}$$

3. Additional rules for *quantifier-free arithmetic* (induction rule will be explained in the full paper).

(A) $$\frac{\boldsymbol{S}M = \boldsymbol{S}N}{M = N} \qquad \frac{\boldsymbol{S}M = \boldsymbol{0}}{E} \qquad \frac{E(\boldsymbol{0}, y) \qquad \begin{array}{c}\cancel{E(x, Fy)}\\ \vdots \\ E(\boldsymbol{S}x, y)\end{array}}{E(x,y)}$$

The (sub)groups L1, L2, CR, CB and A compose into the theories $\boldsymbol{\lambda}\Box$ according to the following table.

$\boldsymbol{\lambda}^\tau$	L1				
$\boldsymbol{\lambda}\mathbf{T}_{(\mathbf{A})}$	L1		CR		(A)
$\boldsymbol{\lambda}\mathbf{TB}_{(\mathbf{A})}$	L1		CR	CB	(A)
$\boldsymbol{\lambda 2}$	L1	L2			
$\boldsymbol{\lambda 2}\mathbf{T}_{(\mathbf{A})}$	L1	L2	CR		(A)

2.6. REMARKS. (i) We use the denotation $\langle n_0, \ldots, n_{k-1} \rangle$ $(k \geq 0)$ for finite sequences of natural numbers; the empty sequence is denoted by $\langle\,\rangle$. Let $s, s', \ldots$ range over such sequences. We presuppose a surjective, primitive recursive encoding of such sequences as natural numbers, with $*$ (concatenation operator), lh (length function) and $\preceq$ (prefix relation) primitive recursive. If $s = \langle n_0, \ldots, n_{k-1} \rangle$ then $[s]$ is the function assigning n_i to $i < k$ and 0 to $i \geq k$; $[s]$ is primitive recursive in s. The denotations $<, \geq, \mathrm{lh}(s), *, \langle x \rangle$, and $[s]$ in the defining rules of the bar recursor (CB above) should be taken as abbreviations of $\boldsymbol{\lambda}\mathbf{T}$-terms representing the corresponding primitive recursive functions and (characteristic functions of) relations explained above, e.g. $Y[s] < \mathrm{lh}(s)$ stands for $\boxed{<}(Y[s])(\mathrm{lh}(s)) = \boldsymbol{1}$. Thus the syntax in CB is explained.

(ii) Bar recursion (of type $\mathbf{0}$) is in fact recursion on trees of finite sequences (of natural numbers). As an example, consider for given Y, G, H the computation of $\boldsymbol{BYGH}\langle\,\rangle$. This involves recursion on the tree $T = \{\sigma \mid \forall \sigma' \prec \sigma\ Y[\sigma'] \geq \mathrm{lh}(\sigma')\}$. The termination condition of the recursion is $Y[s] < \mathrm{lh}(s)$: in a leaf s of T the value is given by Gs. In an inner node s of T the value of $\boldsymbol{BYGH}s$ is defined recursively in terms of all values $\boldsymbol{BYGH}(s * \langle x \rangle)$: by applying H to $\lambda x.\boldsymbol{BYGH}(s * \langle x \rangle)$. Of course Y should be such that T is well-founded, which indicates that it is non-trivial to find a model for bar recursion.

(iii) Let $\boldsymbol{\lambda}\Box$ be one of the systems $\boldsymbol{\lambda}\mathbf{T}, \boldsymbol{\lambda}\mathbf{TB}, \boldsymbol{\lambda}\mathbf{2T}$. The ease in use of the quantifier-free arithmetic of $\boldsymbol{\lambda}\Box_{\mathbf{A}}$ would be greatly improved by the addition of propositional logic. This can be achieved within the equational system since type $\mathbf{0}$ equality can be encoded: there exists a term $\boxed{=}$ of $\boldsymbol{\lambda}\mathbf{T}$ such that in $\boldsymbol{\lambda}\mathbf{T}_{\mathbf{A}}$ both $M =_{\mathbf{0}} N \vdash \boxed{=}MN =_{\mathbf{0}} \boldsymbol{1}$ and $\boxed{=}MN =_{\mathbf{0}} \boldsymbol{1} \vdash M =_{\mathbf{0}} N$ (this cannot be done for $M =_\sigma N$ with $\sigma \neq \mathbf{0}$). Furthermore, the classical truth functions for propositional logic can be encoded in $\boldsymbol{\lambda}\mathbf{T}$ (with $\boldsymbol{1}$ representing 'true' and $\boldsymbol{0}$ representing 'false'). Thus propositional logic can be used (with type $\mathbf{0}$ equations), while the equational character of the theory is preserved. For example, $x > \boldsymbol{0} \to \mathbf{Cond}\, xyz = y$ stands for the equation

$$\boxed{\to}(\boxed{>}x\boldsymbol{0})(\boxed{=}(\mathbf{Cond}\, xyz)y) =_{\mathbf{0}} \boldsymbol{1},$$

where $\boxed{\to}$, $\boxed{>}$, $\boxed{=}$ are terms encoding $\to$, $>$ and $=_{\mathbf{0}}$ as explained previously. This equation is indeed provable in $\boldsymbol{\lambda}\mathbf{T}_{\mathbf{A}}$. Moreover, bounded quantification $(\forall x \leq y\ \cdots, \exists x \leq y\ \cdots)$ can be represented in this way.

(iv) Using the encoding of formulas explained above, the term **Cond** can be used for case distinction (if $\cdots$ then $\cdots$ else $\cdots$). Also bounded minimization $(\mu x \leq y\ \cdots)$ can be implemented in $\boldsymbol{\lambda}\mathbf{T}$.

3. A new PER model

In this section a model construction for $\boldsymbol{\lambda}\mathbf{2T}_{\mathbf{A}}$ is presented. The construction will be used to obtain a certain counter model in section 4. The essence of this counter model is that it contains a specific discontinuous functional of type $\mathbf{3}$.

Well-known model constructions for $\boldsymbol{\lambda}\mathbf{2}$ and $\boldsymbol{\lambda}\mathbf{2T}$ seem to exhibit an inherent continuity. Even a relativized version of the recursion theoretic model HEO_2 (see Girard [1972]) does not work: one can show that the Kreisel-Lacombe-Schoenfield theorem can be relativized.

Our model is obtained using a PER-construction (see Breazu-Tannen and Coquand [1988]), based on an applicative structure consisting of untyped λ-terms, extended with a constant serving as a higher type oracle, modulo some conversion relation.

The idea of adding higher-type oracles to untyped λ-calculus originates from Kleene [1962]. He used the system to prove the equivalence between recursiveness and lambda definability in higher types. Below we describe a variant with one single oracle. The generalized construction, where an entire first order model is added in the form of oracles (to appear in a paper by the first author), however, appears to have interesting applications.

Untyped lambda calculus with higher type oracles

In this subsection the applicative structure will be constructed which forms the basis of a PER-model.

For this subsection, let $\varphi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ be a (type-**2**) functional.

3.1. DEFINITION. Let $\boldsymbol{\varphi}$ be a constant. The set of *$\lambda\varphi$-terms* (notation $\Lambda\varphi$) is defined inductively as follows.

$$\begin{aligned} \boldsymbol{\varphi} &\in \Lambda\varphi, \\ x \in V &\Rightarrow x \in \Lambda\varphi, \\ M, N \in \Lambda\varphi &\Rightarrow (MN) \in \Lambda\varphi, \\ M \in \Lambda\varphi, x \in V &\Rightarrow (\lambda x.M) \in \Lambda\varphi. \end{aligned}$$

We use the same notional conventions for $\lambda\varphi$-terms as we do for λ-terms; see Barendregt [1984]. The set of closed $\lambda\varphi$-terms is denoted as $\Lambda^{\circ}\varphi$.

The principle reduction relation is β-reduction. A second notion of reduction will be added to the system. We suppose the reader is familiar with the concept of reduction relations and induced conversion relations (denoted by $\to_\beta$ and $=_\beta$ for β-reduction).

3.2. DEFINITION. For each $n \in \mathbb{N}$ the *Church numeral* $\ulcorner n \urcorner \in \Lambda\varphi$ is defined by setting

$$\ulcorner n \urcorner \equiv \lambda f x.f^n(x).$$

3.3. DEFINITION. Let $\to_R$ be a notion of reduction (on $\Lambda\varphi$), and $F \in \Lambda\varphi, f : \mathbb{N} \to \mathbb{N}$. Then F is said to *R-define* f (notation $F \rhd_R f$) if for all $n \in \mathbb{N}$

$$F\ulcorner n \urcorner =_R \ulcorner f(n) \urcorner.$$

The aim is to construct a reduction relation $\to_\varphi$ such that for any F and f

$$\boldsymbol{\varphi} F \to_\varphi \ulcorner \varphi(f) \urcorner \qquad \text{if} \qquad F \rhd_{\beta\varphi} f. \tag{$*$}$$

Note that if '$F \rhd_{\beta\varphi} f$' would be replaced by '$F \rhd_\beta f$', the reduction $\to_\varphi$ could immediately be obtained by so-called δ-reduction making some external function internal (see Barendregt [1984]). This corresponds to the notion of

oracle in recursion theory. In $(*)$, however, the behaviour of the oracle is specified in terms of itself.

Because of this impredicativity, the question arises whether or not $\to_\varphi$ can be well defined. This is indeed the case. The idea is to generate $\to_\varphi$ in stages, according to an inductive operator. The details of this transfinite induction can be found in a forthcoming paper by the first author and J.P. van Draanen. The most important property of $\to_{\beta\varphi}$ (crucial in the proof of proposition 4.13) is the following (cf. Kleene [1962]).

3.4. THEOREM. *The reduction relation $\to_{\beta\varphi}$ is Church-Rosser.*

PROOF (Sketch). By a nested transfinite induction using the Hindley-Rosen lemma (see Barendregt [1984]). □

Building a PER model

We consider the set of closed $\lambda\varphi$-terms (modulo $=_{\beta\varphi}$) as an applicative structure.

3.5. DEFINITION. (i) For $M \in \Lambda^\circ\varphi$ write

$$\leqslant M \geqslant = \{M \in \Lambda^\circ\varphi \mid M =_{\beta\varphi} N\}.$$

(ii) $\mathcal{L}_\varphi = \{\leqslant M \geqslant \mid M \in \Lambda^\circ\varphi\}$.

(iii) *Application* in $\mathcal{L}_\varphi$ is defined as usual:

$$\leqslant M \geqslant \cdot \leqslant N \geqslant = \leqslant MN \geqslant.$$

Note that this is a sound definition, i.e. the resulting equivalence class is independent of the choice of the representatives of $\leqslant M \geqslant$ and $\leqslant N \geqslant$.

For the interpretation of $\boldsymbol{\lambda 2T}$-terms in a per structure over $\mathcal{L}_\varphi$ a few facts from classical untyped lambda calculus are needed.

To simplify notation, $=_{\beta\varphi}$ is mostly written as $=$.

3.6. DEFINITION. (i) Define

$$\begin{aligned} \mathbf{true} &\equiv \lambda xy.x, \\ \mathbf{false} &\equiv \lambda xy.y. \end{aligned}$$

Then $\mathbf{true}\,MN = M$ and $\mathbf{false}\,MN = N$, so if B is either equal to **true** or **false** the conditional if B then P else Q can be expressed as BPQ.

(ii) Let $\mathbf{S}^+, \mathbf{P}^-, \mathbf{Zero} \in \Lambda^\circ$ be a successor, a predecessor and a test for zero w.r.t. the numerals $\ulcorner n \urcorner$ (cf. Barendregt [1984]).

(iii) Using the fixed point combinator $\mathbf{Y}$, define

$$\mathbf{Rec} \equiv \mathbf{Y}(\lambda rmnx.\mathsf{if}\ \mathbf{Zero}\,x\ \mathsf{then}\ m\ \mathsf{else}\ n(\mathbf{P}^-x)(rmn(\mathbf{P}^-x))).$$

Now untyped terms with constants from $\boldsymbol{\lambda 2T}$ can be interpreted in $\mathcal{L}_\varphi$. This gives a way to interpret $\boldsymbol{\lambda 2T}$-terms.

3.7. INTERPRETATION. After replacing $\boldsymbol{0}$ by $\ulcorner 0 \urcorner$, $\boldsymbol{S}$ by $\mathbf{S}^+$, and $\boldsymbol{R}_\sigma$ by **Rec** untyped terms are interpreted as in ordinary term models. Then $\boldsymbol{\lambda 2T}$-terms are, as usual in PER-models, interpreted by erasing type information and using this untyped interpretation. The resulting model is called $\mathfrak{P}_\varphi$.

3.8. THEOREM. *$\mathfrak{P}_\varphi$ is a model of $\boldsymbol{\lambda 2T_A}$.*

4. Proof theoretic results

We will compare the first order systems $\boldsymbol{\lambda TB_A}$ and $\boldsymbol{\lambda T_A}$, and show that $\boldsymbol{\lambda TB_A}$ is a nonconservative extension of $\boldsymbol{\lambda T_A}$. This will be done by constructing an equation E in the language of $\boldsymbol{\lambda T_A}$ such that

$$\boldsymbol{\lambda TB_A} \vdash \mathrm{E},$$
$$\boldsymbol{\lambda T_A} \nvdash \mathrm{E}.$$

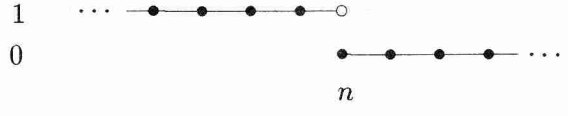
The nonconservativity does not really come as a surprise since it follows from Spector [1962] that $\boldsymbol{\lambda TB_A}$ proves the consistency of Peano Arithmetic, i.e. $\neg \mathrm{Proof}_{\mathbf{PA}}(x, \ulcorner 0 = 1 \urcorner)$ (after suitable encoding). The equation formulated here, however, does not refer to any metamathematical properties and is very simple.

The same equation yields a negative result concerning definability in $\boldsymbol{\lambda 2T_A}$.

4.1. DEFINITION. The term $\boldsymbol{\Sigma} \in \mathrm{Term}_{\mathbf{0}\to\mathbf{0}\to\mathbf{0}}(\boldsymbol{\lambda T})$ is defined by

$$\boldsymbol{\Sigma} \equiv \lambda xy.\mathsf{if}\ y < x\ \mathsf{then}\ \boldsymbol{1}\ \mathsf{else}\ \boldsymbol{0}.$$

The intuition is that $\boldsymbol{\Sigma n}$ is a so called *stepfunction* that can be depicted as follows.



4.2. DEFINITION. The equation $\mathrm{E} \equiv \mathrm{E}(\Phi)$, with Φ a type-**3** variable, is defined by

$$\mathrm{E}(\Phi) \equiv x < \Phi\varphi \to \varphi(\boldsymbol{\Sigma} x) \geq x \quad \wedge \quad \varphi(\boldsymbol{\Sigma}(\Phi\varphi)) < \Phi\varphi.$$

Here φ is a type-**2** variable and x a type-**0** variable. Note that $\mathrm{E}(\Phi)$ can be written in the form $G\Phi\varphi x =_{\mathbf{0}} \boldsymbol{0}$ or $G\Phi =_{\mathbf{2}\to\mathbf{0}\to\mathbf{0}} \lambda\varphi x.\boldsymbol{0}$.

4.3. INTUITION. If Φ satisfies $\mathrm{E}(\Phi)$ then

$$\Phi\varphi = \mu x\ [\varphi(\boldsymbol{\Sigma} x) < x].$$

4.4. DEFINITION. Let $\boldsymbol{\lambda\Box}$, $\boldsymbol{\lambda\boxtimes}$ be two of our systems. We say that E is $\boldsymbol{\lambda\Box}$-*realizable* in $\boldsymbol{\lambda\boxtimes}$ if for some $F \in \mathrm{Term}_{\mathbf{3}}(\boldsymbol{\lambda\Box})$ one has

$$\boldsymbol{\lambda\boxtimes} \vdash \mathrm{E}(F).$$

Our aim is to show that E is $\boldsymbol{\lambda}\mathbf{T}$-realizable in $\boldsymbol{\lambda}\mathbf{T}\mathbf{B}_\mathbf{A}$, but not in $\boldsymbol{\lambda}\mathbf{T}_\mathbf{A}$.

4.5. PROPOSITION. *There exists no* $F \in \mathrm{Term}_3(\boldsymbol{\lambda}\mathbf{T})$ *such that*

$$\boldsymbol{\lambda}\mathbf{T}_\mathbf{A} \vdash \mathrm{E}(F).$$

PROOF. Suppose, towards a contradiction, $\boldsymbol{\lambda}\mathbf{T}_\mathbf{A} \vdash \mathrm{E}(F)$ for some F. Now consider $\mathfrak{M}(\mathbb{N})$, the so-called full type structure of functionals over $\mathbb{N}$. Note that $\mathfrak{M}(\mathbb{N})$ is a model of $\boldsymbol{\lambda}\mathbf{T}_\mathbf{A}$. Define $\delta : \mathbb{N} \to \mathbb{N}^\mathbb{N}$ by

$$\begin{aligned} \delta(n)(x) &= 1 \quad \text{if } x < n, \\ &= 0 \quad \text{if } x \geq n. \end{aligned}$$

Note that $[\![\boldsymbol{\Sigma}]\!] = \delta$. Notation: $\delta_n = \delta(n)$. Let $\varphi : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ be such that for each $n \in \mathbb{N}$

$$\varphi(\delta_n) = n.$$

Then one has

$$\varphi(\delta([\![F]\!]\varphi)) = [\![F]\!]\varphi,$$

contradicting $\mathfrak{M}(\mathbb{N}) \models \mathrm{E}(F)$. $\square$

It can easily be seen that E is $\boldsymbol{\lambda}\mathbf{T}\mathbf{B}$-realizable in $\boldsymbol{\lambda}\mathbf{T}\mathbf{B}$. Indeed, the minimization stated above can be obtained by constructing a 'searching functional' Ψ of type $\mathbf{2}{\to}\mathbf{0}{\to}\mathbf{0}$ such that

$$\begin{aligned} \Psi\varphi s &= 0 && \text{if } \varphi[s] < \mathrm{lh}(s), \\ &= 1 + \Psi\varphi(s * \langle 1 \rangle) && \text{otherwise,} \end{aligned}$$

and finally defining $F \equiv \lambda\varphi.\Psi\varphi\langle\,\rangle$. From the form of the equations for Ψ it is clear that such a Ψ can be constructed using the bar recursor.

4.6. DEFINITION. The term $\mathrm{F_B} \in \mathrm{Term}_3(\boldsymbol{\lambda}\mathbf{T}\mathbf{B})$ is defined as follows.

$$\mathrm{F_B} \equiv \lambda\varphi.\boldsymbol{B}\varphi GH\langle\,\rangle,$$

where

$$\begin{aligned} G &\equiv \lambda s.\boldsymbol{0}, \\ H &\equiv \lambda fs.\boldsymbol{S}(f\boldsymbol{1}). \end{aligned}$$

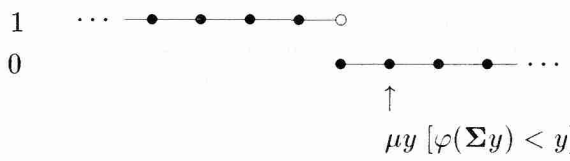
4.7. PROPOSITION. $\boldsymbol{\lambda}\mathbf{T}\mathbf{B}_\mathbf{A} \vdash \mathrm{E}(\mathrm{F_B})$.

It will turn out that E is even $\boldsymbol{\lambda}\mathbf{T}$-realizable in $\boldsymbol{\lambda}\mathbf{T}\mathbf{B}_\mathbf{A}$. Use is made of a trick due to G. Kreisel, also employed by J.A. Bergstra and J. Terlouw, see Barendregt [1984], p. 581.

4.8. DEFINITION. Define $f_\varphi \in \mathrm{Term}_1(\boldsymbol{\lambda}\mathbf{T})$ by

$$f_\varphi \equiv \lambda x.\mathsf{if}\ \forall y \leq x{+}1\ [\varphi(\boldsymbol{\Sigma}y) \geq y]\ \mathsf{then}\ \boldsymbol{1}\ \mathsf{else}\ \boldsymbol{0}.$$

4.9. INTUITION. f_φ is a stepfunction which looks as follows.



So $f_\varphi = \Sigma(\mu y\ [\varphi(\Sigma y) < y] - 1)$ (provided $\exists y\ \varphi(\Sigma y) < y$; otherwise '$f_\varphi = \Sigma_\infty$'). Therefore

$$\varphi(f_\varphi) \geq \mu y\ [\varphi(\Sigma y) < y] - 1$$

and hence

$$\mu y\ [\varphi(\Sigma y) < y] \leq \varphi(f_\varphi) + 1.$$

This gives a primitive recursive upperbound of the minimization expressed in E.

4.10. DEFINITION. The term $\mathrm{F_T} \in \mathrm{Term_3}(\boldsymbol{\lambda T})$ is defined as follows.

$$\mathrm{F_T} \equiv \lambda\varphi.\mu x \leq \varphi(f_\varphi) + 1\ [\varphi(\Sigma x) < x].$$

4.11. PROPOSITION. $\boldsymbol{\lambda TB_A} \vdash \mathrm{E}(\mathrm{F_T})$.

PROOF (Sketch). One shows that $\mathrm{F_T}\varphi = \mathrm{F_B}\varphi$, using the argument lined out in 4.9. Hence by extensionality one has $\mathrm{F_T} = \mathrm{F_B}$. Combining this with proposition 4.7 gives $\mathrm{E}(\mathrm{F_T})$. □

Now we can combine the results obtained thus far.

4.12. THEOREM. $\boldsymbol{\lambda TB_A}$ *is a non-conservative extension of* $\boldsymbol{\lambda T_A}$.

PROOF. Note that $\mathrm{E}(\mathrm{F_T})$ is an equation in the language of $\boldsymbol{\lambda T_A}$. By proposition 4.5

$$\boldsymbol{\lambda T_A} \nvdash \mathrm{E}(\mathrm{F_T}),$$

whereas by proposition 4.11

$$\boldsymbol{\lambda TB_A} \vdash \mathrm{E}(\mathrm{F_T}).\ \square$$

It will now be shown that E is not even $\boldsymbol{\lambda 2T}$-realizable in $\boldsymbol{\lambda 2T_A}$. This suggests that bar recursion is, in some sense, a more powerful extension of $\boldsymbol{\lambda T}$ than the concept of polymorphism.

The idea of the proof is the same as in the proof for $\boldsymbol{\lambda T_A}$ (proposition 4.5), but the counter model is far more involved. The presence of a type-**2** functional like φ, introducting a 'fatal discontinuity' would solve the problem. This can be achieved by applying the construction from section 3.

4.13. PROPOSITION. *There exists no* $F \in \mathrm{Term_3}(\boldsymbol{\lambda 2T})$ *such that*

$$\boldsymbol{\lambda 2T_A} \vdash \mathrm{E}(F).$$

PROOF (Sketch). Take $\varphi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ such that $\varphi(\delta_n) = n$ (like in the proof of proposition 4.5), and consider $\mathfrak{P}_\varphi$. The rest of the proof is analogous to the one for the first-order case, using the Church-Rosser theorem. □

References

BARENDREGT, H.P.

[1984] *The lambda calculus: its syntax and semantics*, 2nd edition, Studies in Logic 103, North-Holland, Amsterdam.

BREAZU-TANNEN, V. and TH. COQUAND

[1988] Extensional models for polymorphism, *Theor. Comput. Sci.* 59, pp. 85–114.

DEKKER, J.C.E. (ed.)

[1962] *Proceedings of symposia in pure mathematics V*, AMS, Providence.

GIRARD, J.-Y.

[1972] Interprétation fonctionelle et élimination des coupures dans l'arithmétique d'ordre supérieur, Thèse de doctorat d'état, Université Paris VII.

GÖDEL, K.

[1958] Über eine bisher noch nicht benutzte Erweiterung des finiten Standpunktes, *Dialectica* 12, pp. 280–287.

KLEENE, S.C.

[1962] Lambda-definable functionals of finite types, *Fundamenta Math.* 50, pp. 281–303.

SPECTOR, C.

[1962] Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles formulated in current intuitionistic mathematics, in: DEKKER [1962], pp. 1–27.

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