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Some Intuitionistic Elementary Equivalences

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Some Intuitionistic Elementary Equivalences

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Abstract: In this paper Fraïssé's method is used to show that the irrationals form an elementary substructure of the reals.

Some intuitionistic elementary equivalences

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The notions of elementary equivalence and elementary embedding can be carried over from classical model theory to intuitionistic (or more generally, constructive) model theory. Whereas one usually fixes some particular notion of model, such as Kripke-model or Beth-model, we will consider "models" in the naive sense, i.e. we will consider structures for first-order languages in an intuitionistic meta-language.

Whereas some theories and model classes are completely trivial from a classical point of view, their intuitionistic counterparts are far from simple. E.g. the theory of equality with infinitely many objects is classically complete, but intuitionistically already $\mathbb{Q}$ and $\mathbb{R}$ are not elementarily equivalent. The same example shows that the intuitionistic theory of dense linear order without endpoints is not complete. As a matter of fact, the underlying logic alone already blocks the traditional completeness results. Here we will consider some special structures, and investigate their elementary equivalence. This paper contains some proofs of results announced at the 1988 logic meeting at Oberwolfach. Recently W. Veldman told me that the elementary embedding of the irrationals in the reals, listed as an open problem in [1], can be solved by an adaptation of the classical techniques of R. Fraïssé. He also stated the elementary equivalence of all spreads "without isolated branches" (i.e. with the property that each node dominates at least two incomparable nodes). In the present paper, I have redone the old material using Fraïssé's method; for completeness sake I have restated the basic notions and shown that the main fact is intuitionistically correct.

The idea to practice intuitionistic model theory for natural models was independently conceived by Veldman and Jansen, cf [4].

We will first show that one half of a theorem by Fraïssé can be derived intuitionistically for intuitionistic logic. Then we will apply the method to establish the elementary embeddings of the irrationals in the reals and the real line in the plane. The basic facts about topology can be found in [3], Ch. 7.

1

We will consider in this paper only relational structures; the terminology differs from that of [2], but the notions are the same.

Definition 1.1 A *local isomorphism* (local iso) of $\mathfrak{A}$ to $\mathfrak{B}$ is an isomorphism of a substructure of $\mathfrak{A}$ onto a substructure of $\mathfrak{B}$

Definition 1.2 A (k,p)-iso of $\mathfrak{A}$ to $\mathfrak{B}$ is inductively defined by

i. f is a (o,p)-iso for all p, if it is a local iso.

ii. f is a (k,p)-iso, for $k>0$, if for every $a_1,\ldots,a_q \in |\mathfrak{A}|$ $(q \leq p)$ there is an extension $\bar{f}$ of f with domain dom $f \cup \{a_1,\ldots,a_q\}$ such that $\bar{f}$ is a $(k-1,p-q)$-iso *and* if there is for each $b_1,\ldots,b_r \in |\mathfrak{B}|$, $(r \leq p)$ an extension $\tilde{f}$ of f^{-1} with domain ran $f \cup b_1,\ldots,b_r\}$ such that $\tilde{f}$ is a $(k-1,p-r)$-iso.

The following facts follow directly from the above definition:

1. f is a (k,p)-iso $\Rightarrow$ f is a (k',p')-iso for $k' \leq k, p' \leq p$
2. f is a (k,p)-iso $\Rightarrow$ f is a local iso
3. f is a local iso $\Rightarrow$ f is a $(k,0)$-iso for all k
4. f is a (p,p)-iso $\Rightarrow$ f is a (k,p)-iso for all k.

Note that we have not required the a_i's or b_j's to be distinct. All sets $\{a_1,\ldots,a_n\}$ will in this paper be just *finitely indexed sets* (i.e. ranges of functions on initial segements of $\mathbb{N}$).

Definition 1.3 ((k,p)-equivalence).
$(\mathfrak{A},a_1,\ldots,a_n) \overset{(k,p)}{\sim} (\mathfrak{B},b_1,\ldots,b_n)$ iff there is a (k,p)-iso from $\mathfrak{A}$ to $\mathfrak{B}$ such that $a_i \mapsto b_i (i=1,\ldots,n)$.

Fact. $\overset{(k,p)}{\sim}$ is an equivalence relation.

Definition 1.4 (characteristic of a formula).

i. if φ is atomic then φ has characteristic $(0,0)$

ii. if $\varphi = \varphi_1 \Box\, \varphi_2$ (where $\Box$ is a propositional connective), and φ_i has characteristic (k_i, p_i), then φ has characteristic

$$(\max(k_1,k_2),\ \max(p_1,p_2)).$$

iii. if $\varphi = Qx_1 \ldots x_n \psi(x_1,\ldots,x_n)$ (where Q is $\forall$ or $\exists$) and ψ has characteristic (k,p), then φ has characteristic $(k+1, p+n)$

The notion of (k,p)-equivalence and characteristic allow us to formulate a theorem about the preservation of validity.

Theorem 1.5 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures with language L and let φ be a formula of L with characteristic at most (k,p). If f is a (k,p)-iso defined on $\{a_1,\ldots,a_n\}$ from $\mathfrak{A}$ to $\mathfrak{B}$, then*

$$\mathfrak{A} \models \varphi(a_1,\ldots,a_n) \Leftrightarrow \mathfrak{B} \models \varphi(f(a_1),\ldots,f(a_n)).$$

Proof. By induction on φ.

i. φ atomic, then the theorem holds because f is a local iso.

ii. $\varphi = \varphi_1 \Box\, \varphi$.
Induction hypothesis: $\mathfrak{A} \models \varphi_i(a_1,\ldots,a_n) \Leftrightarrow \mathfrak{B} \models \varphi_i(f(a_1),\ldots,f(a_n))$ $(i = 1,2)$. We conclude by ordinary intuitionistic logic that

$$\mathfrak{A} \models \varphi(a_1,\ldots,a_n) \Leftrightarrow \mathfrak{B} \models \varphi(f(a_1),\ldots,f(a_n)).$$

iii. $\varphi(x_1,\ldots,x_n) = \forall y_1,\ldots,y_r\ \psi(x_1,\ldots,x_n,y_1,\ldots,y_r)$
Since φ had characteristic at most (k,p), ψ has characteristic at most $(k-1, p-r)$.
Induction hypothesis: f is a $(k-1,p-r)$-iso $\Rightarrow$

$$\mathfrak{A} \models \psi(a_1,\ldots,a_n,b_1,\ldots,b_r) \Leftrightarrow \mathfrak{B} \models \psi(f(a_1),\ldots,f(a_n),f(b_1),\ldots,f(b_r))$$

for arbitrary a_i, $b_j \in |\mathfrak{A}|$.

Now let $\mathfrak{A} \models \varphi(a_1, \ldots, a_n)$, i.e., $\mathfrak{A} \models \psi(a_1, \ldots, a_n, b_1, \ldots, b_r)$ for all $b_j \in |\mathfrak{A}|$.
Consider $\psi(f(a_1), \ldots, f(a_n), b'_1, \ldots, b'_r)$ for $b'_j \in |\mathfrak{B}|$; since f is a (k,p)-iso, we can extend f^{-1} to a $(k-1, p-r)$-iso $\tilde{f}$ on $\{f(a_1), \ldots, f(a_n), b'_1, \ldots, b'_r\}$, and so by induction hypothesis,

$$\mathfrak{B} \models \psi(f(a_1), \ldots, f(a_n), b'_1, \ldots, b'_r) \Leftrightarrow \mathfrak{A} \models \psi(a_1, \ldots, a_n, \tilde{f}(b'_1), \ldots, \tilde{f}(b'_r)).$$

By assumption, the right-hand side holds, so

$$\mathfrak{B} \models \psi(f(a_1), \ldots, f(a_n), b'_1, \ldots, b'_r)$$

for arbitrary $b'_1, \ldots, b'_r$, i.e. $\mathfrak{B} \models \varphi(f(a_1), \ldots, f(a_n))$.
The converse is shown similarly.

iv. $\varphi(x_1, \ldots, x_n) = \exists y_1 \ldots y_r \psi(x_1, \ldots, x_n, y_1, \ldots, y_r)$.
Let $\mathfrak{A} \models \varphi(a_1, \ldots, a_n)$, i.e. $\mathfrak{A} \models \psi(a_1, \ldots, a_n, b_1, \ldots, b_r)$ for certain $b_1, \ldots, b_r \in |\mathfrak{A}|$. Since f is a (k,p)-iso with domain $\{a_1, \ldots, a_n\}$, it can be extended to a $(k-1, p-r)$-iso $\bar{f}$ on $\{a_1, \ldots, a_n, b_1, \ldots, b_r\}$, and so, by induction hypothesis,

$$\mathfrak{A} \models \psi(a_1, \ldots, a_n, b_1, \ldots, b_r) \Leftrightarrow \mathfrak{B} \models \psi(f(a_1), \ldots, f(a_n), \bar{f}(b_1), \ldots, \bar{f}(b_r)).$$

Therefore, $\mathfrak{B} \models \psi(f(a_1), \ldots, f(a_n))$. The converse is shown similarly. $\square$

Corollary 1.6

i. If $\mathfrak{A}$ and $\mathfrak{B}$ are (k,p)-equivalent (i.e. for an empty set), then they have the same valid sentences of characteristic at most (k,p).

ii. If $\mathfrak{A}$ and $\mathfrak{B}$ are (k,p)-equivalent for all k and p, then they are elementarily equivalent.

iii. If $(\mathfrak{A}, a_1, \ldots, a_n)$ and $(\mathfrak{B}, f(a_1), \ldots, f(a_n))$ are (k,p)-equivalent for all k and p and all $a_1, \ldots, a_n \in |\mathfrak{B}|$, then f is an elementary embedding.

Fraïssé's results for classical model theory are considerably stronger, but the arguments are essentially non-constructive.

2

In this section we will show that the irrationals form an elementary substructure of the reals, with respect to the language of order. Since apartness and equality are definable in terms of order, the result extens to the irrationals and reals, considered as apartness structures, or just plain structures.

The main problem is to establish the extension properties of local iso's. We will first demonstrate the technique in a simple case.

Theorem 2.1 *Let f be a local iso from* $\mathbf{R}$ *to* $\mathbf{R}$ *with domain* $\{a_1, \ldots, a_n\}$, *then for any b f can be extended to a local iso $\bar{f}$ on* $\{a_1, \ldots, a_n, b\}$.

Trivial as the theorem may be from a classical viewpoint, the intuitionistic version is not all that easily cracked, mainly because in the absence of the principle of the excluded third, one cannot determine the position of the extra point b.

Proof. Since there are a number of constructive finer points involved in the extension of f, we will spell the proof out in some detail. The image of b under the extension of f is constructed by means of a fundamental sequence, which is produced in stages. The basic idea is to enclose b in a shrinking sequence of intervals and to determine the position of b with respect to the given numbers $a_1, \ldots, a_n$.

Notations. : $A = \{a_1, \ldots, a_n\}$, W_i and V_i are open intervals with center b. $a'_i = f(a_i), i = 1, \ldots, n$.

Since both $\{a_1, \ldots, a_n\}$ and $\{a'_1, \ldots, a'_n\}$ are compact (located) sets in the sence of Brouwer, f is a homeomorphism. So there are two moduli of continuity m and p such that

(1) $|a_i - a_j| < 2^{-m(n)} \rightarrow |a'_i - a'_j| < 2^{-n}$

(2) $|a'_i - a'_j| < 2^{-p(n)} \rightarrow |a_i - a_j| < 2^{-n}$

It is no restriction to assume that m and p are strictly monotone.

We will now indicate the first two steps of the construction, and then proceed to the general case.

Step 1. Let W_1 and V_1 have diameter $2^{-m(1)-1}$, resp. $2^{-m(1)}$. For each a_i we have $a_i \in V_1$ or $a_i \notin W_1$. We test all a_i's and if none of them is in W_1, then b is apart from all a_i's, and hence A is partitioned into a left set A_ℓ and a right set A_r, such that $a_i \in A_\ell \to a_i < b$ and $a_i \in A_r \to b < a_i$. The sets A_ℓ and A_r are decidable subsets of A.

Note that $|a'_i - a'_j| < 2^{-p(m(1)+1)} \to |a_i - a_j| < 2^{-m(1)-1}$
Since $W_1 \cap A = \emptyset$, and W_1 has diameter $2^{-m(1)-1}$, there is a gap between A'_ℓ and A'_r of length at least $2^{-p(m(1)+1)}$. Choose a point b'_1 in this gap and put $b'_k = b'_1$ for all k.

In the second case there is at least one a_i in V_1, choose one such a_{i_1} and put $b'_1 = a'_{i_1}$.

Step 2. Assume that at the first step we dealt with case 2. (otherwise there is nothing left to do).
Let W_2 and V_2 be intervals with center b and diameter $2^{-m(2)-1}$ and $2^{-m(2)}$.
case a. $W_2 \cap A = \emptyset$. Again b determines a partition A_ℓ, A_r of A. Between A_ℓ and A_r is a gap of length at least $2^{-m(2)-1}$. Assume $b < a_{i_1}$($a_{i_1} < b$ is completely analogous). Since A_r is located, $\rho = \inf\{|b - a_i| | a_i \in A_r\}$ exists and is positive. There is a point a_{j_0} with $a_{j_0} - b < \rho + 2^{-m(p(m(2)+1)+1)}$, we use this point to define $b'_2 := a_{j_0} - 2^{-p(m(2)+1)-1}$.
Claim: $A'_\ell < b'_2 < A'_r$ and

$$|b'_2 - b'_1| = |b'_2 - a'_{i_1}| < 2^{-1}$$

Let $a'_j \in A'_r$, then $a_j \in A_r$. By definition b'_2 we have $a'_j > b'_2$ or $a'_j < a'_{j_0}$. In the first case we are done. So consider $a'_j < a'_{j_0}$. Note that $a_{j_0} > a_j \geq b + \rho$, so $a_{j_0} - a_j < 2^{-m(p(m(2)+1)+1)}$, and hence $a'_{j_0} - a'_j < 2^{-p(m(2)+1)-1}$, i.e. $a'_j > b'_2$.
The case A'_ℓ is dealt with similarly: if $a'_j \in A'_\ell$, then $a_j \in A_\ell$. In view of the length of W_2 there is a gap of length at least $2^{-m(2)-1}$ and so there is a gap of length at least $2^{-p(m(2)+1)}$ between a'_j and a'_{j_0}. Now the distance of b'_2 and a'_{j_0} is $2^{-p(m(2)+1)-1)}$, so $a'_j < b'_2$.
Now consider $|b'_2 - b'_1|$:

$$|b'_2 - a'_{i_1}| = |a'_{j_0} - 2^{-p(m(2)+1)-1} - a'_{i_1}| \leq| |a'_{j_0} - a'_{i_1}| + |2^{-p(m(2)+1)-1}| |$$

Now $|a_{j_0} - a_{i_1}| < 2^{-m(1)-1}$, so $|a'_{j_0} - a'_{i_1}| < 2^{-1}$ and hence

$$|b'_2 - a'_{i_1}| < 2^{-1} - 2^{-p(m(2)+1)-1} < 2^{-0}$$

case b. There is an a_i in V_2, choose such an a_{i_2} and define $b'_2 := a'_{i_2}$

$$|a_{i_1} - a_{i_2}| < 2^{-m(1)-1} + 2^{m(2)-1} < 2^{-m(1)-1} + 2^{-m(1)-1} = 2^{-m(1)},$$

and hence

$$|b'_1 - b'_2| = |a'_{a_i} - a'_{i_2}| < 2^{-1}$$

Case b. There is an a_i in V_2, choose such an a_{i_2} and define $b'_2 := a'_{i_2}$.

$$|a_{i_1} - a_{i_2}| < 2^{-m(1)-1} + 2^{m(2)-1} < 2^{-m(1)-1} + 2^{-m(1)-1} = 2^{-m(1)},$$

and hence $|b'_1 - b'_2| = |a'_{i_1} - a'_{i_2}| < 2^{-1}$.

Let us now consider the general case.
Assume that $b'_1, \ldots, b'_s$ have been constructed such that

i. $|b'_i - b'_{i+1}| < 2^{-i+1}, i = 1, \ldots, s-1$

ii. *either* we have established at a step $\leq s$ that b is apart from all a_i's (i.e. we got into case a) and hence the sequence has become constant
or each of the $b'_1, \ldots, b'_s$ is some $a'_i \in A'$ and for $b'_s = a'_{i_s}$ we have $|b - a_{i_s}| < 2^{-m(s)-1}$.

We only need to consider the second case.
Let W_{s+1} and V_{s+1} be intervals with center b and diameters $2^{-m(s+1)-1}$, $2^{-m(s+1)}$ resp.

case a. $W_{s+1} \cap A = \emptyset$.
Now b is apart from all a_i's, assume that $b < a_{i_s}$.

Again A is partitioned into two sets A_ℓ and A_r such that $A_\ell < b < A_r$. This induces a partioning of A' into A'_ℓ and A'_r. A_ℓ and A_r are separated by a gap of length at least $2^{-m(s+1)-1}$, and hence A'_ℓ and A'_r are separated by a gap of length at least $2^{-p(m(s+1)+1)}$.

Let ρ be the distance from b to A_r. Then we can find a point of A_r at distance almost ρ from b: $a_{j_0} \in A_r$ with $a_{j_0} - b < \rho + 2^{-m(p(m(s+1)+1)+1}$
Put $b'_{s+1} := a'_{j_0} - 2^{-p(m(s+1)+1)-1}$
Claim:

i. $A'_\ell < b'_{s+1} < A'_r$

ii. $|b'_{s+1} - b'_s| < 2^{-s+1}$

For (i) we treat A'_s and A'_r separately. Let $a'_j \in A'_r$, i.e., $a_j \in A_r$. Observe that $a'_j > b'_{s+1}$ or $a'_j < a'_{j_0}$. In the first case we are done, in the second case we have $b+\rho \leq a_j < a_{j_0}$ and hence $a_{j_0} - a_j \leq a_{j_0} - b - \rho < 2^{-m(p(m(s+1)+1)+1)}$.
So $|a'_{j_0} - a'_j| < 2^{-p(m(s+1)+1)-1}$, i.e. $a'_j > b'_{s+1}$.
Now let $a'_j \in A'_\ell$, i.e. $a_j \in A_\ell$. Since $W_{s+1} \cap A = \emptyset$, we have

$$a_{j_0} - a_j > 2^{-m(s+1)-1},$$

and hence

$$a'_{j_0} - a'_j \geq 2^{-p(m(s+1)+1)}.$$

So, by the definition of b'_{s+1}, $a'_j < b'_{s+1}$.
(ii). We have by induction hypothesis $a_{i_s} - b < 2^{-m(s)-1} < 2^{-m(s)}$, and so

$$|a_{i_i} - a_{j_0}| < 2^{-m(s)}, \text{ hence } |a'_{j_0} - a'_{i_s}| < 2^{-s}.$$

So

$$|b'_{s+1} - b'_s| = |a'_{j_0} - 2^{-p(m(s+1)+1)+1)} - a'_{i_s}| < 2^{-s} + 2^{-p(m(s+1)+1)-1}$$

$$< 2^{-s} + 2^{-s-3} < 2^{-s+1}$$

This finishes case a.

case b. There is a $a_{i_{s+1}} \in V_{s+1}$.

$$|a_{i_{s+1}} - a_{i_s}| < 2^{-m(s+1)-1} + 2^{-m(s)-1} < 2^{-m(s)},$$

so

$$|b'_{s+1} - b'_s| = |a'_{i_{s+1}} - a'_{i_s}| < 2^{-s} < 2^{-s+1}.$$

Now we can finish the proof: the sequence (b_n') converges to a point b'. The definitions above ensure that $a_i \gtrless b \Leftrightarrow a_i' \gtrless b'$, so we have constructed an extention of f that takes b to b' and is an order preserving map. □

By iterating the argument, one easily sees that f can be extended to an $\bar{f}$ on an arbitary finitely indexed set.

We will now apply the above techniques to the problem of $I \prec \mathbf{R}$, where I is the set of strong irrationals i.e. those irrational numbers which are apart from all rationals. We first prove an auxilliary result.

Lemma 2.2 *Every local iso f from $\mathbf{R}$ to I, defined on $\{a_1, \ldots, a_n\}$ can for every b be extended to a local iso on $\{a_1, \ldots, a_n, b\}$.*

Proof. We essentially copy the proof of theorem 2.1, but we carry out a spoiling argument with respect to the rationals. I.e. we modify the construction of (b_n'), such that at step n the n-th rational is avoided. This can be done in a positive way, i.e. b' can be apart from each rational.
The simplest way to do this is to recast the above proof in terms of shrinking nested segments, then the spoiling argument is literally Poincaré's original proof of the uncountability of the reals. □

The result is also correct for the irrationals themselves, i.e. the real numbers which are not rational. In that case we use the facts from the construction of b' in theorem 2.1. Whenever, during the construction of the sequence b_n' we get into case a, we choose an irrational number. Now suppose that b' is rational, then case a never occurs, i.e. each neighbourhood V_n of b contains an element of A. Thus b is a closure point of the compact set A, and therefore belongs to A, i.e. $f(b)$ is already defined and belongs to A'. Therefore $f(b) = b'$ is irrational. Contradiction.

Corollary 2.3 *For every local iso from $\mathbf{R}$ to I, defined on $\{a_1, \ldots, a_n\}$ and every set $\{b_1, \ldots, b_p\}$ there is an extension of $\bar{f}$ of f to $\{a_1, \ldots, a_n, b_1, \ldots, b_p\}$ which is a local iso.*

Corollary 2.4 *Each local iso on $\{a_1, \ldots, a_n\}$ is a $(1,p)$-iso for all p.*

Corollary 2.5 *I and $\mathbf{R}$ are $(1,p)$ equivalent for each $\{a_1, \ldots, a_n\} \subseteq I$.*

In order to apply the Fraïssé criterium, we have to show that $(\mathbf{R}, a_1, \ldots, a_n) \overset{(k,p)}{\sim} (I, f(a_1), \ldots, f(a_n))$ for all f and $a_1, \ldots, a_n$, for all k and p. In the case of I and $\mathbf{R}$ this can be accomplished by showing that $(1,p)$-equivalence suffices.

Lemma 2.6 *Let f be a $(1,p)$-iso from $\mathbf{R}$ to I on $\{a_1, \ldots, a_n\}$, then f is also a (k,p)-iso for all k.*

Proof. We have to show that for arbitary $b_1, \ldots, b_q$ and $q \leq p$, there is a $(1, p-q)$-iso $\bar{f}$ that extends f to $\{a_1, \ldots, a_n, b_1, \ldots, b_q\}$. Since f is a $(1,p)$-iso, we know that there is a local iso $\bar{f}$ satisfying the requirements. By cor. 2.4 $\bar{f}$ is also a $(1, p-q)$-iso, hence f is a $(2,p)$-iso. An iteration of the argument yields the required result. □

So now we can conclude

Theorem 2.7 $I \prec \mathbf{R}$.

We observe that the result can be read in two ways: I is the set of strongly irrational numbers (i.e. those irrationals apart from all rationals), or the "negative" irrationals (i.e. the complement of the rationals). The proof covers both cases, and actually all intermediate families of irrationals.

3

The basic idea of the above proofs is that finite local iso's can always be extended to finite local iso's. The same technique can be used in structures that resemble I and $\mathbf{R}$. We list some of those:

Theorem. *Consider the language of apartness:*

i. The Cantor set is an elementary substructure of Baire space (with respect to $<, \#, =$).

ii. All spreads "without isolated" branches are elementary substructures of the univeral spread (with respect to $\#, =$) (Veldman).

iii. $\mathbf{R} \prec \mathbf{R}^2$ *(with respect to $\#, =$).*

iv. $\mathbf{R} - \{0\} \prec \mathbf{R}$ *(with respect to $<, \#, =$).*

This list can easily be extended, but for the time being it seems more useful to look for natural elementary equivalence classes.

The above treatment has a minor weakness; it seems that Brouwer's continuity theorem is essential, so the proof does not work for, e.g., Bishop's mathematics. It would be interesting to know the precise role of the continuity theorem in this context.

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