Towards the Interpretability Logic of $I\Delta_0+EXP$

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Abstract:
We provide principles for the Interpretability Logic of $I\Delta_0$+EXP.

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0 Introduction

Among the different interpretability logics corresponding to (classes of) arthmetical theories, the interpretability logic of $I\Delta_0$+EXP (to which we will refer as IL_{exp}), takes a special place. Though we have no explicit axiomatization for IL_{exp}, we do have a complete description of the theory. Visser shows, in [VIS], that relative interpretability over $I\Delta_0$+EXP can be characterized in terms of cut-free provability. From his observation that Löb's logic is the provability logic for cut-free provability in $I\Delta_0$+EXP it follows that there is an embedding of IL_{exp} in Löb's logic. Thus, validity of IL_{exp}-principles can be decided using the characterization and finite Kripke models for L. The characterization result and the arithmetical completenesss of Löb's logic completely reduces the problem of determining IL_{exp} to a purely modal question.
It can be easily verified that $ILP \subseteq IL_{exp}$. After Visser established the arithmetical completeness of ILP for finitely axiomatizable theories extending $I\Delta_0$+SUPEREXP it was thought, $I\Delta_0$+EXP being finitely axiomatizable, that ILP might be the interpretability logic of this theory as well. However, Visser and de Jongh found a principle that is valid in IL_{exp} and not derivable from ILP [VIS, appendix].
In this paper we discuss a subsystem of IL_{exp} which is an extension of ILP with X and E. Unlike the usual axioms of interpretability logic, X and E are rather axiom schemata than proper axioms, in two ways. First, they indicate infinite lists of axioms E_1, E_2, ... and X_1, X_2, ... Secondly, the axioms E_n and X_n are formulated using two kinds of variables: the usual propositional variables, which may be substituted for by arbitrary formulae in the language of interpretability logic, and special variables for which only formulae of special classes may be substituted. It will be shown that the system ILPXE is not finitely axiomatizable. Concerning the question whether ILPXE equals IL_{exp} or is a proper subsystem of it, we do not have conclusive arguments.

We employ the following notational conventions :
$\neg$, $\Box$, $\Diamond$, Δ, ∇, bind equally strong; $\wedge$, $\vee$, bind equally strong; $\rightarrow$, $\leftrightarrow$, bind equally strong;
$\Box$ binds stronger than $\rhd$; $\rhd$ binds stronger than $\wedge$; $\wedge$ binds stronger than $\rightarrow$.

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1 Relevant facts

1.1 Löb's logic and $I\Delta_0$+EXP

Löb's logic L is arithmetically sound & complete w.r.t all theories T with the following properties: (i) T has a Σ_1- provability predicate, (ii) T extends $I\Delta_0$+EXP, (iii) T does not prove $Prov^n(\bot)$ for any n. So, for such T we have
$L \vdash A$ iff for all arithmetic interpretations * which translate $\Box$ with provability from T, $I\Delta_0+EXP \vdash A^*$.
Visser observes, in [VIS], that the same holds if we let arithmetic interpretations * translate the $\Box$ with cut free provability from T:
The transformation of an ordinary T-proof into a cut free proof from T is a superexponential process. That is, if x is the original proof, then the result of the cut elimination process will be bounded by itexp($|x|$,$\varrho(x)$), where $|x|$ is the binary length of x, and $\varrho(x)$ is the cut rank of x.
We will write $\Box_T$ for ordinary provability from T and Δ_T for cut free provability from T. Let φ and ψ be sentences in the language of S.
So in general $I\Delta_0$+EXP will not prove $\Box_T\varphi \to \Delta_T\varphi$, but does prove $\Delta_T\varphi \to \Box_T\varphi$.
Clearly we have $I\Delta_0+EXP \vdash \varphi \Rightarrow I\Delta_0+EXP \vdash \Delta_T\varphi$ (Necesitation).
L1: The usual Σ-completeness argument yields $I\Delta_0+EXP \vdash \Delta_T\varphi \to \Box_T\Delta_T\varphi$. However, inspection of this argument shows that the cuts in the proof of $\Delta_T\varphi$ can be eleminated in $I\Delta_0$+EXP, so $I\Delta_0+EXP \vdash \Delta_T\varphi \to \Delta_T\Delta_T\varphi$ holds.
L2: From $I\Delta_0+EXP \vdash \Box_T(\varphi \to \psi) \to (\Box_T\varphi \to \Box_T\psi)$ and $I\Delta_0+EXP \vdash \Delta_T\varphi \to \Box_T\varphi$, we get $I\Delta_0+EXP \vdash \Delta_T(\varphi \to \psi) \to (\Delta_T\varphi \to \Box_T\psi)$. Here the cut formula in the proof of ψ is standard, so the cut elimination necessary to get $\Delta_T\psi$ from $\Box_T\psi$ is only multi-exponential. Hence $I\Delta_0+EXP \vdash \Delta_T(\varphi \to \psi) \to (\Delta_T\varphi \to \Delta_T\psi)$.
L3: $I\Delta_0$+EXP has diagonalization, so with L1 and L2, also Löb's axiom is true for Δ_T: $I\Delta_0+EXP \vdash \Delta_T(\Delta_T\varphi \to \varphi) \to \Delta_T\varphi$.

1.2 The Friedman-Visser characterization

In the following,

$A \rhd B$ will stand for $I\Delta_0$+EXP+A interpretes $I\Delta_0$+EXP+B;
ΔA for $\Delta_{I\Delta_0+EXP}A$; ∇A for $\neg\Delta\neg A$;
$\Box A$ for $\Box_{I\Delta_0+EXP}A$; $\Diamond A$ for $\neg\Box\neg A$.

In [VIS], Visser gives the following Friedman-style characterization of relative interpretability over $I\Delta_0$+EXP:

Theorem 1.2.1 $I\Delta_0+EXP \vdash A \rhd B \leftrightarrow \Delta(\nabla A \rightarrow \nabla B)$.

Corollary 1.2.2
(a) $I\Delta_0+EXP \vdash \Box A \leftrightarrow \Delta\Delta A$;
(b) $I\Delta_0+EXP \vdash \Delta A \rightarrow \Box A$.

This theorem, combined with the the fact that L is the provability logic of cut free provability in $I\Delta_0$+EXP, gives us a complete characterization of the interpretability logic of $I\Delta_0$+EXP.
We define a translation t which translates formulae of $L(\Box, \rhd)$ into formulae of $L(\Delta)$ according to the Visser-Friedman characterization, as follows:

Definition 1.2.3
$\top^t = \top$ and $\bot^t = \bot$;
$p^t = p$, for all propositional variables p;
$(\Box\varphi)^t = \Delta\Delta\varphi^t$;
$(\varphi \rhd \psi)^t = \Delta(\nabla\varphi^t \rightarrow \nabla\psi^t)$.

Trivially, we have the following lemma:

Lemma 1.2.4 For all $\varphi \in L(\Box, \rhd)$, $IL_{exp} \vdash \varphi$ iff $L_\Delta \vdash \varphi^t$.

This lemma suggests the following semantics for the interpretability logic of $I\Delta_0$+EXP:

Definition 1.2.5 An *IL_{exp} Kripke model* M is a quadruple $(W,R,b,\Vdash)$, where (W,R,b) is a finite Kripke model for L, i.e. W is a finite set, R is a transitive irreflexive binary relation on W, $b \in W$ and for all $x \in W$, if $x \neq b$ then bRx, and $\Vdash$ is a forcing relation on (W,R,b) with accessability relations for $\Box$ and $\rhd$ defined as follows:
$x \Vdash A \rhd B$ iff $\forall y,z\,(xRyRz \wedge z \Vdash A \rightarrow \exists v\,(yRv \wedge v \Vdash B))$;
$x \Vdash \Box A$ iff $\forall y,z\,(xRyRz \rightarrow z \Vdash A)$.

It will be claer that the IL_{exp} Kripke models provide us with semantics for whichIL_{exp} is sound and complete. Also, by arithmetical completeness of L for cut free provability

in $I\Delta_0$+EXP, once we have an axiomatization A for IL_{exp} that is sound and complete with respect to these semantics, we have arithmetical completeness of A.
An immediate consequence of this is:

Corollary 1.2.6 $I\Delta_0+EXP \nvdash \Box A \rightarrow \Delta A$.
Proof Consider the following Kripke model $M=(W,R,\Vdash)$:

$W=\{b,x,y\}$,

$R=\{\langle b,x\rangle\langle b,y\rangle\langle x,y\rangle\}$

$t\Vdash p$ iff $t=y$.

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2 ILP and IL_{exp}

2.1 ILP $\subseteq$ IL_{exp}

Using the Friedman-Visser characterization one can easily see that all theorems of ILP are theorems of IL_{exp}.

Theorem 2.1.2 $IL_{exp} \vdash$ ILP.

Proof We will show that $L_\Delta \vdash (ILP)^t$, where ^{t} is the translation defined in Definition 1.2.3. By the Friedman-Visser characterization,the theorem immediately follows from this. First we will show that the translation of Löb's axiom is a theorem of L_Δ. Consider $(\Box(\Box A \to A) \to \Box A))^t$, i.e. $\Delta\Delta(\Delta\Delta A^t \to A^t) \to \Delta\Delta A^t$.

Reason in L_Δ :

$\Delta(\Delta\Delta A^t \to A^t) \to \Delta(\Delta A^t \to A^t)$ (since $\Delta A^t \to \Delta\Delta A^t$);

$\Delta(\Delta\Delta A^t \to A^t) \to \Delta A^t$ (by Löb's axiom), so

$\Delta\Delta(\Delta\Delta A^t \to A^t) \to \Delta\Delta A^t$ (by necessitation).

Next we show that the persistency axiom is a theorem of L_Δ. Consider

$(A \rhd B \to \Box(A \rhd B))^t = \Delta(\nabla A^t \to \nabla B^t) \to \Delta\Delta\Delta(\nabla A^t \to \nabla B^t)$.

Apply L2 twice to $\Delta(\nabla A^t \to \nabla^t B)$.

We leave checking of other axioms and rules to the reader.

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2.2 Conservativity of IL_{exp} over ILP

In this paragraph, we will give a modal proof of the conservativity of IL_{exp} over ILP for formulae in $L(\Box)$, and show that IL_{exp} is conservative over ILP for a restricted, and semantically defined, class of formulae of $L(\Box, \rhd)$. In order to do the latter, we introduce a Friedman style semantics for which ILP is sound and complete. (See paragraph 3.4 for some negative results on conservativity of IL_{exp}.)

Definition 2.2.1 A *structured Friedman frame* F is a quadruple (W,R,N,b), where W is a finite set (the worlds of F), R is a transitive, irreflexive relation on W, N is a subset of W, b is a world in W such that $\forall x(x \in W \wedge x \neq b \rightarrow bRx)$ (so b is the unique root of F).

If $x \in N$ we will also write N(x), and say that x is a normal world, if $x \notin N$ we say that x is a structural world, and write S(x) or $x \in S$.

Definition 2.2.2 A *structured Friedman model* M is given by a structured Friedman frame F together with a forcing relation $\Vdash$ which is only defined on normal worlds and which satisfies

$x \Vdash \Box A$ iff $\forall y \in S\ \forall z \in N\ (xRyRz \rightarrow z \Vdash A)$;

$x \Vdash A \rhd B$ iff $\forall y \in S\ \forall z \in N\ (xRyRz \wedge z \Vdash A \rightarrow \exists t \in N\ (yRt \wedge t \Vdash B))$.

Definition 2.2.3

(a) A world x of a structured Friedman frame F is said to be *in level n* ($n \geq 0$) if

$n = \max\{ k : \exists y_0,\ldots,y_k \in W\ (y_0=b\ \&\ y_k=x\ \&\ y_iRy_{i+1}$ for $0 \leq i < k\}$.

(So, x is in level n if x is maximally n R-steps away from b.)

(b) A structured Friedman frame F is *levelled* if

1) $\forall x \in W$ (N(x) iff x is in an even level), and

2) All blind worlds of F are in an even level.

(c) A *levelled Friedman model M* is a Friedman model M on a levelled structured Friedman frame.

Lemma 2.2.4 ILP is sound and complete with respect to levelled Friedman models.

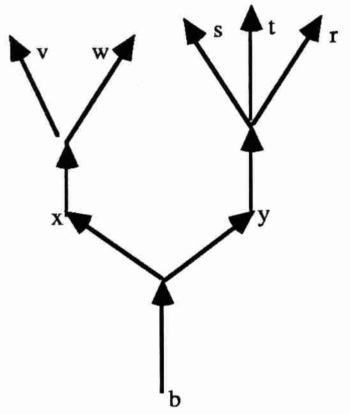
Proof Cf. [KA].

Lemma 2.2.5 For all $\varphi \in L(\Box)$, $IL_{exp} \vdash \varphi$ iff $ILP \vdash \varphi$.

Proof The lemma immediately follows from the arithmetical completeness theorems for L, ILP, and IL_{exp}. The following is a semantical proof of the left to right part of this statement.

Let $\varphi \in L(\Box)$, and $ILP \nvdash \varphi$. Let $M=(W,R,N,b,\Vdash)$ be a levelled Friedman counter model for φ, in which $b \Vdash \neg\varphi$. From M we will construct, in two stages, an IL_{exp} conter model to φ. First we construct an IL_{exp} frame $F'=(W',R',b)$ from (W,R,N,b), then we will define a forcing relation $\Vdash'$ on F', using $\Vdash$.

First stage. W' will consist of all worlds x in W such that N(x) plus, for all such x, except b, a copy of x, called x_p. R' will be defined as follows: $xR'y$ iff $(xRy \vee x=y_p \vee \exists z\ (xRz \wedge y= z_p))$. See the figure below:



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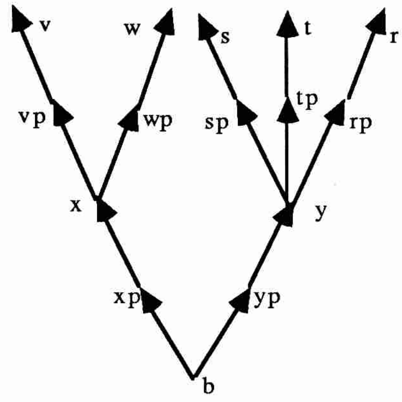

F'

Note that the following hold for worlds x,y,z in W':
(a) If N(x) and N(y), then xR'y iff xRy;
(b) Each world x has one immediate R'-predecessor, and if N(x), then this immediate predecessor is x_p;
(c) If N(x) and xR'yR'z, then (i) if N(z), then xRRz; (ii) if $z=v_p$, then xRRv;
(d) If $x_pR'yR'z$, and N(z), then xRRz and xR'R'z;
(e) If $x_pR'yR'z$, and $z=v_p$, then xRRv and xR'R'v.

The forcing relation $\Vdash'$ will be defined as follows:
$x \Vdash' p$ iff N(x) and $x \Vdash p$ or $x=y_p$ and $y \Vdash p$.

Claim For all $\varphi \in L(\Box)$, and all worlds x in $W \cap N \backslash \{b\}$, $x \Vdash \varphi \leftrightarrow x \Vdash' \varphi \leftrightarrow x_p \Vdash' \varphi$; and $b \Vdash \varphi \leftrightarrow b \Vdash' \varphi$.
It is easily seen that the claim holds for all blind worlds, and for all propositional variables. Suppose it holds for all successors of a world x.
Suppose $x \Vdash \Box\varphi$. Let y,z be worlds of W', such that xR'yR'z. Then either N(z) or $z=v_p$. If N(z), then by (c), xRRz, so $z \Vdash \varphi$, so, by supposition, $z \Vdash' \varphi$; if $z=v_p$, then by (c), xRRv, so $v \Vdash \varphi$, so by supposition, $v_p \Vdash' \varphi$. So $x \Vdash' \Box\varphi$.
Suppose $x \Vdash' \Box\varphi$. Let y,z be worlds of W, such that xRyRz, and N(z). Then there is a world y' in W', such that xR'y'R'z. Then $z \Vdash' \varphi$, so $z \Vdash \varphi$. So $x \Vdash \Box\varphi$.
Suppose $x \Vdash' \Box\varphi$. Let y,z be worlds of W', such that $x_pR'yR'z$. Now either N(z), so, by (d), xR'R'z, so $z \Vdash' \varphi$; or $z=v_p$, so, by (c), xR'R'v, so $v \Vdash' \varphi$, so by supposition $z \Vdash' \varphi$. So $x_p \Vdash' \Box\varphi$.
Suppose $x_p \Vdash' \Box\varphi$. $x_pR'x$, so $x \Vdash' \Box\varphi$.
This shows that the claim holds, and concludes the proof.

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Definition 2.2.6 We define for formulae φ in $L(\Box, \rhd)$ the $\Box/\rhd$-depth $D(\varphi)$ as follows:

(i) $D(p) = 0$ for all propositional variables p;

(ii) $D(\Box\varphi)= D(\varphi)+1$ for all φ;

(iii) $D(\varphi \rhd \psi)= \max(D(\varphi),D(\psi))+ 1$ for all φ and ψ;

(iv) $D(\neg\varphi)=D(\varphi)$;

(v) $D(\varphi \wedge \psi) = D(\varphi \vee \psi) = \max(D(\varphi),D(\psi))$.

The following lemma shows that IL_{exp} is conservative over ILP for a certain (semantically characterized) class of formulae.

Lemma 2.2.7 If φ is a formula of $L(\Box, \rhd)$ such that there is a levelled Friedman model M in which the maximum of the levels is smaller than or equal to 4 and in which $b \nVdash \varphi$, then $IL_{exp} \nvdash \varphi$.

Proof Let φ be a formula of $L(\Box, \rhd)$ for which there is a levelled Friedman counter model $M=(W,R,N,b,\Vdash)$, in which $b \Vdash \neg\varphi$, and the maximum of the levels of M is ≤ 4.

We can transform M into a Friedman model for IL_{exp} by defining a new forcing relation $\Vdash'$ on (W,R,b) as follows:

a) if x is a normal world of (W,R,s,b), then $x \Vdash' p$ iff $x \Vdash p$;

b) if x is a structural world of level 1, then $x \Vdash' p$ for all p (or $x \Vdash' \neg p$ for all p, or ...);

c) if x is a structural world of level 3, then :

choose one particular y of level 4, henceforth referred to as y(x), such that xRy(x), and

let $x \Vdash' p$ iff $y(x) \Vdash p$, for all p.

We will show that $\Vdash'$ has the following properties:

1) for all normal worlds x in levels 2 and 4 of (W,R,s,b), $x \Vdash' \varphi$ iff $x \Vdash \varphi$, for all φ.

2) for all structural worlds x in level 3 of (W,R,s,b), $x \Vdash' \varphi$ iff $y(x) \Vdash \varphi$, for all φ.

We will first show (1) and (2) and then use these to show that the following holds:

3) $b \Vdash' \varphi$ iff $b \Vdash \varphi$, for all φ.

(1) Because the worlds of level 4 are blind and (according to condion *a* above) the '-forcing relation for propositional variables equals $\Vdash$ for propositional variables on worlds of level 4, it is clear that , for all x of level 4, $x \Vdash \varphi$ iff $x \Vdash' \varphi$.

As for worlds x of level 2, '-forcing of a formula φ is completely determined by '-forcing of propositional variables in such x itself and '-forcing on the worlds of level 4 which are accessible form x. By definition, the '-forcing of propositional variables in a world x of level 2 is equal to $\Vdash$, and we already know that $y \Vdash \varphi$ iff $y \Vdash' \varphi$ for all worlds y of level 4.

(2) is proved by induction on φ. Let x be a world of level 3.
We have defined $\Vdash'$ on x exactly so that, for propositional variables p, $x \Vdash' p$ iff $y(x) \Vdash p$.
(IH) Suppose $x \Vdash' \varphi$ iff $y(x) \Vdash \varphi$ and $x \Vdash' \psi$ iff $y(x) \Vdash \psi$.
Then
($\neg$) $x \Vdash' \neg\varphi$ iff $y(x) \Vdash \neg\varphi$, by IH;
($\wedge$) Suppose $x \Vdash' \varphi \wedge \psi$. Then $x \Vdash' \varphi$ and $x \Vdash' \psi$, so, by IH, $y(x) \Vdash \varphi$ and $y(x) \Vdash \psi$.
Suppose $y(x) \Vdash \varphi \wedge \psi$. Then, again by IH, $x \Vdash' \varphi \wedge \psi$.
($\rhd$) This case is immediately true by the fact that all worlds of level 3 and 4 force all formulae of the form $\varphi \rhd \psi$.

3) Again, this is proved by induction on φ. By *a*, $b \Vdash' p$ iff $b \Vdash p$, for all propositional variables p. (IH) Suppose $b \Vdash' \varphi$ iff $b \Vdash \varphi$ and $b \Vdash' \psi$ iff $b \Vdash \psi$. The cases $\neg$ and $\wedge$ are trivial, so we only treat the case $\rhd$.
Suppose $b \Vdash' \varphi \rhd \psi$. Let bRtRz and $z \Vdash \varphi$ and z of level 2. Then by (1), $z \Vdash' \varphi$, so there must be a y such that tRx and $x \Vdash' \psi$. If x is of level 2 or 4, then by (1), $x \Vdash \psi$. If x is of level 3, then we can apply (2), and find that $y(x) \Vdash' \psi$. Again by (1), $y(x) \Vdash \psi$, and tRy(x). So $b \Vdash \varphi \rhd \psi$.
For the converse, suppose $b \Vdash \varphi \rhd \psi$. Let bRtRz and $z \Vdash' \varphi$. If z is of level 3, then there is y(z) with zRy(z) and $y(z) \Vdash \varphi$, so we can find, by supposition, an x in level 2 or 4 such that tRx and $t \Vdash \psi$, so by (1), $t \Vdash' \psi$. The case in which z is of level 2 or 4 is again an easy application of (1). This completes the proof of statement (3).

Thus we found an IL_{exp} model $M' = (W,R,b,\Vdash')$ such that $b \nVdash' \varphi$.
This completes the proof of Lemma 2.2.7.

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3 The axiom schema E

ILP is not all there is to IL_{exp}. De Jongh and Visser [VIS] first discovered a sentence showing that IL_{exp} strictly extends ILP. We will show that at least two different axiom schemata, which are mutually independent over ILP, are valid in IL_{exp}. This section is devoted to the treatment of the axiom schema E. In Section 4, the schema X will be treated and the relative independency of E and X over ILP.

3.1 Σ- and Π-formulae

We define two classes of formulae in the language $L(\Box, \rhd)$:

Definition 3.1.1 The class of *Σ-formulae* of $L(\Box, \rhd)$ is defined as follows:
(i) $\top$ and $\bot$ are in Σ;
(ii) for all φ and ψ, $\varphi \rhd \psi$ is in Σ;
(iii) for all φ , then $\Box\varphi$ is in Σ;
(iv) if φ and ψ are in Σ, the so are $\varphi \vee \psi$, $\varphi \wedge \psi$;
(v) no other formulae are in in Σ.

In the following, we will consider all formulae which are equivalent to a formula of the Σ-class, as belonging to this class.
The class of Π-formulae consists of formulae which are equivalent to a negation of a Σ-formula:

Definition 3.1.2 The class of *Π-formulae* of $L(\Box, \rhd)$ is defined as follows:
(i) $\top$ and $\bot$ are in Π;
(ii) for all φ and ψ, $\neg(\varphi \rhd \psi)$ is in Σ;
(iii for all φ, then $\Diamond\varphi$ is in Π;
(iv) if φ and ψ are in Π, then so are $\varphi \vee \psi$, $\varphi \wedge \psi$;
(v) no other formulae are in in Π.

The following lemma sums up the characteristics of the behaviour of Σ- and the Π-formulae in Kripke models for IL_{exp} and Veltman models for ILP.

Lemma 3.1.3 Let P be a Π-formula and S a Σ-formula, x be a world of a Kripke model for IL_{exp}. The following hold:
(i) $x \Vdash P \Rightarrow \forall y\,(\forall z\,(xRz \rightarrow yRz) \rightarrow y \Vdash z)$ (*jump-over* of Π-formulae);
(ii) $x \Vdash P \Rightarrow \forall y\,(yRx \rightarrow y \Vdash z)$ (*downwards preservation* of Π-formulae);

(iii) $x \Vdash S \Rightarrow \forall y\,(\forall z(yRz \to xRz) \to y \Vdash S)$ (*jump-over* of Σ-formulae);
(iv) $x \Vdash S \Rightarrow \forall y\,(xRy \to y \Vdash S)$ (*upwards preservation* of Σ-formulae).
In ILP Veltman models, we have the following:
(v) $x \Vdash S \Rightarrow \forall y\,(\forall u,z\,(yRu \to (xRu \wedge (uSz \wedge xRz \to yRz))) \to y \Vdash S$;
(vi) $x \Vdash P \Rightarrow \forall y\,(\forall u,z\,(xRu \to (yRu \wedge (uSz \wedge yRz \to xRz))) \to y \Vdash P$.

Proof The proofs (by easy induction on P and S) of (i), (iii), (v), and (vi), are left to the reader. By transitivity of R, (i) implies (ii), and (iii) implies (iv) .

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Throughout Section 3, $P_1,\ldots,P_n$ will be Π-formulae and S will be some Σ-formula.

3.2. The axiom schema E

Definition 3.2.1 E_n is the following axiom schema:

$$(\Diamond A \wedge A \rhd P_1 \wedge \ldots \wedge A \rhd P_n \wedge S) \rhd (\Box\neg A \wedge P_1 \wedge \ldots \wedge P_n \wedge S).$$

We will refer to E_n as E if we do not want to specify the index n.
In this paragraph we will prove the following theorem:

Theorem 3.2.2 $IL_{exp} \vdash E$.

We will give two proofs of this theorem. The first one semantic (Lemma 3.2.4), the second syntactic (Lemma 3.2.6). In the first proof, we use a distance function on IL_{exp} frames, defined below, which gives, for each pair of worlds, the maximal number of worlds lying in between. We use xR_0y to express that either x=y or xRy.

Definition 3.2.3 d_F is a partial function on pairs of worlds in a Kripke frame F, defined by

$d_F(x,y)=\sup\{1+d_F(z,y): xRzRy\}$ if xRw;
$d_F(x,y)$=is undefined otherwise.

Lemma 3.2.4 E is valid on all IL_{exp} frames.
Proof Let F=(W,R,b) be a finite Kripke frame with irreflexive, transitive acessibility relation R, let M be $(F,\Vdash)$. We will show that all instances of E are forced in all worlds of M. We consider the following instance of E:

$$(\Diamond A \wedge A \rhd P_1 \wedge \ldots \wedge A \rhd P_n \wedge S) \rhd (\Box\neg A \wedge P_1 \wedge \ldots \wedge P_n \wedge S).$$

Let x be a world of M, and suppose there are t, y such that xRtRy and

$y \Vdash \Diamond A \wedge A \rhd P_1 \wedge \ldots \wedge A \rhd P_n \wedge S.$ (1)

We will show that there exists a y' such that tRy' and

$y' \Vdash \Box \neg A \wedge P_1 \wedge \ldots \wedge P_n \wedge S.$

As $y \Vdash \Diamond A$ we know there are worlds w', z', such that

yRw'Rz' and $z' \Vdash A$. (2)

From the properties of the frame F it follows that there are w,z satisfying (2) such that $d_F(w,z)=1$ and $d_F(y,z)=\max\{d_F(y,z'): z' \Vdash A\}$. It follows that

$w \Vdash \Box \neg A.$ (3)

But $y \Vdash A \rhd P_1 \wedge \ldots \wedge A \rhd P_n$, yRwRz and $z \Vdash A$; so there must be u_i (for $1 \leq i \leq n$), such that wRu_i and $u_i \Vdash P_i$ for $1 \leq i \leq k$. By downwards preservation of the P_i we get

$w \Vdash P_1 \wedge \ldots \wedge P_n.$ (4)

By (1) and upwards preservation of S we get

$w \Vdash S.$ (5)

Combining (3) – (5) we find that w is the y' we were looking for.

⊠

By the soundness of IL_{exp} frames for IL_{exp}, Lemma 3.2.5 implies that all instances of E are theorems of IL_{exp}. Next we will give a syntactic proof of this fact, by showing that the ^{t}-translation of every instance of E is derivable in L_Δ. We will use the following properties of translated Σ- and Π-formulae in L_Δ:

Lemma 3.2.5 For all S in Σ and all P in Π,

(a) $L_\Delta \vdash S^t \rightarrow \Delta S^t$;

(b) $L_\Delta \vdash \nabla P^t \rightarrow P^t$.

Proof Because every Π-formula is the negation of a Σ-formula, (b) immediately follows from (a), by contraposition. To prove (a), note that L_Δ proves the following:

$\top \rightarrow \Delta \top$;

$\bot \rightarrow \Delta \bot$;

$(\Box \varphi)^t \rightarrow \Delta(\Box \varphi)^t$, for all φ;

$(\varphi \rhd \psi)^t \rightarrow \Delta(\varphi \rhd \psi)^t$, for all φ and ψ.

Now assume that L_Δ proves $\varphi^t \rightarrow \Delta \varphi^t$ and $\psi^t \rightarrow \Delta \psi^t$. Then L_Δ proves $(\varphi \wedge \psi)^t \rightarrow \Delta(\varphi \wedge \psi)^t$ and $(\varphi \vee \psi)^t \rightarrow \Delta(\varphi \vee \psi)^t$. This concludes the proof.

⊠

Lemma 3.2.6

$L_\Delta \vdash \Delta[\nabla[\nabla\nabla A^t \wedge \Delta(\nabla A^t \rightarrow \nabla P_1{}^t) \wedge \ldots \wedge \Delta(\nabla A \rightarrow \nabla P_n{}^t) \wedge S^t] \rightarrow$

$\nabla[\Delta\Delta\neg A^t \wedge P_1{}^t \wedge ... \wedge P_n{}^t \wedge S^t]]$.

Proof Reason in L_Δ. First, note that contraposition of Löb's axiom for $\neg B$ implies $\nabla B \rightarrow \nabla(B \wedge \Delta\neg B)$. Substitution of ∇A^t for B yields

$\nabla\nabla A^t \rightarrow \nabla(\nabla A^t \wedge \Delta\Delta\neg A^t)$. (1)

Suppose

$\nabla\nabla A^t \wedge \Delta(\nabla A^t \rightarrow \nabla P_1{}^t) \wedge ... \wedge \Delta(\nabla A \rightarrow \nabla P_n{}^t) \wedge S^t$. (2)

Then

$\nabla\nabla A^t \wedge \Delta((\nabla A^t \rightarrow (\nabla P_1{}^t \wedge ... \wedge \nabla P_n{}^t)) \wedge S^t)$.

Using (1), we then have

$\nabla(\nabla A^t \wedge \Delta\Delta\neg A^t) \wedge \Delta((\nabla A^t \rightarrow (\nabla P_1{}^t \wedge ... \wedge \nabla P_n{}^t)) \wedge S^t)$.

This implies

$\nabla(\Delta\Delta\neg A^t \wedge \nabla P_1{}^t \wedge ... \wedge \nabla P_n{}^t \wedge S^t)$,

which, by Lemma 3.2.6, yields

$\nabla(\Delta\Delta\neg A^t \wedge P_1{}^t \wedge ... \wedge P_n{}^t \wedge S^t)$, (3)

Thus,

$\nabla\nabla A^t \wedge \Delta(\nabla A^t \rightarrow \nabla P_1{}^t) \wedge ... \wedge \Delta(\nabla A \rightarrow \nabla P_n{}^t) \wedge S^t$
$\quad \rightarrow \nabla(\Delta\Delta\neg A^t \wedge P_1{}^t \wedge ... \wedge P_n{}^t \wedge S^t)$

Using necessitation, this yields

$\nabla[\nabla\nabla A^t \wedge \Delta(\nabla A^t \rightarrow \nabla P_1{}^t) \wedge ... \wedge \Delta(\nabla A \rightarrow \nabla P_n{}^t) \wedge S^t]$
$\quad \rightarrow \nabla\nabla(\Delta\Delta\neg A^t \wedge P_1{}^t \wedge ... \wedge P_n{}^t \wedge S^t)$,

which, by L2, gives

$\nabla[\nabla\nabla A^t \wedge \Delta(\nabla A^t \rightarrow \nabla P_1{}^t) \wedge ... \wedge \Delta(\nabla A \rightarrow \nabla P_n{}^t) \wedge S^t]$
$\quad \rightarrow \nabla(\Delta\Delta\neg A^t \wedge P_1{}^t \wedge ... \wedge P_n{}^t \wedge S^t)$,

Now use necessitation to conclude the proof.

⊠

3.3 Some facts about E

In this paragraph we show that E_1 is derivable in ILP (Lemma 3.3.1), E_2 is not derivable in ILP (Lemma 3.3.2), for n<m, $ILPE_n \nvdash E_m$ (Lemma 3.3.3). The latter implies that ILPE cannot be finitely axiomatized.

Lemma 3.3.1 ILP $\vdash (\Diamond A \wedge A \rhd P_1 \wedge S) \rhd (\Box\neg A \wedge P_1 \wedge S)$.

Proof Reason in ILP:

$\Box(\Box\Box\neg A \rightarrow \Box\neg A) \rightarrow \Box\Box\neg A$ (1)

$((\Box\Box\neg A \rightarrow \Box\neg A) \wedge \Box(\Box\Box\neg A \rightarrow \Box\neg A)) \rightarrow \Box\neg A$ (2)

By contraposition on (2),

$\Diamond A \to ((\Diamond A \wedge \Box\Box\neg A) \vee \Diamond(\Diamond A \wedge \Box\Box\neg A))$ (3)

$\Diamond A \wedge A \rhd P_1 \to$

$((\Diamond A \wedge \Box\Box\neg A \wedge A \rhd P_1) \vee \Diamond(\Diamond A \wedge \Box\Box\neg A) \wedge A \rhd P_1)$ (4)

Use P and J4, to get

$\Diamond A \wedge A \rhd P_1 \to ((\Box\Box\neg A \wedge \Diamond P_1) \vee \Diamond(\Box\Box\neg A \wedge \Diamond P_1))$ (5)

This gives

$\Diamond A \wedge A \rhd P_1 \to ((\Diamond(\Box\neg A \wedge P_1) \vee \Diamond\Diamond(\Box\neg A \wedge P_1))$ (6)

So we get, by Σ_1-completeness,

$\Diamond A \wedge A \rhd P_1 \to \Diamond(\Box\neg A \wedge P_1)$ (7)

Application of L1 gives

$(\Diamond A \wedge A \rhd P_1) \rhd \Diamond(\Box\neg A \wedge P_1)$ (8)

Application of L5 and transitivity of $\rhd$ yields

$(\Diamond A \wedge A \rhd P_1) \rhd (\Box\neg A \wedge P_1)$. (8)

It is a simple application of L2 to the reasoning above, to get

$(\Diamond A \wedge A \rhd P_1 \wedge S) \rhd (\Box\neg A \wedge P_1 \wedge S)$.

⊠

Lemma 3.3.2.

ILP $\nvdash (\Diamond A \wedge A \rhd \Diamond B \wedge A \rhd \Diamond C \wedge \Box D) \rhd (\Box\neg A \wedge \Diamond B \wedge \Diamond C \wedge \Box D)$.

Proof The following is an ILP countermodel for $(\Diamond A \wedge A \rhd \Diamond B \wedge A \rhd \Diamond C \wedge \Box D) \rhd (\Box\neg A \wedge \Diamond B \wedge \Diamond C \wedge \Box D)$:

Let F= (W,R,S) with

W= {x,y,z,s,t,u,v}

R as follows: xRy, yRz, yRs,yRt, sRu,tRv

S the smallest reflexive extension of R containing zSs, zSt.

Define a forcing relation $\Vdash$ on F such that

$z \Vdash A$, and A is forced only there,

$u \Vdash B$, and B is forced only there,

$v \Vdash C$, and C is forced only there,

$y \Vdash \Box D$.

See the figure below.

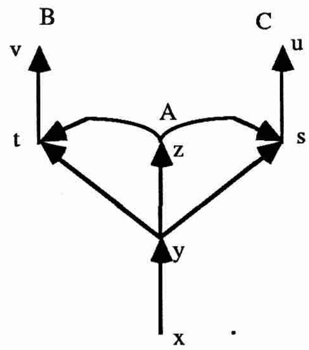


fig 3.3.2

Then $y \Vdash \Diamond A \wedge A \rhd \Diamond B \wedge A \rhd \Diamond C \wedge \Box D$.
Now y is the only world in the model which forces $\Diamond B \wedge \Diamond C$ and is S-accessible from y. But as $y \Vdash \Diamond A$, it does not force $\Box \neg A$.

⊠

Lemma 3.3.3 For $n<m$, $ILPE_n \nvdash E_m$.
Proof Note that $ILPE_{m+1} \vdash E_m$. So the general case $n<m$ can be reduced to showing, for each n, that $ILPE_{n+1} \nvdash E_n$. We will only show the case $n=2$. For other n, essentially the same trick can be used.
We define an ILP Veltman frame $F=\langle W,R,S \rangle$ such that $F \Vdash E_2$ and $F \nVdash E_3$.
$W=\{b, v, z, u_0, u_1, u_2, w_0, w_1, w_2, t_0, t_1, t_2\}$
Let R be the smallest transitive irreflexive relation on W containing

bRv, vRz,
vRu_i for all i,
u_iRw_i for all i,
bRt_i for all i,
t_0Rw_0, t_0Rw_1, t_1Rw_1, t_1Rw_2, t_2Rw_2, t_2Rw_0,

Let S be the smallest reflective extension of R such that also

zSu_i for all i,
vSt_i for all i.

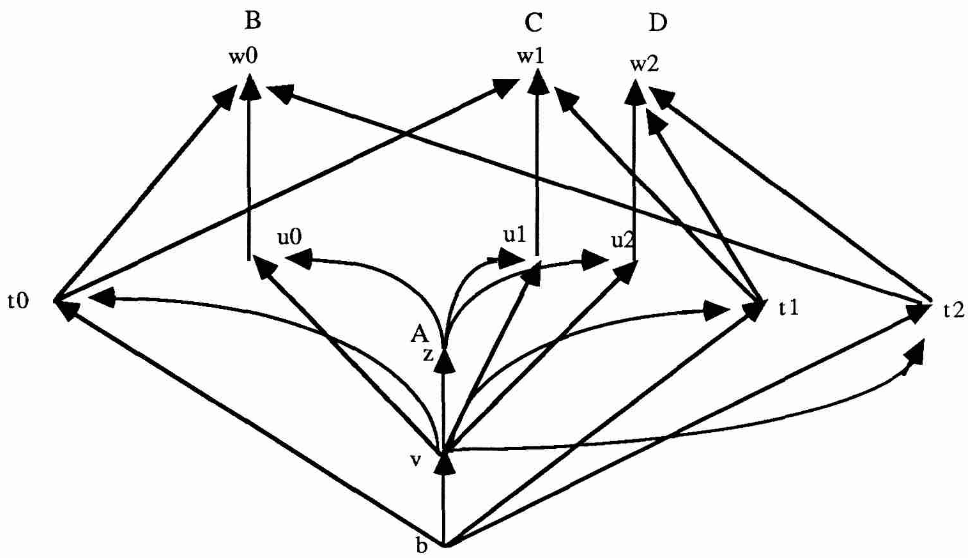


Fig 3.3.3

We show that $F \models E_2$:

Let $\Vdash$ be a forcing relation on F.

Note the following: If $ILP \vdash P \leftrightarrow \top$, then $ILP \vdash (\Diamond A \wedge A \rhd P \wedge A \rhd Q \wedge S) \rhd (\Box\neg A \wedge P \wedge Q \wedge S) \leftrightarrow (\Diamond A \wedge A \rhd Q \wedge S) \rhd (\Box\neg A \wedge Q \wedge S)$. The left part of this equivalence is as we saw in Lemma 3.3.1, already provable in ILP. So we need not consider instances of E_2 for which $ILP \vdash (P \leftrightarrow \top) \vee (Q \leftrightarrow \top)$. If $ILP \vdash P \leftrightarrow \bot$, then $ILP \vdash (\Diamond A \wedge A \rhd P \wedge A \rhd Q \wedge S) \rhd (\Box\neg A \wedge P \wedge Q \wedge S) \leftrightarrow \top$. So we need not consider instances of E_2 for which $ILP \vdash (P \leftrightarrow \bot) \vee (Q \leftrightarrow \bot)$. So we assume $ILP \nvdash (P \leftrightarrow \top) \vee (Q \leftrightarrow \top) \vee (P \leftrightarrow \bot) \vee (Q \leftrightarrow \bot)$.

Suppose x and y are such that xRy and $y \Vdash \Diamond A \wedge A \rhd P \wedge A \rhd Q \wedge S$.

Then $x \Vdash \Diamond\Diamond A$, so x must be either b or v. Because $y \Vdash \Diamond A \wedge A \rhd P$, there must be an a such that yRa and $a \Vdash P$; and because of the assumptions about P, there must be a b such that aRb. Clearly, x must be b and y must be v.

So we have

$$v \Vdash \Diamond A \wedge A \rhd P \wedge A \rhd Q \wedge S. \qquad (1)$$

Note that the frame F has, by Lemma 3.1.3(vi), the following propertie:

$$\text{if } \varphi \text{ is P or Q, then } u_i \Vdash \varphi \Rightarrow t_i \Vdash \varphi \wedge t_{i-1 (\mathrm{mod}\ 3)} \Vdash \varphi \quad (0 \le i \le 2). \qquad (2)$$

We will show that there is a d such that vSd and $d \Vdash \Box\neg A \wedge P \wedge Q \wedge S$.

From the same observations which led us to (1) it follows that A cannot be forced in either of the w_i. This implies that

$$\Box\neg A \text{ is forced in z, in all of the } u_i \text{, and in all of the } t_i. \qquad (3)$$

However, A must be forced in z or in one of the u_i.

If A is forced in one of the u_i, then for this i, by (1), $u_i \Vdash P \wedge Q$. By (3), $u_i \Vdash \Box\neg A$. Also, S is forced in v and, being a Σ-formula, upwards preserved, so $u_i \Vdash S$. Then u_i is the d we were looking for.

If A is not forced in one of the u_i, then A is forced in z. By (1), we will then find both P and Q in worlds *f* and *g* such that z*Sf* and z*Sg*. Note that *f* and *g* cannot be equal to z. So *f* is u_i for an $i \leq 3$, and *g* is u_j for a $j \leq 2$. Now by (2),
if $\{i,j\} \subseteq \{0,1\}$, then $t_0 \Vdash P \wedge Q$,
if $\{i,j\} \subseteq \{1,2\}$, then $t_1 \Vdash P \wedge Q$,
if $\{i,j\} \subseteq \{2,1\}$, then $t_2 \Vdash P \wedge Q$.
As $v \Vdash S$, $t_i \Vdash S$ for all i, by (3), $t_i \Vdash \Box \neg A$ for all i.
So in this case, one of the t_i is the *d* we were looking for.
This shows that $F \models E_2$.

Next we show that $F \not\models E_3$:
$\{x \in W : x \Vdash A\} = \{z\}$;
$\{x \in W : x \Vdash B\} = \{w_0\}$;
$\{x \in W : x \Vdash C\} = \{w_1\}$;
$\{x \in W : x \Vdash D\} = \{w_2\}$.
We will show that

$b \nVdash (\Diamond A \wedge A \rhd \Diamond B \wedge A \rhd \Diamond C \wedge A \rhd \Diamond D) \rhd (\Box \neg A \wedge \Diamond B \wedge \Diamond C \wedge \Diamond D)$.

$v \Vdash \Diamond A \wedge A \rhd \Diamond B \wedge A \rhd \Diamond C \wedge A \rhd \Diamond D$; (1)
$\Diamond B \wedge \Diamond C \wedge \Diamond D$ is only forced in v and b (of which b is not accessible from v); (2)
$v \nVdash \Box \neg A$. (3)
(1) – (3) show that there is no *d* such that v*Sd* and $d \Vdash \Box \neg A \wedge \Diamond B \wedge \Diamond C \wedge \Diamond D$.
This shows that that $F \not\models E_3$.

⊠

3.4 On rules in $\mathbf{IL_{exp}}$

In [RIJ] it is shown that IL, ILP, and ILM have the following property:
Let ILS be either of these three theories, then
$ILS \vdash A \rhd B$ iff $ILS \vdash A \to (B \vee \Diamond B)$.
IL_{exp} does not have this property. The following countermodel $M = (W,R,b,\Vdash)$ shows this:

$W=\{x,y,z,u,v,s,t,r,p\}$;
$R=\{\langle x,y\rangle, \langle y,z\rangle, \langle y,u\rangle, \langle u,s\rangle, \langle s,r\rangle, \langle y,v\rangle, \langle v,t\rangle, \langle t,p\rangle\}$;
$z \Vdash A$, $z \Vdash B$, $z \Vdash C$.

In M, which is an IL_{exp} model,
$x \Vdash (\Diamond A \wedge A \rhd \Diamond B \wedge A \rhd \Diamond C) \rhd (\Box\neg A \wedge \Diamond B \wedge \Diamond C)$.
Also, $x \Vdash \Diamond A \wedge A \rhd \Diamond B \wedge A \rhd \Diamond C$.
But clearly, $x \nVdash (\Box\neg A \wedge \Diamond B \wedge \Diamond C) \vee \Diamond(\Box\neg A \wedge \Diamond B \wedge \Diamond C)$.

We do however have the following:
For A,B in $\mathcal{L}(\Box,\rhd)$
$IL_{exp} \vdash A \rhd B$ iff $L_\Delta \vdash A^t \to (B^t \vee \nabla B^t)$.
The right to left implication is trivial.
The converse is proved semantically, by a trick known as 'Smorynski's trick'. Assume that $L_\Delta \nvdash A^t \to (B^t \vee \nabla B^t)$. Then by modal completeness there is a Kripke model M in which the bottem node b forces $\neg(A^t \to (B^t \vee \nabla B^t))$, so b forces $A^t \wedge \neg B^t \wedge \Delta\neg B^t$.
From this model M we can construct a Kripke model N in which the bottem node does not force $A \rhd B$: Add two worlds to the frame of M, say x and y, and take for the accessibility relation of N the smallest irreflexive transitive extension R' of $R \cup \{\langle x,y\rangle, \langle y,b\rangle\}$. Let, in N x and y force all (or some, or ...) propositional variables, and let for all p and for all worlds z in N other that x and y, $z \Vdash_N p$ iff $z \Vdash_M p$. Clearly then $x \nVdash_N A \rhd B$.

Consider the following rule $\mathcal{S}$:
$\mathcal{S}$ $\vdash(\Diamond A \wedge B) \rhd (\Box\neg A \wedge C) \Rightarrow \vdash(\Diamond A \wedge B \wedge S) \rhd (\Box\neg A \wedge C \wedge S)$,
for $S \in \Sigma$.

Lemma 3.4.1 a) ILP is closed under $\mathcal{S}$; b) IL_{exp} is closed under $\mathcal{S}$.
Proof a) Suppose $ILP \nvdash (\Diamond A \wedge B \wedge S) \rhd (\Box\neg A \wedge C \wedge S)$. Then there is an ILP Veltman model $M=(W,R,S,b,\Vdash)$, such that $b \nVdash (\Diamond A \wedge B \wedge S) \rhd (\Box\neg A \wedge C \wedge S)$. So there is a world y for which bRy and $y \Vdash \Diamond A \wedge B \wedge S$, and for all z such that ySz

and xRz, $z \Vdash \Diamond A \vee \neg C \vee \neg S$. By cutting out the part of the model which is not above y or between b and y, we get a countermodel to $(\Diamond A \wedge B) \rhd (\Box\neg A \wedge C)$:

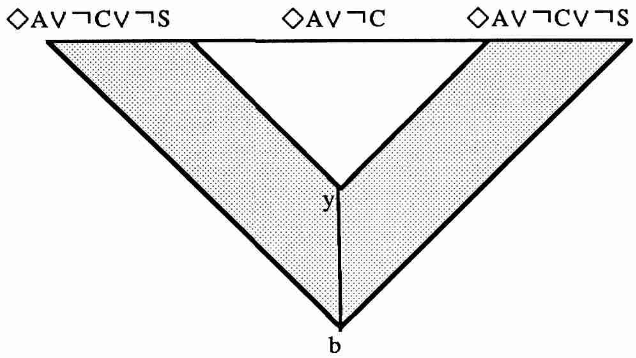


fig. 3.4.1

Let the model $M=(W',R',S',b,\Vdash')$ be defined as follows:

$W'=\{t\in W : t=b \vee t=y \vee bRtRy \vee yRt\}$; $R'=R\cap(W'\times W')$; $S'=S\cap(W'\times W')$; $t\Vdash' p$ iff $t\Vdash p$, for all $t\in W'$ and all propositional variables p.

Then $\forall t\,(y=t \vee yR't \rightarrow (t\Vdash'\varphi \leftrightarrow t\Vdash\varphi))$. So $y\Vdash' \Diamond A \wedge B \wedge S$. Consider a world z such that yS'z and bR'z. Such a z must be either y itself or yRz. So $z\Vdash' \Diamond A \vee \neg C \vee \neg S$.

As $y\Vdash S$ and S is preserved upwards, $z\Vdash' \Diamond A \vee \neg C$.

So $b\nVdash'(\Diamond A \wedge B \wedge) \rhd (\Box\neg A \wedge C)$.

b) Can be proved in the same manner as (a) was proved.

⊠

Let the schema E^- be defined as follows:

$(\Diamond A \wedge A\rhd P_1 \wedge \ldots \wedge A\rhd P_n) \rhd (\Box\neg A \wedge P_1 \wedge \ldots \wedge P_n)$.

Lemma 3.4.1 tells us that, in approaching IL_{exp}, we can either consider ILPE or consider $ILPE^-$ plus the rule $\mathcal{S}$ - both logics are part of IL_{exp}. Clearly, $ILPE^-$ plus the rule $\mathcal{S}$ proves all axioms of ILPE; we do not know whether ILPE is closed under $\mathcal{S}$.

3.5 More on conservativity

Definition 3.5.1 We define for formulae φ in $L(\Box, \rhd)$ the $\rhd$-depth $ID(\varphi)$ as follows:

(i) $ID(p) = 0$ for all propositional variables p;

(ii) $ID(\Box\varphi) = D(\varphi)$ for all φ;

(iii) $ID(\varphi \rhd \psi) = \max\{ID(\varphi), ID(\psi)\} + 1$ for all φ and ψ;

(iv) $ID(\neg\varphi) = ID(\varphi)$;

(v) $ID(\varphi \wedge \psi) = ID(\varphi \vee \psi) = \max\{ID(\varphi), ID(\psi)\}$.

Consider the schema K, where, like in E, the P_i are Π-formulae:

$$\Diamond\Diamond A \wedge A \rhd P_1 \wedge \ldots \wedge A \rhd P_n \rightarrow \Diamond(\Box\neg A \wedge P_1 \wedge \ldots \wedge P_n).$$

Lemma 3.5.2 (a) $IL_{exp} \vdash K$; (b) $ILP \vdash K_1$; (c) $ILP \nvdash K$.

Proof

(a) $ILP \vdash \Diamond\Diamond A \wedge A \rhd P_1 \wedge \ldots \wedge A \rhd P_n$
$\rightarrow \Diamond(\Diamond A \wedge A \rhd P_1 \wedge \ldots \wedge A \rhd P_n)$.

Application of J4 to E_n shows that $IL_{exp} \vdash K$.

(b) Left to the reader.

(c) $ILP \nvdash \Diamond\Diamond a \wedge a \rhd \Diamond p \wedge a \rhd \Diamond q \rightarrow \Diamond(\Box\neg a \wedge \Diamond p \wedge \Diamond q)$.

A countermodel is $M=(F, \Vdash)$, where F is the frame in the proof of Lemma 3.3.2, and $\Vdash$ is defined as follows:

$3 \Vdash a$ and a is only forced there;

$6 \Vdash p$ and p is only forced there;

$7 \Vdash q$ and q is only forced there;

Clearly $1 \Vdash \Diamond\Diamond a \wedge a \rhd \Diamond p \wedge a \rhd \Diamond q \wedge \Box(\Diamond p \wedge \Diamond q \rightarrow \Diamond a)$.

⊠

Lemma 3.5.2 shows that (1) IL_{exp} is not conservative over ILP with regard to formulae φ for which $ID(\varphi) \leq 1$; and that (2) IL_{exp} is not conservative over ILP with regard to formulae φ for which $D(\varphi) \leq 2$.

4 The axiom schema X

4.1. The axiom schema X

Definition 4.1.1 $X_{n,k}$ is the following axiom schema:

$$(P_1 \wedge \ldots \wedge P_n) \rhd (\Box\neg A \to D_1) \wedge \ldots \wedge (P_1 \wedge \ldots \wedge P_n) \rhd (\Box\neg A \to D_k)$$
$$\to (\Diamond A \wedge S) \rhd (A \rhd P_1 \wedge \ldots \wedge A \rhd P_n \to D_1 \wedge \ldots \wedge D_k \wedge S).$$

where $P_1,\ldots,P_n$, $D_1,\ldots,D_k$ are Π-formulae, and S is a Σ-formula.

In its basic form, the schema X was found by Marc Jumelet.
In this section we will always suppose that $P_1,\ldots,P_n$, $D_1,\ldots,D_k$ are Π-formulae, and that S is a Σ-formula.

Lemma 4.1.2 $IL_{exp} \vdash X_{n,k}$, for all n, k.
Proof Suppose we have a finite IL_{exp} Kripke model $M=(W,R,b,\Vdash)$, such that
$b \nVdash (P_1 \wedge \ldots \wedge P_n) \rhd (\Box\neg A \to D_1)\} \wedge \ldots \wedge (P_1 \wedge \ldots \wedge P_n) \rhd (\Box\neg A \to D_k)$
$\to (\Diamond A \wedge S) \rhd (A \rhd P_1 \wedge \ldots \wedge A \rhd P_n \to D_1 \wedge \ldots \wedge D_k \wedge S)$
for some P_1, P_2, D_1, D_2 in Π and S in Σ.
So,
$b \Vdash (P_1 \wedge \ldots \wedge P_n) \rhd (\Box\neg A \to D_1) \wedge \ldots \wedge (P_1 \wedge \ldots \wedge P_n) \rhd (\Box\neg A \to D_k)$ (2)
and there are worlds x and y such that
$bRxRy$, (3a)
$y \Vdash \Diamond A \wedge S$, (3b)
$\forall z\, \{xRz \to z \Vdash A \rhd P_1 \wedge \ldots \wedge A \rhd P_n \wedge (\neg D_1 \vee \ldots \vee \neg D_n \vee \neg S)$. (3c)
In particular, from (3b&c), $y \Vdash \neg D_1 \vee \ldots \vee \neg D_n$. Suppose, without loss of generality, that
$y \Vdash \neg D_1$. Note that, because D_1 is a Π-formula, $\neg D_1$ is upwards preserved, i.e.
$\forall t\, \{yRt \to t \Vdash \neg D_1\}$ (4)

By properties of W and R, there must be a world w such that yRw or y=w, $w \Vdash \Diamond A$, and $\forall t\{wRt \to t \Vdash \Box\neg A\}$. Let t and z witness this, i.e.,
$wRtRz$, (5a)
$z \Vdash A$, (5b)
$t \Vdash \Box\neg A$. (5c)

By (3c), there must be $u_1,\ldots,u_n$, such that
tRu_i and $u_i \Vdash P_i$ for $1 \le i \le n$. (6)

From (6) and the fact that the P_i are downwards preserved, we get

$t \Vdash P_1 \wedge \ldots \wedge P_n$. (7)

By (2) and (7), there must be $v_1, \ldots, v_n$, such that

wRv_i and $v_i \Vdash \Box\neg A \to D_i$ for $1 \leq i \leq n$. (8)

By (4), $v_i \Vdash \neg D_1$, for $1 \leq i \leq n$, so $v_1 \Vdash \Diamond A$, which contradicts (5c).

This completes the proof.

⊠

The following corollarium says that if we consider $X_{n,1}$ we can drop the condition that D_1 is a Π-formula.

Corallarium 4.1.3

$$IL_{exp} \vdash (P_1 \wedge \ldots \wedge P_n) \rhd (\Box\neg A \to B)$$
$$\to (\Diamond A \wedge S) \rhd (A \rhd P_1 \wedge \ldots \wedge A \rhd P_n \to B \wedge S)$$

Proof The proof goes along the same lines as the proof of Lemma 4.1.2. In this case however, because we do not get a disjunction in (3c), we do not need upward preservativion of the different disjuncts. Instead, we can simply remark that $\forall z \, \{xRz \to z \Vdash A \rhd P_1 \wedge \ldots \wedge A \rhd P_n \wedge \neg B\}$, then proceed along (5), (6) and (7) and find that there must be $v_1, \ldots, v_n$, such that wRv_i and $v_i \Vdash \Diamond A \vee B$ for $1 \leq i \leq n$. By our remark, $v_i \Vdash \neg B$, for $1 \leq i \leq n$, but also $v_i \Vdash \Box\neg A$ by (5c), for $1 \leq i \leq n$. Contradiction.

⊠

Lemma 4.1.4 $ILP \vdash X_{1,k}$ for $k \geq 1$.

Proof Suppose $M=(W,R,N,b,\Vdash)$ is a levelled Friedman counter model to an instance of $X_{1,k}$, say (w.l.o.g.) $b \Vdash P \rhd (\Box\neg A \to D_1) \wedge \ldots \wedge P \rhd (\Box\neg A \to D_k)$ (1)

and $b \nVdash (\Diamond A \wedge S) \rhd (A \rhd P \to D_1 \wedge \ldots \wedge D_k \wedge S)$. (2)

By (2), there exist y, x such that bRyRx, N(x), and

$x \Vdash \Diamond A \wedge S$ and $\forall z \, (N(z) \wedge yRz \to z \Vdash A \rhd P \wedge (\neg D_1 \vee \ldots \vee \neg D_k \vee \neg S)$.

In particular,

$x \Vdash A \rhd P$ and $x \Vdash \neg D_1 \vee \ldots \vee \neg D_k$. (3)

Suppose $x \Vdash \neg D_1$. (4)

By finiteness of W, there is an x' such that N(x) and x'=x or xRRx' and $x' \Vdash \Box\Box\neg A$ and there are u, v, such that N(v), x'RuRv and $v \Vdash A$. Also, $x' \Vdash \neg D_1$. By (3), there is a w such that N(w), uRw and $w \Vdash P$. By (1), there must be a t such that N(t) and uRt and $t \Vdash \Diamond A \vee D_1$. On the other hand for such a t we find that by our choice of x', $t \Vdash \Box\neg A$, and by (4) and upward preservation of $\neg D_1$, $t \Vdash \neg D_1$. Contradiction.

⊠

Remark that also in this case, if k=1 we can drop the condition that D is a Π-formula. So we find

Corollarium 4.1.5 For every Π-formula P,

$ILP \vdash P \rhd (\Box\neg A \to B) \to (\Diamond A \wedge S) \rhd (A \rhd P \to B \wedge S)$.

Proof Like the proofs of Lemma 4.1.4 and Corollarium 4.1.3.

⊠

Lemma 4.1.6

(a) $(P_1 \wedge ... \wedge P_n) \rhd (\Box\neg A \to \top) \wedge ... \wedge (P_1 \wedge ... \wedge P_n) \rhd (\Box\neg A \to D_k)$
$\to (\Diamond A \wedge S) \rhd (A \rhd P_1 \wedge ... \wedge A \rhd P_n \to \top \wedge ... \wedge D_k \wedge S)$

is in fact an instance of $X_{n,k-1}$;

(b) $(P_1 \wedge ... \wedge \top) \rhd (\Box\neg A \to D_1) \wedge ... \wedge (P_1 \wedge ... \wedge P_n) \rhd (\Box\neg A \to D_k)$
$\to (\Diamond A \wedge S) \rhd (A \rhd \top \wedge ... \wedge A \rhd P_n \to D_1 \wedge ... \wedge D_k \wedge S)$

is in fact an instance of $X_{n-1,k}$;

(c) $(P_1 \wedge ... \wedge \bot) \rhd (\Box\neg A \to D_1) \wedge ... \wedge (P_1 \wedge ... \wedge P_n) \rhd (\Box\neg A \to D_k)$
$\to (\Diamond A \wedge S) \rhd (A \rhd \bot \wedge ... \wedge A \rhd P_n \to D_1 \wedge ... \wedge D_k \wedge S)$

is equivalent to $\top$ (already in IL).

(d) $(P_1 \wedge ... \wedge P_n) \rhd (\Box\neg A \to D_1) \wedge ... \wedge (P_1 \wedge ... \wedge P_n) \rhd (\Box\neg A \to \bot)$
$\to (\Diamond A \wedge S) \rhd (A \rhd P_1 \wedge ... \wedge A \rhd P_n \to \bot \wedge ... \wedge D_k \wedge S)$

is already provable in ILP.

Proof

(a) follows from $(\varphi \to \top) \leftrightarrow \top$ and $\top \wedge \varphi \leftrightarrow \varphi$.

(b) follows immediately from $\varphi \wedge \top \leftrightarrow \varphi$ and $A \rhd \top \leftrightarrow \top$;

(c) follows from $A \rhd \bot \leftrightarrow \Box\neg A$ and $\varphi \rhd \varphi$;

(d) by reasoning on ILP Veltman or Friedman models.

⊠

Lemma 4.1.7 $ILPX_{n,k} \nvdash X_{n+1,k}$, for $n \geq 1, k \geq 1$.

Proof We show how to prove this lemma for the case n=2, k=1. The example generalizes to other cases. Consider an ILP Veltman frame F=$\langle W,R,S \rangle$, with W, R and S as follows:

$W = \{b, x, z, u_1, u_2, u_3, w_1, w_2, w_3, t_1, t_2, t_3\}$.

R is the smallest transitive irreflexive relation on W containing

bRx, xRz,

vRu_i for all i,

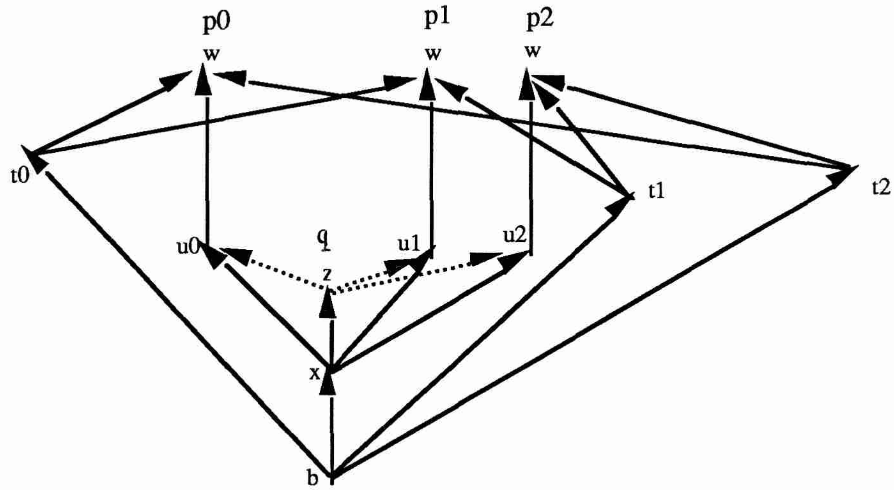
u_iRw_i for all i,

bRt_i for all i,

t_1Rw_1, t_1Rw_2, t_2Rw_2, t_2Rw_3, t_3Rw_3, t_3Rw_1,

Let S be the smallest reflexive extension of R such that also zSu_i for all i.

See the figure below.



To see that $F \models X_{2,k}$, note that for $0 \leq i \leq 2$, $F \cap \{b,x,z,u_i,u_{i+1},w_i,w_{i+1},t_i\}$ is isomorfic to the frame in the proof of Lemma 4.1.5.

Let $\Vdash$ be the following forcing relation on F: $w_i \Vdash p_i$, and p_i is forced nowhere else, for $0 \leq i \leq 2$, and q is only forced in z. Then $\Diamond p_0 \wedge \Diamond p_1 \wedge \Diamond p_2$ is only forced in x, and $x \Vdash \Diamond q$, so $(\Diamond p_0 \wedge \Diamond p_1 \wedge \Diamond p_2) \rhd (\Box \neg q \rightarrow \Diamond s)$ is forced in every world. Also, $q \rhd \Diamond p_0 \wedge q \rhd \Diamond p_1 \wedge q \rhd \Diamond p_2$ is forced in every world. But everywhere $\Box \neg s$ is is forced. So

$$b \nVdash (\Diamond p_0 \wedge \Diamond p_1 \wedge \Diamond p_2) \rhd (\Box \neg q \rightarrow \Diamond s) \rightarrow$$
$$\Diamond q \rhd (q \rhd \Diamond p_0 \wedge q \rhd \Diamond p_1 \wedge q \rhd \Diamond p_2 \rightarrow \Diamond s).$$

So $F \nvDash X_{3,1}$.

⊠

Lemma 4.1.8 $ILPX_{n,k} \nvdash X_{n,k+1}$.

Proof We show that the lemma holds for n=2, k=1.

Consider the following Veltman frame F=(W,R,S) for ILP:

W={1,2,3,4,5,6,7,8};

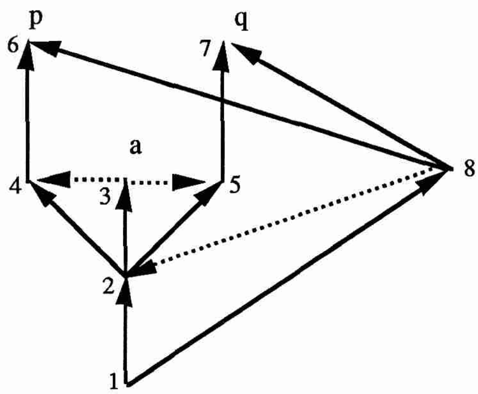
R is the smallest transitive extension on W of

{1R2, 1R8, 2R3, 2R4, 2R5, 4R6, 5R7, 8R6, 8R7};

S is the smallest transitive, reflexive extension of R∪{ 8S2, 3S4, 3S5}.

Let M=(F,$\Vdash$), with $\Vdash$ defined as follows:

$x \Vdash a$ iff x=3; $x \Vdash p$ iff x=6; $x \Vdash q$ iff x=7; t is nowhere forced. See the figure:



We will show that $F \models X_{2,1}$, and

$M \not\models (\Diamond p \wedge \Diamond q) \rhd (\Diamond a \vee \Diamond d_1) \wedge (\Diamond p \wedge \Diamond q) \rhd (\Diamond a \vee \Diamond d_2)$
$\rightarrow \Diamond a \rhd (a \rhd \Diamond p \wedge a \rhd \Diamond q \rightarrow \Diamond d_1 \wedge \Diamond d_2)$.

The only worlds in which $\Diamond q \wedge \Diamond r$ is forced are 2 and 8. But $2 \Vdash \Diamond a$, and 2S2 and 2S8 so $1 \Vdash (\Diamond p \wedge \Diamond q) \rhd (\Diamond a \vee \Diamond d_1) \wedge (\Diamond p \wedge \Diamond q) \rhd (\Diamond a \vee \Diamond d_2)$.

Every world of M forces $a \rhd \Diamond q \wedge a \rhd \Diamond r$, whereas $\Diamond d_1 \wedge \Diamond d_2$ is nowhere forced. So none of the worlds forces $a \rhd \Diamond p \wedge a \rhd \Diamond q \rightarrow \Diamond d_1 \wedge \Diamond d_2$, while $2 \Vdash \Diamond p$. So $1 \nVdash \Diamond a \rhd (a \rhd \Diamond p \wedge a \rhd \Diamond q \rightarrow \Diamond d_1 \wedge \Diamond d_2)$. Thus,

$1 \nVdash (\Diamond p \wedge \Diamond q) \rhd (\Diamond a \vee \Diamond d_1) \wedge (\Diamond p \wedge \Diamond q) \rhd (\Diamond a \vee \Diamond d_2)$
$\rightarrow \Diamond a \rhd (a \rhd \Diamond p \wedge a \rhd \Diamond q \rightarrow \Diamond d_1 \wedge \Diamond d_2)$.

So $F \not\models X_{2,2}$.

Let φ be an instance of $X_{2,1}$, i.e. φ is

$(P_1 \wedge P_2) \rhd (\Box \neg A \rightarrow D) \rightarrow (\Diamond A) \rhd (A \rhd P_1 \wedge A \rhd P_2 \rightarrow D)$.

Because none of the worlds among 3, 4, 5, 6, 7, 8 can force $\Diamond \Diamond A$, any of them forces φ. Suppose that $2 \nVdash \varphi$, i.e.

$2 \Vdash (P_1 \wedge P_2) \rhd (\Box \neg A \rightarrow D)$, (1)

and there is a world x such that

2Rx and $x \Vdash \Diamond A$, (2)

$\forall y\, (xSy \wedge 2Ry \rightarrow y \Vdash A \rhd P_1 \wedge A \rhd P_2 \wedge \neg D)$. (3)

Then x must be 4 or 5. Suppose x=4. Then, by (2), $6 \Vdash A$, by (3), $6 \Vdash P_1 \wedge P_2$. By (1), $6 \Vdash \Box \neg A \rightarrow D$, but by (3) and because 6 is blind, $6 \Vdash \Box \neg A \wedge \neg D$. So x cannot be 4. By the same considerations, x cannot be 5. So $2 \Vdash \varphi$.

Suppose that $1 \nVdash \varphi$, i.e.

$1 \Vdash (P_1 \wedge P_2) \rhd (\Box \neg A \rightarrow D)$, (4)

and there is a world x such that

1Rx and $x \Vdash \Diamond A$, (5)

$\forall y\, (xSy \wedge 1Ry \rightarrow y \Vdash A \rhd P_1 \wedge A \rhd P_2 \wedge \neg D)$. (6)

Like above, x cannot be 4 or 5, so $4 \Vdash \Box\neg A$ and $5 \Vdash \Box\neg A$. (7)

Suppose $4 \Vdash A$. Then, by (6) and downwards preservation of P_1 and P_2, $4 \Vdash P_1 \wedge P_2$. By jump over, $8 \Vdash P_1 \wedge P_2$. Then by (4), we must find $\Diamond A \vee D$ in either 8, or 7, or 6. By (7), all of those force $\Box\neg A$. By (7), 7 and 8 force $\neg D$, so also $8 \Vdash \neg D$. So $4 \nVdash A$. By reasons of symmetry, $5 \nVdash A$.

Suppose $3 \Vdash A$ and $4 \Vdash \neg A \wedge \Box\neg A$ and $5 \Vdash \neg A \wedge \Box\neg A$. Then by (6) and jumpover, $8 \Vdash P_1 \wedge P_2$, which, like above leads to a contradiction.

So x can not be 2. Suppose x=8. Then, by (1), either $6 \Vdash A$ or (the symmetrical case) $7 \Vdash A$. If $6 \Vdash A$, then by (6), $6 \Vdash P_1 \wedge P_2$. Then, by (4), $6 \Vdash \Box\neg A \rightarrow D$. But by (6), $6 \Vdash \neg D$.

So $1 \Vdash \varphi$.

⊠

In the next two lemmata it is shown show that the schemata X and E are independent over ILP.

Lemma 4.1.9 ILPX $\nvdash$ E.

Proof Consider the following Veltman frame F = (W,R,S), (see the figure below) with

W={1,2,3,4,5,6,7,8};

R is the smallest transitive extension on W of

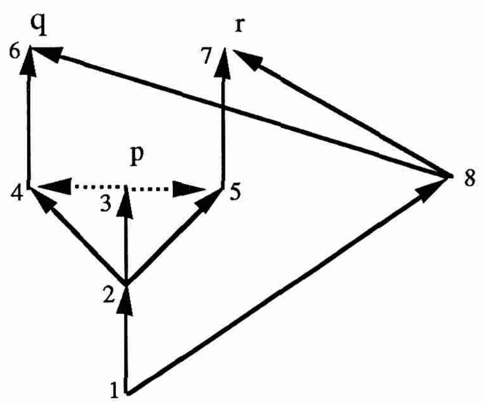
{1R2, 1R8, 2R3, 2R4, 2R5, 4R6, 5R7, 8R6, 8R7};

S is the smallest transitive, reflexive extension of R ∪ {3S4, 3S5}.

Let M = (F, $\Vdash$) be a model on F with $x \Vdash p$ iff x=3, $x \Vdash q$ iff x=6, $x \Vdash r$ iff x=7.

M is a countermodel for E: In M, $2 \Vdash \Diamond p \wedge p \rhd \Diamond q \wedge p \rhd \Diamond r$; $\Box\neg p \wedge \Diamond q \wedge \Diamond r$ is only forced in 8, but we do not have 2S8.

Thus $b \nVdash (\Diamond p \wedge p \rhd \Diamond q \wedge p \rhd \Diamond r) \rhd (\Box\neg p \wedge \Diamond q \wedge \Diamond r)$.



$F \vDash X$: We consider an instance of $X_{n,k}$,

$$(P_1 \wedge \ldots \wedge P_n) \rhd (\Box \neg A \to D_1) \wedge \ldots \wedge (P_1 \wedge \ldots \wedge P_n) \rhd (\Box \neg A \to D_k)$$
$$\to (\Diamond A \wedge W) \rhd (A \rhd P_1 \wedge \ldots \wedge A \rhd P_n \to D_1 \wedge \ldots \wedge D_k \wedge W,$$

with W a Σ-formula.

Let $\Vdash$ be a forcing relation on F.

To show that $X_{n,k}$ is forced in every world x of $M=(F,\Vdash)$, we treat three different cases:

(a) If $x>2$, then $x \Vdash \Box\Box\neg A$, so $x \Vdash X_{n,k}$.

(b) $x=2$. Suppose

$$2 \Vdash (P_1 \wedge \ldots \wedge P_n) \rhd (\Box \neg A \to D_1) \wedge \ldots \wedge (P_1 \wedge \ldots \wedge P_n) \rhd (\Box \neg A \to D_k) \quad (1)$$

and $2 \nVdash (\Diamond A \wedge W) \rhd ((A \rhd P_1 \wedge \ldots \wedge A \rhd P_n) \to D_1 \wedge \ldots \wedge D_k \wedge W)$.

Then either $4 \Vdash \Diamond A \wedge W$ and (2)

$$\forall z\, (4Sz \to z \Vdash A \rhd P_1 \wedge \ldots \wedge A \rhd P_n \wedge (\neg D_1 \vee \ldots \vee \neg D_k \vee \neg W), \quad (3)$$

or (2) and (3) are true for 5. We treat the case in which they hold for 4. From (2) $6 \Vdash A \wedge S$, so from (3),

$$6 \Vdash P_1 \wedge \ldots \wedge P_n \wedge (\neg D_1 \vee \ldots \vee \neg D_k). \quad (4)$$

Now either $6 \nVdash P_1 \wedge \ldots \wedge P_n$ in which case we have a contradiction, or $6 \Vdash P_1 \wedge \ldots \wedge P_n$ in which case we get, by (1) $6 \Vdash (\Diamond A \vee D_1) \wedge \ldots \wedge (\Diamond A \vee D_n)$. The latter implies $6 \Vdash D_1 \wedge \ldots \wedge D_n$, which contradicts (4).

(c) $x=1$. Suppose

$$1 \Vdash (P_1 \wedge \ldots \wedge P_n) \rhd (\Box \neg A \to D_1) \wedge \ldots \wedge (P_1 \wedge \ldots \wedge P_n) \rhd (\Box \neg A \to D_k) \quad (5)$$

and $1 \nVdash (\Diamond A \wedge W) \rhd (A \rhd P_1 \wedge \ldots \wedge A \rhd P_n \to D_1 \wedge \ldots \wedge D_k \wedge W)$ (6)

and

$$\forall z>1 (\, z \nVdash (P_1 \wedge \ldots \wedge P_n) \rhd (\Box \neg A \to D_1) \wedge \ldots \wedge (P_1 \wedge \ldots \wedge P_n) \rhd (\Box \neg A \to D_k)$$
$$\vee\ z \Vdash (\Diamond A \wedge W) \rhd (A \rhd P_1 \wedge \ldots \wedge A \rhd P_n \to D_1 \wedge \ldots \wedge D_k \wedge W).$$

We distinguish two different cases:

(c1) $2 \Vdash (\Diamond A \wedge W)$ and (7)

$$\forall z\, (2Sz \to z \Vdash A \rhd P_1 \wedge \ldots \wedge A \rhd P_n \wedge (\neg D_1 \vee \ldots \vee \neg D_k \vee \neg W) \quad (8)$$

and

(c2) (7) and (8) hold for 8.

c2: Then $6 \Vdash A \wedge W$, so by (8), $6 \Vdash P_1 \wedge \ldots \wedge P_n$, so by (5), $6 \Vdash D_1 \wedge \ldots \wedge D_k$, which contradicts (8); or the same holds for 7.

c1: Again, there are two possibilities. Either $6 \Vdash A$ and $4 \Vdash \neg A$ (or the same holds for 7 and 5, which is, by symmetry, the same case), but this contradicts our supposition. Or

$3 \Vdash A$, or $4 \Vdash A \wedge \Box\neg A$ and $5 \Vdash \Box\neg A$ (or, alternatively, 4 and 5 are interchanged here, which is essentially the same case).

Suppose $4 \Vdash A \wedge \Box\neg A$ and $5 \Vdash \Box\neg A$. Then by (8), and downward preservation of Π-formulae, $4 \Vdash P_1 \wedge ... \wedge P_n$. Then by (5) and the fact that if 4Sz then $z \Vdash \Box\neg A$, and by downwards preservation of $D_1,...D_k$, $4 \Vdash D_1 \wedge ... \wedge D_k$. Also, by (7), $4 \Vdash S$. But this contradicts (8).

Suppose that $3 \Vdash A$, and that both 4 and 5 force $\neg A \wedge \Box\neg A$. Then by (8) and the jumpover of $P_1,...,P_n$, we find that $8 \Vdash P_1 \wedge ... \wedge P_n$. So, by (5), we find $u_1,...u_k$ among {6,7,8}, such that $u_i \Vdash \Box\neg A \rightarrow D_i$, for $1 \leq i \leq k$. From the supposition we know that any of the worlds 6, 7, and 8 force $\Box\neg A$, so that $u_i \Vdash D_i$. By jumpover and downward preservation we find that $2 \Vdash D_1 \wedge ... \wedge D_n$. Also $2 \Vdash W$. But this contradicts (8).

⊠

Lemma 4.1.10 ILPE $\nVdash X_{n,k}$ for $n \geq 2$.

Proof Consider the following ILP Veltman frame F=(W,R,S):

W={1,2,3,4,5,6,7,8};

R is the smallest transitive extension on W of

{1R2, 1R8, 2R3, 2R4, 2R5, 4R6, 5R7, 8R6, 8R7};

S is the smallest transitive, reflexive extension of $R \cup$ {2S8, 8S2, 3S4, 3S5}.

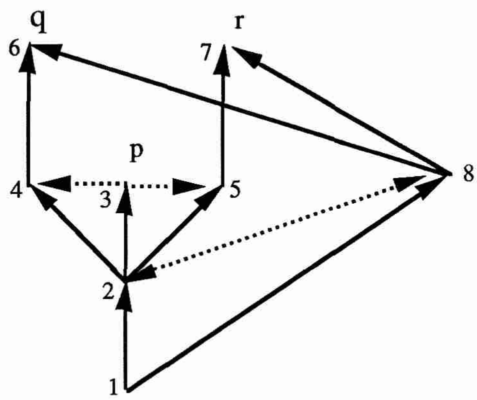
Let $M=(F, \Vdash)$, with $\Vdash$ defined as follows:

$x \Vdash p$ iff $x=3$; $x \Vdash q$ iff $x=6$; $x \Vdash r$ iff $x=7$; t is nowhere forced.

The only worlds in which $\Diamond q \wedge \Diamond r$ is forced are 2 and 8. But $2 \Vdash \Diamond p$, so $2 \Vdash \Box\neg p \rightarrow \Diamond t$, and we have 2S2 and 8S2. So $1 \Vdash (\Diamond q \wedge \Diamond r) \rhd (\Box\neg p \rightarrow \Diamond t)$.

Every world of M forces $p \rhd \Diamond q \wedge p \rhd \Diamond r$, whereas $\Diamond t$ is forced nowhere. So none of the worlds forces $p \rhd \Diamond q \wedge p \rhd \Diamond r \rightarrow \Diamond t$, while $2 \Vdash \Diamond p$. So $1 \nVdash \Diamond p \rhd (p \rhd \Diamond q \wedge p \rhd \Diamond r \rightarrow \Diamond t)$. Thus,

$$1 \nVdash (\Diamond q \wedge \Diamond r) \rhd (\Box\neg p \rightarrow \Diamond t) \rightarrow \Diamond p \rhd (p \rhd \Diamond q \wedge p \rhd \Diamond r \rightarrow \Diamond t).$$



Let $M=(F,\Vdash)$, for some forcing relation $\models$ on F. We show that $M\models E$.

Being blind worlds, 3, 6 and 7 force E. 4 and 5 force E because 6 and 7 do not force $\Diamond A$.

Suppose 2Rx and $x\Vdash \Diamond A \wedge\ A \rhd P_1 \wedge ... \wedge A \rhd P_n$. As 3 does not force $\Diamond A$, x cannot be 3. So x is 4 (or, the symmetrical case, which is treated similarly, x is 5). Then $6\Vdash \Box\neg A \wedge P_1 \wedge ... \wedge P_n$. So $2\Vdash E$.

Suppose 1Rx and $x\Vdash \Diamond A \wedge\ A \rhd P_1 \wedge ... \wedge A \rhd P_n$. We distinguish four cases.

Case1: x is 2 and $6\Vdash A$ (or, the symmetical case, $7\Vdash A$). Then again we find $6\Vdash \Box\neg A \wedge P_1 \wedge ... \wedge P_n$.

Case 2: x is 2 and $4\Vdash A \wedge \Box\neg A$ and $5\Vdash \Box\neg A$ (or 4 and 5 are interchanged). Then we find, by our supposition, that $4\Vdash P_1 \wedge ... \wedge P_n$. This implies, by jumping over, $8\Vdash P_1 \wedge ... \wedge P_n$. Also by jumping over, $8\Vdash \Box\neg A$. (Use 2S8.)

Case 3: x is 2 and $3\Vdash A$ and 4 and 5 force $\neg A \wedge \Box\neg A$. Then we find some of the P_i in 4, the others in 5. Jumping them over, we must find $8\Vdash P_1 \wedge ... \wedge P_n$. Also by jumping over, $8\Vdash \Box\neg A$.

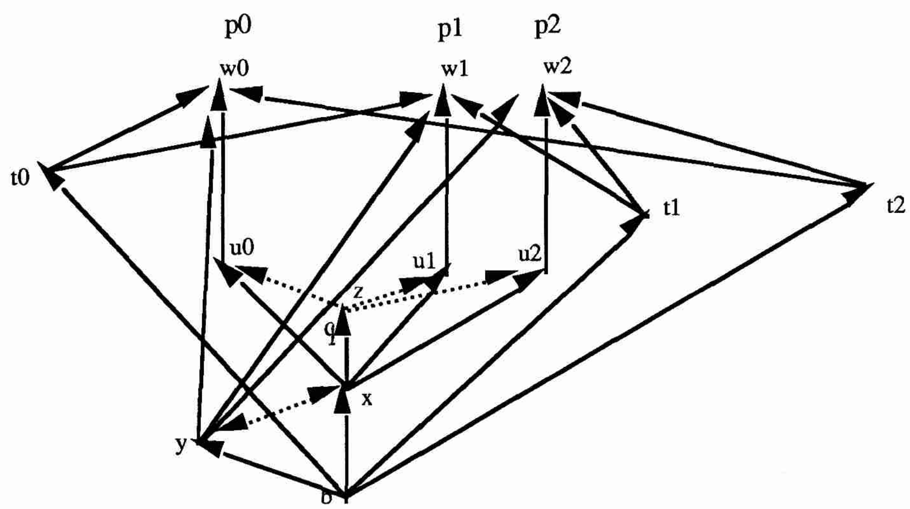
Case 4: x is 8. Then we find A in 6 (or 7), so $6\Vdash P_1 \wedge ... \wedge P_n$. Being blind, $6\Vdash \Box\neg A$. (Use 8S6.)

This completes the proof.

⊠

Lemma 4.1.11 $ILPEX_{n,k} \nvdash X_{n+1,k}$, for $n\geq 1$, $k\geq 1$.

Proof We will show that the lemma holds for n=2, k=1. Let W, R, S, be as in the proof Lemma 4.1.7. Let F'=(W',R',S') be defined by $W'=W\cup\{y\}$, R' is the transitive closure of $R\cup\{bRy,\ yRw_0, yRw_1,\ yRw_2\}$, S' is the reflexive transitive closure of $S\cup\{xSy,\ ySx\}$. See the figure below.



Reason along the lines of the proofs of Lemma 4.1.7 and 4.1.10 to see that $F'\models ILPEX_{2,k}$. A forcing relation defined exactly like the forcing relation in the proof of

Lemma 4.1.7 yields a model M on F' such that

$$M \not\models (\Diamond p_0 \wedge \Diamond p_1 \wedge \Diamond p_2) \rhd (\Box \neg q \rightarrow \Diamond s) \rightarrow$$
$$\Diamond q \rhd (q \rhd \Diamond p_0 \wedge q \rhd \Diamond p_1 \wedge \, q \rhd \Diamond p_2 \rightarrow \Diamond s).$$

⊠

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