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Logic Group
Preprint Series
No. 52
October 1989

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MSC-1980 classification 03B45/03F30
Key words and phrases: Provability Logic,
Σ_1, fragments of arithmetic

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ABSTRACT: In this paper we show that $I\Delta_0+\Omega_1$ verifies the sentential Σ_1^0-conservativity of schematical, sentential Σ_1^0-completeness. (This means that for any finite set of Σ_1^0-sentences S we can prove in $I\Delta_0+\Omega_1$ that the statement expressing the completeness of S w.r.t. $I\Delta_0+\Omega_1$ is conservative over $I\Delta_0+\Omega_1$ w.r.t. Σ_1^0-sentences.) Some consequences are discussed. We formulate a system of provability logic based on the verifiable sentential Σ_1^0-conservativity of schematical, sentential Σ_1^0-completeness.

1 Introduction

As is well known it is a difficult question whether $I\Delta_0+\Omega_1$ proves Σ_1^0-Completeness. From Buss[86], chapter 8 we can extract the following point: let A(x) be any coNP-complete Π_1^b formula. Suppose $I\Delta_0+\Omega_1$ proves: $\forall x(A(x)\rightarrow\Box_{I\Delta_0+\Omega_1}A(x))$. Then by Parikh's Theorem for some polynomial P(x) $I\Delta_0+\Omega_1$ proves: $\forall x(A(x)\rightarrow\exists|y|<P(|x|)\ \mathrm{Proof}_{I\Delta_0+\Omega_1}(y,A(x)))$. Hence *in the standard model* we have: $\forall x(A(x)\leftrightarrow\exists|y|<P(|x|)\ \mathrm{Proof}_{I\Delta_0+\Omega_1}(y,A(x)))$. In other words A(x) is equivalent to a Σ_1^b-predicate. Ergo NP=coNP. On the other hand if $I\Delta_0+\Omega_1$ *proves* a suitable schematic version of NP=coNP, then -as is easily seen- $I\Delta_0+\Omega_1$ proves Σ_1^0-Completeness.

Verbrugge (see Verbrugge[89]) shows that for A(x) in the above argument we may also take a formula of the form: $\Box_{I\Delta_0+\Omega_1}B(x)<\Box_{I\Delta_0+\Omega_1}C(x)$. Such a formula is $\exists\Pi_1^b$. This means that if Completeness for Rosser-ordered provabilities (with parameter) were provable in $I\Delta_0+\Omega_1$, then again NP=coNP.

In Paris & Wilkie[87] it is shown that all principles of Löb's Logic are valid in $I\Delta_0+\Omega_1$. Solovay's proof of the arithmetical completeness of Löb's Logic, however, uses essentially the verifiability of schematical, sentential Σ_1^0-completeness (in fact: completeness for Rosser-ordered provabilities) in the arithmetical theory (see Verbrugge[89]). As a consequence the question of arithmetical completeness of Löb's Logic for interpretations in $I\Delta_0+\Omega_1$ is still open.

In this paper we show that for any finite set S $I\Delta_0+\Omega_1$ verifies that the statement expressing the completeness of S w.r.t. $I\Delta_0+\Omega_1$ is conservative over $I\Delta_0+\Omega_1$ w.r.t. Σ^0_1-sentences. In other words: $I\Delta_0+\Omega_1$ verifies the sentential Σ^0_1-conservativity of schematical, sentential Σ^0_1-completeness over $I\Delta_0+\Omega_1$. This fact gives rise to a rather natural system of provability logic. Let us add to the language of Löb's Logic propositional variables s,s',... for Σ^0_1-sentences. If we consider interpretations in a theory U extending $I\Delta_0$+EXP (with Σ^0_1 provability predicate) the resulting arithmetically valid & arithmetically complete logic is Löb's Logic+$\{s\rightarrow\Box s | s \text{ a } \Sigma\text{-variable}\}$. (The proof is surprisingly easy: see Visser[81].) If we consider interpretations for the extended language in $I\Delta_0+\Omega_1$ we can (by our present lights) only justify the system Löb's Logic+$\{\Box(\bigwedge\{s\rightarrow\Box s | s \text{ in } S\}\rightarrow s^*)\rightarrow\Box s^* \mid s^* \text{ is a } \Sigma\text{-variable, } S \text{ is a finite set of } \Sigma\text{-variables}\}$.

This logic is useful, for example, if one wants to formalize metamathematical reasoning involving the Rosser-ordering in $I\Delta_0+\Omega_1$ (see the forthcoming work of Carbone on provable fixed points).

Acknowledgement: the present result was found in direct interaction with the research by Allessandra Carbone and Dick de Jongh on provable Fixed Points. That research in its turn was inspired by ideas of Franco Montagna and by earlier work by De Jongh & Montagna.

2 Prerequisites

The reader should be acquainted with Buss[86], Paris & Wilkie[87}, Smorynski[85].

3 Programming cuts

Let U be an arithmetical theory. A U-cut will be in this paper: a formula I(x), having only x free, such that U proves that $0\in I$, that I is closed under successor, addition, multiplication and ω_1, and that I is downwards closed w.r.t. <. If we speak simply about a cut, we mean: $I\Delta_0+\Omega_1$-cut. We write A^I for the result of relativizing all quantifiers in A to I.

Let I and J be $I\Delta_0+\Omega_1$-cuts. Define:

$I\leq J \quad :\Leftrightarrow I\Delta_0+\Omega_1\vdash \forall x\, x\in I \rightarrow x\in J.$

$I=J \quad :\Leftrightarrow I\leq J$ and $J\leq I$.

$x\in \mathbb{ID} \quad :\Leftrightarrow x=x.$

$x\in I\circ J \quad :\leftrightarrow x\in J \wedge (x\in I)^J.$

$x\in I[A]J \quad :\leftrightarrow (A\wedge x\in I) \vee (\neg A\wedge x\in J)$. (Here A is a sentence.)

We enumerate some elementary facts about cuts. The proofs are left to the diligent reader.

1) $I \circ J$ is a cut.
The proof uses that $I\Delta_0+\Omega_1 \vdash$ (I is a cut)^J. Note that this would not work if we were considering $I\Delta_0$+EXP and $I\Delta_0$+EXP-cuts and instead of $I\Delta_0+\Omega_1$ and $I\Delta_0+\Omega_1$-cuts.
2) ID is a cut. Cuts are closed under union and intersection and (.)[A](.).
3) = is a congruence relation w.r.t. $\cap$, $\cup$, $\circ$ and (.)[A](.) and $\leq$ is a po w.r.t. cuts modulo =.
4) ID is the identity w.r.t. $\circ$. Moreover ID is the maximum w.r.t. $\leq$.
5) $I \circ J \leq J$.
6) $I \leq I' \Rightarrow (I \circ J) \leq (I' \circ J)$.
7) $(I \cap I') \circ J = (I \circ J) \cap (I' \circ J)$.
8) $(I \cup I') \circ J = (I \circ J) \cup (I' \circ J)$.
9) For A a sentence: $I\Delta_0+\Omega_1 \vdash A^{I \circ J} \leftrightarrow (A^I)^J$.
10) $\circ$ is associative.
11) $I\Delta_0+\Omega_1 \vdash B^{I[A]J} \leftrightarrow ((A \wedge B^I) \vee (\neg A \wedge B^J))$
12) $I \circ (J[A]K) = (I \circ J)[A](I \circ K)$.
13) $(I[A]J) \circ K = (I \circ K)[A^K](J \circ K)$.

4 On schematical, sentential Σ_1^0-completeness

Define $C(S) := \bigwedge\!\!\!\bigwedge \{s \rightarrow \Box s \mid s \text{ in } S\}$, where S is a finite set of Σ_1^0-sentences and where $\Box$ is provability in $I\Delta_0+\Omega_1$. We have: for every S there is a cut J(S) such that: $I\Delta_0+\Omega_1 \vdash C(S)^{J(S)}$.

Proof: Let I be such that for any Σ_1^0-sentence s: $I\Delta_0+\Omega_1 \vdash s^I \rightarrow \Box s$.

The proof is by induction on the cardinality of S. Put $J(\emptyset)$:=ID. Note that $C(\emptyset)=\top$. Suppose $S := S^* \cup \{s^*\}$, where $s^* \notin S^*$. Put $J(S) := (ID[s^* \rightarrow \Box s^*](J(S^*) \circ I)) \circ J(S^*)$. (Evidently our construction as it stands doesn't give a unique result. It can be made unique e.g. by using some ordering on Σ_1^0-sentences.)

By the Induction Hypothesis $I\Delta_0+\Omega_1 \vdash C(S^*)^{J(S^*)}$. Note that also: $I\Delta_0+\Omega_1 \vdash C(S^*)^{J(S^*) \circ I \circ J(S^*)}$, beause $I\Delta_0+\Omega_1$ is again valid on $J(S^*) \circ I \circ J(S^*)$.

Reason in $I\Delta_0+\Omega_1$ and reason 'inside' $J(S^*)$: we have $C(S^*)$ and $C(S^*)^{J(S^*) \circ I}$. In case $s^* \rightarrow \Box s^*$, clearly C(S) and ipso facto $C(S)^{ID}$. Otherwise it follows that $\neg \Box s^*$ and hence $(\neg s^*)^I$ (since $s^{*I} \rightarrow \Box s^*$). By the downwards persistence of Π_1^0-sentences, also $(\neg s^*)^{J(S^*) \circ I}$ and thus

$(s^* \rightarrow \Box s^*)^{J(S^*)\circ I}$. Combining this with $C(S^*)^{J(S^*)\circ I}$ we find: $C(S)^{J(S^*)\circ I}$. So we may conclude: $C(S)^{ID[s^* \rightarrow \Box s^*](J(S^*)\circ I)}$. $\Box$

There is an alternative proof that is conceptually very simple: (in $I\Delta_0+\Omega_1$) consider the set of true elements of S. Go inside I. Either inside I the same elements of S are true or less (because we can only lose witnesses). In the first case we are done: for any s in S we have: if s then s^I then $\Box s$. In case we have less, repeat the procedure inside I. This can go on no more than n times, because after each step S is left with strictly less truths and S contains only n elements. So in all cases we finish with C(S)! Below I give the alternative proof in a slightly more formal style.

Alternative proof: Let I be as before. Suppose the cardinality of S is n. Define $FIX(S):= \bigwedge \{s \leftrightarrow s^I | s \in S\}$. Let $J_0(S):=ID$ and $J_{k+1}(S):=ID[FIX(S)](J_k(S)\circ I)$ and $J(S):=J_n(S)$. Reasoning in $I\Delta_0+\Omega_1$ one easily sees that each time the right hand side is chosen strictly less elements of S will be true. If this happens n times no elements will be left and C(S) is trivially true. Otherwise at some stage k FIX(S) is true. Clearly FIX(S) implies C(S). $\Box$

Remark: Let K be any $I\Delta_0+\Omega_1$-cut. Define $K^0:=ID$, $K^{n+1}:=K\circ K^n$. It is a nice exercise to show that for the $J_k(S)$ of the alternative proof we have: $J_k(S)=(ID[FIX(S)]I)^k$. (Hint: use 10 and 12 of section 3).

The sentential Σ_1^0-conservativity of schematical, sentential Σ_1^0-completeness:

for all S and s: $I\Delta_0+\Omega_1 \vdash \Box(C(S) \rightarrow s) \rightarrow \Box s$.

Proof: Reason in $I\Delta_0+\Omega_1$: Suppose $\Box(C(S) \rightarrow s)$. Then $\Box s^{J(S)}$. Ergo: $\Box s$. $\Box$

Remarks:

i) It is an open question whether $I\Delta_0+\Omega_1$ verifies the Σ_1^0-conservativity of full sentential Σ_1^0-completeness. As is easily seen it is sufficient to show: $I\Delta_0+\Omega_1 \vdash \forall S,s\ \Box(C(S) \rightarrow s) \rightarrow \Box s$. I conjecture that this is the case. My reasons for believing this conjecture are given in a footnote(1).

ii) Can we get e.g.: $I\Delta_0+\Omega_1 \vdash \Box(\forall x C(S(x)) \rightarrow \forall x\ s(x)) \rightarrow \Box \forall x\ s(x)$, Where S(x) is a finite set of Σ_1^0-formulas having only x free and s(x) is a Σ_1^0-formula having only x free? We can see that this is a difficult problem by the following argument, due to Dick de Jongh: let A(x) be a coNP-complete Π_1^b formula. Let $S(x):=\{A(x)\}$ and $s(x):=C(S(x))$. From the principle under consideration it would follow that $\Box \forall x(A(x) \rightarrow \Box A(x))$. The considerations in the introduction show that we cannot hope for an easy proof of this fact.

Corollary: Let L be Löb's Logic. Let I be an $I\Delta_0+\Omega_1$-cut. An interpretation $(.)^*$ of the modal language is an I-interpretation if $\Box A$ is interpreted as $\Box_{I\Delta_0+\Omega_1}A^I$. We have:

$$L\vdash A \Leftrightarrow \text{for all } I\Delta_0+\Omega_1\text{-cuts I and all I-interpretations } (.)^*\text{: } I\Delta_0+\Omega_1\vdash(A)^{*I}.$$

Sketch of the proof: The soundness side is trivial. Suppose $L\nvdash A$. Let K be a countermodel with extra node 0 added below. Say the domain of K is $\{0,\ldots,n\}$. Define:

$$h(0):=0,$$

$$h(n+1):=i \text{ if } h(n)Ri \text{ and } \mathrm{Proof}_{I\Delta_0+\Omega_1}(n,(L\neq\underline{i})^J),\ h(n+1):=h(n) \text{ otherwise},$$

$$L=i :\Leftrightarrow \exists x\ h(x)=i \wedge \forall y,z((h(y)=i\wedge z>y)\rightarrow h(z)=i),$$

$$J:=J(\{\exists x\ h(x)=1,\ldots,\exists x\ h(x)=n\}).$$

It is easily seen that this definition can be made to work in $I\Delta_0+\Omega_1$, using the Fixed Point Lemma to get both L and J. Note that L and J only occur as codes in the definition of h. Let me briefly indicate why h is provably total in $I\Delta_0+\Omega_1$: first the function $\lambda A,J.A^J$ can be formalized and proved total in $I\Delta_0+\Omega_1$: the reason is that the recursion in its definition is over subformulas. (This fact is verified in detail in Kalsbeek[88].) Using this we can show that the function that assigns (a code for) $(L\neq\underline{d})^J$ to d,H, where H is a code for a formula defining h, is definable in $I\Delta_0+\Omega_1$ and provably total. Define $FCF(\sigma)$ (for: "σ codes a Finitely Changing Function") by:

$$FCF(\sigma) :\Leftrightarrow ((\sigma)_0)_0=0 \wedge (\forall u<\mathrm{lth}(\sigma)\exists v,w<\sigma\ (\sigma)_u=\langle v,w\rangle) \wedge \forall u,v<\mathrm{lth}(\sigma)$$
$$(u<v\rightarrow((\sigma)_u)_0<((\sigma)_v)_0).$$

Define further (for σ such that $FCF(\sigma)$):

$$\sigma(x)=y :\Leftrightarrow \exists u<\mathrm{lth}(\sigma)\exists v<\sigma\ (v\leq x \wedge (\sigma)_u=\langle v,y\rangle \wedge \forall w<\mathrm{lth}(\sigma)(u<w\rightarrow x<((\sigma)_w)_0).$$

It is easily seen that under this definition σ represents a function, when $FCF(\sigma)$.

Let $B(x):=\langle\langle x,0\rangle,\langle x,1\rangle,\ldots,\langle x,n\rangle\rangle$; for a decent coding of sequences B(x) is of order x^k for some standard k. We can write the equivalence proved by the Fixed Point Lemma as follows (where e.g "$\exists x,y<z$" is short for: $\exists x<z\exists y<z$):

$$
\begin{aligned}
h(x)=y \ \leftrightarrow\ & \exists\sigma<B(x)\\
&\quad (FCF(\sigma) \wedge \sigma(x)=y \wedge \sigma(0)=0 \wedge \forall z\leq x\exists u\leq z\exists d\leq n\\
&\qquad (\sigma(u)=\sigma(z)=d \wedge\\
&\qquad\quad (u=0 \vee \exists v<u\exists e\leq n\\
&\qquad\qquad (u=v+1 \wedge \mathrm{Proof}_{I\Delta_0+\Omega_1}(v,(L\neq\underline{d})^J) \wedge \sigma(v)=e \wedge eRd \wedge\forall w<z\\
&\qquad\qquad\quad (v<w\rightarrow\forall f\leq n\\
&\qquad\qquad\qquad (\neg eRf\vee\neg\mathrm{Proof}_{I\Delta_0+\Omega_1}(w,(L\neq\underline{f})^J)))))))
\end{aligned}
$$

The existence of L is trivial, the range of h being standardly finite.

From this point on Solovay's proof goes through as usual with Proof(p,A^J) replacing Proof(p,A). (See Smorynski[85] for an exposition.) $\square$

Corollary: $I\Delta_0+\Omega_1+\neg EXP+\{s\to\Box s \mid s \text{ is a } \Sigma_1^0\text{-sentence}\}$ is locally interpretable in $I\Delta_0+\Omega_1$.

Proof: Let S be any finite set of Σ_1^0-sentences. Let's write $A \rhd B$ for: $I\Delta_0+\Omega_1+B$ is interpretable in $I\Delta_0+\Omega_1+A$. We use the principles for interpretability of the system ILW verified in Visser[88].

We have by our theorem $\top \rhd C(S)$ and (because the interpretation is a cut) $\Box(\top \rhd C(S))$; hence $\Box(\Diamond\top \to \Diamond C(S))$. By a result of Paris and Wilkie: $EXP \rhd \Diamond\top$, so $EXP \rhd \Diamond C(S)$. By the principles W and J5:

$$EXP \rhd (\Diamond C(S)\wedge\Box\neg EXP) \rhd \Diamond(C(S)\wedge\neg EXP) \rhd (C(S)\wedge\neg EXP).$$

Also: $(C(S)\wedge\neg EXP) \rhd (C(S)\wedge\neg EXP)$. Hence by J3:

$$\top \rhd C(S) \rhd (EXP\vee(C(S)\wedge\neg EXP)) \rhd (C(S)\wedge\neg EXP). \quad \square$$

Remark: The fact that $\top \rhd \neg EXP$ was first proved by Solovay in 1986. This was unknown to me when writing Visser[88]. Solovay's proof is quite different from ours.

4 The s-System

Let s-L be Löb's Logic in a language with two sorts of propositional variables: the usual $p,q,r,p', \ldots,p_1,p_2,\ldots$ and $s,s',\ldots,s_1,s_2,\ldots$. The s-variables stand for Σ_1^0-sentences. Let Σ be the smallest class of formulas in the enriched language such that formulas of the form $\bot,\top,\Box A,s$ are in Σ, and if B,C are in Σ, then so are $(B\vee C)$ and $(B\wedge C)$. s-L has the following additional rules:

s-Principle $\vdash \Box(C(S)\to s) \to \Box s$.

Substitution $\vdash A(p_1, \ldots,p_n,s_1,\ldots,s_n) \Rightarrow \vdash A(B_1, \ldots,B_n,\sigma_1,\ldots,\sigma_m)$, for any formulas $B_1,\ldots,B_n$ and for any $\sigma_1,\ldots,\sigma_m$ in Σ.

Equivalently we could take instead of s-Principle plus Substitution:

s^+-Principle $\vdash \Box(C(X)\to\sigma) \to \Box\sigma$, for X a finite subset of Σ and $\sigma\in\Sigma$.

An interpretation (.)* of the language of s-L is a function from the elements of this language to sentences of the language of arithmetic, which satisfies the following conditions:

- $(s)^* \in \Sigma_1^0$, $(\perp)^*=\perp$, $(\top)^*=\top$,
- (.)* commutes with the propositional connectives,
- $(\Box A)^*=\Box_{I\Delta_0+\Omega_1}A^*$.

As is easily seen s-L is arithmetically valid for interpretations in this sense, i.e.:

$$\text{s-L}\vdash A \Rightarrow \forall(.)^*\ I\Delta_0+\Omega_1\vdash A^*.$$

Evidently the closure of s-L under the rule: $\vdash\Box A \Rightarrow \vdash A$, is also arithmetically valid. I conjecture that s-L is already closed under this rule.

We give some theorems in s-L:

S1 $\vdash \Box\bigvee S \to \Box\bigvee S^+$, where $S^+:=\{s\wedge\Box s | s\in S\}$

S2 $\vdash (\Box(\Box A\to\bigvee S)\wedge\Box(\bigvee S^+\to A))\to\Box A$

S3 $\vdash \Box(C(S)\to(A\to s))\to\Box^+(\Box A\to\Box s)$, where $\Box^+C:=(C\wedge\Box C)$

S4 $\vdash \Box(C(S)\to(\Box s\to s))\to\Box s$

Proofs: S1 is trivial. For S2: suppose (in the s-System) $\Box(\Box A\to\bigvee S)$, then $\Box(\Box\Box A\to\Box\bigvee S)$. Hence by S1: $\Box(\Box\Box A\to\Box\bigvee S^+)$. Suppose further $\Box(\bigvee S^+\to A)$, then $\Box(\Box\bigvee S^+\to\Box A)$. Combining: $\Box(\Box\Box A\to\Box A)$, and thus $\Box\Box A$. We may conclude $\Box\bigvee S$, hence $\Box\bigvee S^+$, hence $\Box A$.

Ad S3: From $\Box(C(S)\to(A\to s))$, we have $\Box(A\to(C(S)\to s))$. Hence $\Box^+(\Box A\to\Box(C(S)\to s))$. Hence $\Box^+(\Box A\to\Box s)$.

Ad S4: suppose $\Box(C(S)\to(\Box s\to s))$. By S3: $\Box(\Box\Box s\to\Box s)$, hence $\Box\Box s$. Ergo $\Box(C(S)\to s)$ and thus $\Box s$. $\Box$

Remark: It is now easy to specify a reasonable system for Rosser logic valid in $I\Delta_0+\Omega_1$. Take Svejdar's system Z (see Svejdar[83]). The validity of Z for interpretations in $I\Delta_0+\Omega_1$ is verified in detail in Verbrugge[89]. Now add to it the Σ^*-substitution instances of the s-Principle, where Σ^* is the smallest class such that formulas of the form $\perp,\top,\Box A$, $\Box B<\Box C$, $\Box B\leq\Box C$ are in Σ^*, and if B,C are in Σ^*, then so are $(B\vee C)$ and $(B\wedge C)$. Call the resulting system Z-s.(2) Note that Z-s is not valid for the interpretations studied by Svejdar. Z-s is studied by Carbone & De Jongh. They show that the theorem by Montagna and De Jongh on provable fixed points is true for Z-s. See the forthcoming paper by Carbone: *Provable Fixed Points in* $I\Delta_0+\Omega_1$. (For the original result by De Jongh & Montagna see: De Jongh & Montagna[88].)

Footnotes:

1) I sketch a Lakatosian Thought Experiment of which I hope it could be converted into a real proof.

To formalize our argument in $I\Delta_0+\Omega_1$ we should provide bounds for the cut J(S) and for the $I\Delta_0+\Omega_1$-proofs involved. If we follow the first proof of section 4 it seems to me that J(S) will grow too fast in S. So let's look at the alternative proof. By the remark following the proof we can use as a definition of J(S): $(ID[FIX(S)]I)^n$. Let K be any cut and add a unary predicate variable X to the language. Let $(x\in K)^X$ be defined in the obvious way. By a method due to Ferrante and Rackov we can rewrite $(x\in X\wedge(x\in K)^X)$ to a formula P(x,X) with only one occurrence of X. (One needs a language with $\leftrightarrow$.) Let us define K0X using P(x,X) rather than the obvious formula. We can convert a proof of "K is a cut" into a proof of "X is a cut $\rightarrow$ K0X is a cut". Using these facts we can show that the length of K^n is linear in n. Since n is the number of elements of S 2^n exists in $I\Delta_0+\Omega_1$ and hence K^n will exist in $I\Delta_0+\Omega_1$. Furthermore one can show that the $I\Delta_0+\Omega_1$-proof that K^n is a cut exists in $I\Delta_0+\Omega_1$.

Take K:=ID[FIX(S)]I. Our induction hypothesis is: for k (with $0\leq k\leq n$) we have an $I\Delta_0+\Omega_1$-proof of: in K^k we have: FIX(S) or at least k elements of S are false. If we treat this naively then we explicate "at least k elements of S are false" by a big disjunction of conjunctions of negations of elements of S. It is easily seen that this big disjunction is so big that generally it won't exist in $I\Delta_0+\Omega_1$. The alternative is to use a Σ_1^0-truthpredicate. The only problem is that such a truthpredicate is not available in $I\Delta_0+\Omega_1$. However we can save ourselves by a trick: we can choose our cut I in such a way that there is (outside I) a Σ_1^0-truthpredicate for I. I.e. there is a predicate T such that $I\Delta_0+\Omega_1$ proves: for all s there is an $I\Delta_0+\Omega_1$-proof of $s^I\leftrightarrow T(s)$. Now we do our whole construction inside I using T to formulate "at least k elements of S are false". (Note that we have to convert T in truthpredicates for different cuts for the different k. This is easily done by extracting the wittness for s from the witness for T and by demanding that the witness for s is in the desired cut.)

A different way to avoid the big disjunction is to say: there is a 0,1-sequence σ of length n such that 0 occurs at least k times in σ and $\bigwedge\!\!\!\bigwedge\{s_i\leftrightarrow(\sigma)_i=1 \mid 0\leq i\leq n\}$.

2) "s-" before a system signals the presence of special variables for Σ_1^0-sentences and that our system contains the s-Principle . "-s" behind a system means that we have substitution instances of the s-Principle for a suitable class of formulas (the 'Σ-formulas' of the system).

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