Uniform Normalisation beyond Orthogonality

`Category 1: Regular research paper describing new results`

Zurab Khasidashvili[1], Mizuhito Ogawa[2], and Vincent van Oostrom[3]

[1] Department of Mathematics and Computer Science, Bar-Ilan University
Ramat-Gan 52900, Israel
khasidz@cs.bu.ac.il

[2] Japan Science and Technology Corporation, PRESTO and
NTT Communication Science Laboratories
3-1 Morinosato-Wakamiya, Atsugi, Kanagawa 243-0198, Japan
mizuhito@theory.brl.ntt.co.jp

[3] Department of Philosophy, Utrecht University
P.O. Box 80089, 3508 TB Utrecht, The Netherlands
oostrom@phil.uu.nl

Abstract. A rewrite system is called uniformly normalising if all its steps are perpetual, i.e. are such that if $s \to t$ and s has an infinite reduction, then t has one too. For such systems termination (SN) is equivalent to normalisation (WN). A well-known fact is uniform normalisation of *orthogonal non-erasing* term rewrite systems, e.g. the λI-calculus. In the present paper both restrictions are analysed. Orthogonality is seen to pertain to the linear part and non-erasingness to the non-linear part of rewrite steps. Based on this analysis, a modular proof method for uniform normalisation is presented which allows to go beyond orthogonality. The method is shown applicable to biclosed first- and second-order term rewrite systems as well as to a λ-calculus with explicit substitutions.

1 Introduction

Two classical results in the study of uniform normalisation are:

- the λI-calculus is uniformly normalising [7, p. 20, 7 XXV], and
- non-erasing steps are perpetual in orthogonal TRSs [14, Thm. II.5.9.6].

In previous work we have put these results and many variations on them in a unifying framework [13]. At the heart of that paper is the result (Thm. 3.16) that a term s not in normal form contains a redex which is external for *any* rewrite sequence from s.[1] The method presented here, is based instead on the existence of external redexes for a *given* sequence. This allows for generalisation of results from orthogonal to left-linear rewrite systems, in particular to rewrite systems having critical pairs.

[1] According to [11, p. 404], a redex at position p is external for a sequence if in the sequence no redex is contracted above p to which the redex did not contribute.

Uniform normalisation results for abstract rewrite systems (ARSs), first-order term rewrite systems (TRSs), second-order term rewrite systems (P_2RS) and λx^- are presented in Sect. 2, 3, 4, and 5, respectively.

2 Abstract rewriting

In this section the reader is assumed to be familiar with *abstract rewrite systems* (ARSs) as can be found in e.g. [15, Chap. 1] or [1, Chap. 2].

Definition 1. *Let a be an object of an abstract rewrite system. a is* terminating *(strongly normalising,* SN*) if no infinite rewrite sequences are possible from it. We use* ∞ *to denote the complement of* SN*. a is* normalising *(weakly normalising,* WN*) if some rewrite sequence to normal form is possible from it.*

Definition 2. *A rewrite step* $s \to t$ *is* critical *if* $s \in \infty$ *and* $t \in \mathsf{SN}$*, and* perpetual *otherwise. A rewrite system is* uniformly normalising *if there are no critical steps.*

First, note that a rewrite system is uniformly normalising iff $\mathsf{WN} \subseteq \mathsf{SN}$ holds. Moreover, uniform normalisation holds for deterministic rewrite systems.

Definition 3. *A* fork *in a rewrite system is pair of steps* $t_1 \leftarrow s \to t_2$*. It is called* trivial *if* $t_1 = t_2$*. A rewrite system is* deterministic *if all forks are trivial, and* non-deterministic *otherwise.*

To analyse uniform normalisation for non-deterministic rewrite systems it thus seems worthwhile to study their non-trivial forks.

Definition 4. *A rewrite system is* linear orthogonal *if every fork* $t_1 \leftarrow s \to t_2$ *is either trivial or* square*, that is,* $t_1 \to s' \leftarrow t_2$ *for some* s' *[1, Exc. 2.33].*

We will show the *fundamental theorem of perpetuality*:

Theorem 1 (FTP). *Steps are perpetual in linear orthogonal rewrite systems.*

Corollary 1. *Linear orthogonal rewrite systems are uniformly normalising.*

In the next section we will show (Lem. 1) that the abstract rewrite system associated to a term rewrite system which is linear and orthogonal, is linear orthogonal. Linear orthogonality is a weakening of the diamond property [1, Def. 2.7.8], and a strengthening of subcommutativity [15, Def. 1.1.(v)] and of the balanced weak Church-Rosser property [25, Def. 3.1], whence:

Proof. (of Theorem 1) Suppose $s \in \infty$ and $s \to t$. We need to show $t \in \infty$. By the first assumption, there exists an infinite rewrite sequence $S : s_0 \to s_1 \to s_2 \to \ldots$, with $s_0 = s$. Let $t_0 = t$ be the first object of the rewrite sequence T. By orthogonality we can find for every non-trivial fork $s_{i+1} \leftarrow s_i \to t_i$ a next object t_{i+1} of T such that $s_{i+1} \to t_{i+1} \leftarrow t_i$. Consider a maximal sequence T thus constructed. If T is infinite we are done. If T is finite, it has a final object, say t_n, and a fork $s_{n+1} \leftarrow s_n \to t_n$ exists which is trivial, i.e. $s_{n+1} = t_n$. Hence, T and the infinite sequence S from s_{n+1} on can be concatenated. □

FTP can be brought beyond linear orthogonality. Let $\to^{=}$ and $\twoheadrightarrow$ denote the reflexive and reflexive-transitive closure of $\to$, respectively.

Definition 5. *A fork $t_1 \leftarrow s \to t_2$ is* closed *if $t_1 \twoheadrightarrow t_2$. A rewrite system is* linear biclosed *if all forks are either closed or square.*

By replacing the appeal to triviality by an appeal to closedness in the proof of FTP, i.e. by replacing $s_{n+1} = t_n$ by $s_{n+1} \twoheadleftarrow t_n$, we get:

Corollary 2. *Linear biclosed rewrite systems are uniformly normalising.*

Note that linear biclosed rewrite systems are confluent since strongly confluent ones are [1, Lem. 2.7.4], but the latter need not be uniformly normalising.

Example 1. Call a rewrite system *strongly confluent* [1, Def. 2.7.3], if for every fork $t_1 \leftarrow s \to t_2$ it holds that $t_1 \twoheadrightarrow s' \leftarrow^{=} t_2$ for some s'. The rewrite system $\{a \to a, a \to b\}$ is strongly confluent but not uniformly normalising.

Although trivial, the results in this section can be applied to various concrete linear rewrite systems, for instance to interaction nets [16]. In the next section we show they pertain to linear TRSs and can be adapted to non-linear TRSs.

3 First-order term rewriting

In this section first the uniform normalisation results of Section 2 are instantiated to linear term rewriting. Next, the *fundamental theorem of perpetuality for first-order term rewrite systems* is established:

Theorem 2 ($\mathbf{F_1TP}$). *Non-erasing steps are perpetual in orthogonal TRSs.*

Corollary 3. *Non-erasing orthogonal TRSs are uniformly normalising.*

Finally, it is shown that orthogonality can be relaxed to biclosedness. The chief purpose of this section is to illustrate our proof method based on standardisation. Except maybe for the final results, the results obtained are not novel (cf. [15, Lem. 8.11.3.2] and [9, Sect. 3.3]).

In this section the reader is assumed to be familiar with *first-order term rewrite systems* (TRSs) as can be found in e.g. [15] or [1]. Some aberrations and additional concepts are summarised in the following definition.

Definition 6.
- *A term is* linear *if any variable occurs at most once in it. Let $\varrho : l \to r$ be a TRS rule. It is* left-linear *(*right-linear*) if l (r) is linear. It is* linear *if $Var(l) = Var(r)$ and both sides are linear. A TRS is (left-,right) linear if all its rules are.*
- *Let $\varrho : l \to r$ be a rule. A variable $x \in Var(l)$ is* erased *by ϱ if it does not occur in r. The rule ϱ is erasing if it erases some variable. A rewrite step is erasing if the applied rule is. A TRS is erasing if some step is.*
- *Let $\varrho : l \to r$ and $\vartheta : g \to d$ be rules which have been renamed apart. Let p be a non-variable position in l. ϱ is said to* overlap ϑ at p *if a unifier σ of $l_{|p}$ and g does exist. If σ is a most general such unifier, then both $\langle l[d]_p^{\sigma}, r^{\sigma}\rangle$ and $\langle r^{\sigma}, l[d]_p^{\sigma}\rangle$ are* critical pairs at p between ϱ and ϑ.

– *If for all such critical pairs $\langle t_1, t_2 \rangle$ of a left-linear TRS $\mathcal{R}$ it holds that:*
$\exists s'\, t_1 \twoheadrightarrow s' \leftarrow^{=} t_2$, *then $\mathcal{R}$ is* strongly closed *[10, p. 812]*
$t_1 \twoheadrightarrow t_2$, *then $\mathcal{R}$ is* biclosed *[22, p. 70]*
$t_1 = t_2$, *then $\mathcal{R}$ is* weakly orthogonal
$t_1 = t_2$ *and* $p = \epsilon$, *then $\mathcal{R}$ is* almost orthogonal
$t_1 = t_2$, $p = \epsilon$ *and* $\varrho = \vartheta$, *then $\mathcal{R}$ is* orthogonal

Some remarks are in order. First, our critical pairs for a TRS are the critical pairs $\langle s, t \rangle$ of [1, Def. 6.2.1] extended with their opposites ($\langle t, s \rangle$) and the trivial critical pairs between a rule with itself at the head ($\langle r, r \rangle$ for every rule $l \to r$). Next, linearity in our sense implies linearity in the sense of [1, Def. 6.3.1], but not vice versa. Linearity of a step $s = C[l^\sigma] \to C[r^\sigma] = t$ as defined here captures the idea that every symbol in the context-part C or the substitution-part σ in s has a *unique* descendant in t, whereas linearity in the sense of [1, Def. 6.3.1] only guarantees that there is *at most one* descendant in t. Remark:

orth. $\Longrightarrow$ almost orth. $\Longrightarrow$ weakly orth. $\Longrightarrow$ biclosed $\Longrightarrow$ strongly closed

3.1 Linear term rewriting

In this subsection the results of Section 2 for abstract rewriting are instantiated to linear term rewriting. First, remark that strong closedness implies that all critical pairs are strongly joinable in the sense of [1, Def. 6.3.2], so linear strongly closed TRSs are strongly confluent by [1, Lem. 6.3.3], hence confluent by [1, Lem. 2.7.4]. Therefore, by the final remark above, a linear TRS satisfying any of the above mentioned critical pair criteria is confluent.

Lemma 1. *If $\mathcal{R}$ is a linear orthogonal TRS, $\to_{\mathcal{R}}$ is a linear orthogonal ARS.*

Proof. The proof is based on the standard critical pair analysis of a fork $t_1 \leftarrow_{\mathcal{R}} s \to_{\mathcal{R}} t_2$ as in [1, Sect. 6.2]. Actually, it is directly obtained from the proof of Lem. 6.3.3 mentioned above, by noting that:

Case 1 (parallel) establishes that the fork is square (joinable into a diamond),
Case 2.1 (nested) also yields that the fork is square,[2] and
Case 2.2 (overlap) can occur only if the steps in the fork arise by applying the same rule at the same position, by orthogonality, so the fork is trivial. □

From Lem. 1 and Cor. 1 we obtain a special case of Corollary 3.

Corollary 4. *Linear orthogonal TRSs are uniformly normalising.*

As in Section 2, we can go beyond orthogonality.

Lemma 2. *If $\mathcal{R}$ is a linear biclosed TRS, $\to_{\mathcal{R}}$ is a linear biclosed ARS.*

Proof. The analysis in the proof of Lemma 1 needs to be adapted as follows:

Case 2.2 , the instance of a critical pair, is closed by biclosedness of critical pairs and the fact that rewriting is closed under substitution. □

Corollary 5. *Linear biclosed TRSs are uniformly normalising.*

[2] Note that the case $x \notin Var(r_1)$ cannot happen, due to our notion of linearity.

3.2 Non-linear term rewriting

In this subsection the results of the previous subsection are adapted to non-linear TRSs, leading to a proof of F_1TP (Theorem 2). The adaptation is non-trivial, since uniform normalisation may fail for orthogonal non-linear TRSs.

Example 2. The term $e(a)$ in the TRS $\{a \to a, e(x) \to b\}$ witnesses that orthogonal TRSs need not be uniformly normalising.

Non-linearity of a TRS may be caused by non-left-linearity. Although non-left-linearity in itself is not fatal for uniform normalisation of TRSs (see [9, Chap. 3], e.g. Cor. 3.2.9), it will be in case of second-order rewriting (cf. Ex. 3) and our method cannot deal with it. Hence: We assume TRSs to be left-linear.
Under this assumption, non-linearity may only be caused by some symbol having multiple descendants after a step. The problem in Ex. 2 is seen to arise from the fork $e(a) \leftarrow e(a) \to b$ which is not balancedly joinable: it is neither trivial ($e(a) \neq b$) nor square ($\nexists s'\, e(a) \to s' \leftarrow b$). Erasingness is the only problem. To

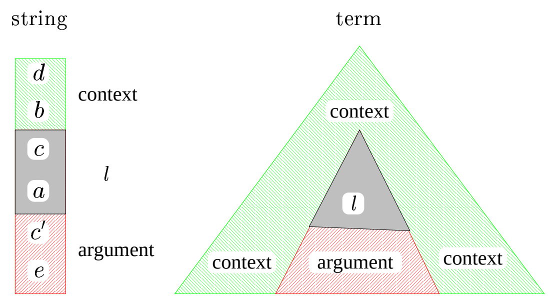

Fig. 1. Split

prove F_1TP, we will make use of the apparent asymmetry in the non-linearity of term rewrite steps: an occurrence of a left-hand side of a rule $l \to r$ *splits* the surrounding into two parts (see Figure 1):

– the *context*-part above or parallel to [1, Def. 3.1.3] l, and
– the *argument*-part, below l.

Observe that term rewrite steps in the context-part might replicate the occurrence of the left-hand side l, whereas steps in the argument-part cannot do so. To deal with such replicating steps in the context-part, we will actually prove a strengthening of F_1TP for parallel steps instead of ordinary steps.

Definition 7. *Let $\varrho : l \to r$ be a TRS rule. s* parallel rewrites *to t* using *ϱ, $s \twoheadrightarrow\!\!\!\!\!\!\!+\,_\varrho t$ [10, p. 814],*[3] *if it holds that $s = C[l^{\sigma_1}, \ldots, l^{\sigma_k}]$ and $t = C[r^{\sigma_1}, \ldots, r^{\sigma_k}]$, for some $k \geq 0$. The step is* erasing *if the rule is. The* context*(*argument*)-part of the step is the part above or parallel to all (below some) occurrences of l.*

[3] Actually our notion is a restriction of his, since we allow only one rule.

To reduce F_1TP to FTP it suffices to reduce to the case where the infinite reduction does not take place (entirely) in the context-part, since then the steps either have overlap or are in the, linear, argument-part. To that end, we want to transform the infinite sequence into an infinite sequence where the steps in the context-part precede the steps in the argument-part.

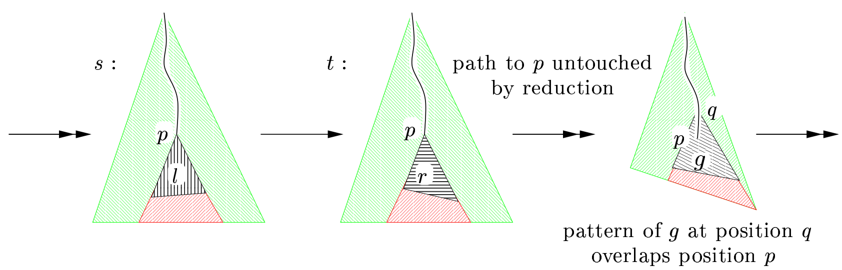

Fig. 2. Standard

Definition 8. *A rewrite sequence is* standard *(see Figure 2) if for any step $C[l^\sigma]_p \to C[r^\sigma]_p$ in the sequence, p is in the pattern of the first step after that step which is above p. That is, if $D[g^\tau]_q$ displays the occurrence of the first redex with $p = qo$, we have that o is a non-variable position in g.*

Theorem 3 (STD). *Any rewrite sequence in a TRS can be transformed into a standard one. The transformation preserves infiniteness.*

Proof. The first part of the theorem was shown to hold for orthogonal TRSs in [11, Thm. 3.19] and extended to left-linear TRSs possibly having critical pairs in [8]. That standardisation preserves infiniteness follows from the fact that at some moment along an infinite sequence $S : s_0 \to s_1 \to \ldots$ a redex at minimal position p w.r.t. the prefix order $\leq$ [1, Def. 3.1.3] must be contracted. Say this happens the first time in step $s_i \to_p s_{i+1}$. Permute all steps parallel to p in S after this step resulting in $S_0; S_1$, where S_0 contains only steps below p and ends with a step at position p, and S_1 is infinite. Standardise S_0 into T_0, note that it is non-empty and observe that concatenating T_0 with *any* standardisation of S_1 will yield a standard sequence by the choice of p. Repeat the process on the infinite sequence S_1. □

Proof. (of Theorem 2) Suppose $s \in \infty$ and $s \mathrel{\#\!\!\!\rightarrow}^{k}_{\varrho} t$ is non-erasing, contracting k redexes w.r.t. rule $\varrho : l \to r$ in parallel. We need to show $t \in \infty$. If $k = 0$, then $t = s \in \infty$ so we are done. Otherwise, there exists by the first assumption an infinite rewrite sequence $S : s_0 \to_{q_0} s_1 \to_{q_1} s_2 \to \ldots$, with $s_0 = s$ and $s_i \to_{q_i} s_{i+1}$ contracting a redex at position q_i w.r.t. rule $\vartheta_i : g_i \to d_i$. By STD S may be assumed standard. Consider the relative positions of the redexes in the fork $s_1 \leftarrow_{q_0} s \mathrel{\#\!\!\!\rightarrow}_{\varrho} t$.

(context) If g_0 occurs entirely in the context-part of the parallel step, then by the Parallel Moves lemma [1, Lem. 6.4.4] the fork is joinable into $s_1 \mathrel{+\!\!\!\!|\!\!\!\!\rightarrow}_{\varrho}^{k'} t_1 \leftarrow_{q_0} t_0$. Since $t_0 \to t_1$, $s_1 \in \infty$, and $s_1 \mathrel{+\!\!\!\!|\!\!\!\!\rightarrow}_{\varrho} t_1$ is non-erasing, repeating the process will yield an infinite sequence from $t_0 = t$ as desired.

(non-context) Otherwise g_0 must be below one or overlap at least one contracted left-hand side l, say the one at position p. Hence, $s \mathrel{+\!\!\!\!|\!\!\!\!\rightarrow}^{k} t$ can be decomposed as $s \to_p s' \mathrel{+\!\!\!\!|\!\!\!\!\rightarrow}^{k-1} t$. We claim $s' \in \infty$. The proof is as for FTP, employing standardness to exclude replication of the pivotal l-redex. Construct a maximal sequence T as follows. Let $t_0 = s'$ be the first object of T. If g_0 overlaps the l at position p, then T is empty. Otherwise, g_0 must be below that l and we set $o_0 = q_0$.

– Suppose the fork $s_{i+1} \leftarrow_{q_i} s_i \to_p t_i$ is such that the contracted redexes do not have overlap. As an invariant we will use that o_i records the outermost position below l (at p) and above q_0 where a redex was contracted in the sequence S up to step i, hence $p \leq o_{i+1} \leq o_i \leq q_0$. Then $q_i < p$ is not possible, since by the non-overlap assumption g_i would be entirely above p, hence above o_i as well, violating standardness of S. Hence, q_i is parallel to or below l (at p). By another appeal to the Parallel Moves lemma the fork can be joined via $s_{i+1} \to_p t_{i+1} \mathrel{\leftarrow\!\!\!\!|\!\!\!\!+}^{k} t_i$, where $k > 0$ by non-erasingness of $s_i \to t_i$ (†). The invariant is maintained by setting o_{i+1} to q_i if $q_i < o_i$, and to o_i otherwise.

If T is infinite we are done. If T is finite, it has a final object, say t_n, and a fork $s_{n+1} \leftarrow_{q_i,\vartheta_i} s_n \to_p t_n$ such that the redexes have overlap (‡). By the orthogonality assumption we must have $q_n = p$ and $\vartheta_n = \varrho$, hence $s_{n+1} = t_n$. By concatenating T and the infinite sequence S from s_{n+1}, the claim ($s' \in \infty$) is then proven. From the claim, we may repeat the process with an infinite standard sequence from s' and $s' \mathrel{+\!\!\!\!|\!\!\!\!\rightarrow}^{k-1} t$.

Observe that the (context)-case is the only case producing a rewrite step from t, but it must eventually always apply since the other case decreases k by 1. □

By replacing the appeal to orthogonality by an appeal to biclosedness in the proof of $\mathrm{F_iTP}$, i.e. by replacing $s_{n+1} = t_n$ by $s_{n+1} \twoheadleftarrow t_n$, we get:

Theorem 4. *Non-erasing steps are perpetual in biclosed TRSs.*

Corollary 6. *Non-erasing biclosed TRSs are uniformly normalising.*

Note that we are beyond orthogonality since biclosed TRSs need not be confluent. The example is as for strongly closed TRSs [10, p. 814], but note that the latter need not be uniformly normalising (Ex. 1)! Next, we show [15, Lem. 8.11.3.2].

Definition 9. *A step $C[l^\sigma] \to C[r^\sigma]$ is* ∞-erasing, *if it erases all ∞-variables, that is, if $x \in Var(r)$ then $x^\sigma \in \mathsf{SN}$.*

Theorem 5. *Non-∞-erasing rewrite steps are perpetual in biclosed TRSs.*

Proof. Replace in the proof of Theorem 4 everywhere non-erasingness by non-∞-erasingness. The only thing which fails is the statement resulting from (†):

– By another appeal to the Parallel Moves Lemma the fork can be joined via $s_{i+1} \rightarrow_p t_{i+1} \leftarrow\!\!\!\!\mid\!\!\mid\!\!\!-^k t_i$, where $k > 0$ by non-∞-erasingness of $s_i \rightarrow t_i$.

We split this case into two new ones depending on whether some argument (instance of variable) to l is ∞ or not.

– In the former case, $t_i \in \infty$ follows directly from non-∞-erasingness.
– In the latter case, $s_i \rightarrow s_{i+1}$ may take place in an erased argument, and $s_{i+1} \rightarrow_p t_{i+1} = t_i$. But since all arguments to l are SN, this can happen only finitely often and eventually the first case applies. □

Gramlich [9] presents fur/other weakenings of the assumptions. For instance, he shows that the left-linearity assumption can be omitted. Since our method cannot deal with this we refer the interested reader to op.cit..

4 Second-order term rewriting

In this section, the *fundamental theorem of perpetuality for second-order term rewrite systems* is established, by generalising the method of Section 3.

Theorem 6 ($\mathbf{F_2TP}$). *Non-erasing steps are perpetual in orthogonal P_2RSs.*

Corollary 7. *Non-erasing orthogonal P_2RSs are uniformly normalising.*

For ERSs and CRSs these results can be found as [12, Thm. 60] and [14, Cor. II.5.9.4], respectively.

In this section the reader is assumed to be familiar with *second-order term rewrite systems* be it in the form of combinatory reduction systems (CRSs [14]), expression reduction systems (ERSs [13]), or higher-order pattern rewrite systems (PRSs [17]). We employ PRSs as defined in op.cit., but will write $x.s$ instead of $\lambda x.s$, thereby freeing the λ for usage as a function symbol.

Definition 10. – *The* order *of a rewrite rule is the maximal order of the free variables in it. The order of a PRS is the maximal order of the rules in it. P_nRS abbreviates n^{th}-order PRS.*
– *A rewrite rule $l \rightarrow r$ is* fully-extended *if for every occurrence $Z(t_1, \ldots, t_n)$ in l of a free variable Z, $t_1, \ldots, t_n$ is the list of variables bound above it.*
– *A rewrite step $s = C[l^\sigma] \rightarrow C[r^\sigma] = t$ is* non-erasing *if every symbol from C and σ in s descends [20, Sect. 3.1.1] to some symbol in t.*[4]

The adaptation is non-trivial since uniform normalisation may fail for orthogonal, but third-order or non-left-linear or non-fully-extended systems.

Example 3. (third-order) [13, Ex. 7.1] Consider the 3rd-order PRS in Tab. 1. It is the standard PRS-presentation of the $\lambda\beta$-calculus [17] extended by a rule. $@ : o{\rightarrow}o{\rightarrow}o$ and $\lambda : (o{\rightarrow}o){\rightarrow}o$ are the function symbols and $M : o{\rightarrow}o$ and $N : o$ are the free-variables of the first (β-)rule. We have made @ an implicit binary infix operation and have written $\lambda x.s$ for $\lambda(x.s)$, for the λ-calculus to take a more familiar form. If Ω abbreviates $(\lambda x.xx)(\lambda x.xx)$, the step $fxy.(\lambda u.x(u))y \rightarrow_\beta fxy.x(y)$ is non-erasing but critical.

[4] A TRS step is non-erasing in this sense iff it is non-erasing in the sense of Def. 6.

third-order	non-fully-extended	non-left-linear
$(\lambda z.M(z))N \to M(N)$	$M(z)\langle z := N\rangle \to M(N)$	$M(x)\langle x := N\rangle \to M(N)$
$fxy.Z(u.x(u), y) \to Z(u.c, \Omega)$	$gxy.Z(y) \to Z(a)$	$g(x.Z(x), x.Z(x)) \to Z(a)$
	$e(x, y) \to c$	$e(x) \to c$
	$f(a) \to f(a)$	$f(a) \to f(a)$

Table 1. Three counterexamples against uniform normalisation of PRSs

(non-fully-extended) [13, Ex. 5.9] Consider the non-fully-extended P_2RS in Tab. 1. The step $gxy.e(z,x)\langle z := f(y)\rangle \to gxy.e(f(y), x)$ is non-erasaing but critical.

(non-left-linear) Consider the non-left-linear P_2RS in Tab. 1. The rewrite step $g(y.e(x)\langle x{:=}f(y)\rangle, y.c\langle x{:=}f(y)\rangle) \to g(y.e(f(y)), y.c\langle x{:=}f(y)\rangle)$ from s to t is non-erasing but critical; t is terminating, but we have the infinite reduction

$$s \to g(y.c\langle x := f(y)\rangle, y.c\langle x := f(y)\rangle) \to c\langle x := f(a)\rangle \to \dots$$

In each item, the second rule causes failure of uniform normalisation.

Hence, for uniform normalisation to hold some restrictions need to be imposed: We assume PRSs to be left-linear and fully-extended P_2RSs. For TRSs the fully-extendedness condition is vacuous, hence the assumption reduces to left-linearity as in Sect. 3 The restriction to P_2RSs entails no restriction w.r.t. the other formats, since both CRSs and ERSs can be embedded into P_2RSs, by coding metavariables in rules as free variables of type $o{\to}\dots{\to}o{\to}o$ [23]. To adapt the proof of F_1TP to P_2RSs, we review its two main ingredients. The first one was a notion of simultaneous reduction, extending one-step reduction such that:

- The residual of a non-erasing step after a context-step is non-erasing.

The second ingredient was STD. It guarantees the following property:

- Any redex pattern l which is entirely above a contracted redex is external to the sequence S; in particular, l cannot be replicated along S, it can only be eliminated by contraction of an overlapping redex in S.

Since the residual of a parallel reduction after a step above it is usually not parallel, we switch from $\mathrel{+\!\!\!+\!\!\!\rightarrow}$ to $\mathrel{-\!\!\circ\!\!\rightarrow}$, where the latter is the (one-rule restriction of the) simultaneous reduction relation of [21, Def. 3.4].

Definition 11. *Let $\varrho : l \to r$ be a rewrite rule. Write $s \mathrel{-\!\!\circ\!\!\rightarrow}_\varrho t$ if it holds that $s = C[l^{\sigma_1}, \dots, l^{\sigma_k}]$ and $t = C[r^{\tau_1}, \dots, r^{\tau_k}]$, where $\sigma_i \mathrel{-\!\!\circ\!\!\rightarrow}_\varrho \tau_i$ for all $1 \le i \le k$.*

The context-part of such a step is the part above or parallel to all occurrences of l. Two technical results on $\mathrel{-\!\!\circ\!\!\rightarrow}$-steps are needed.

Lemma 3 (Finiteness of Developments). *(FD [20, Thm. 3.1.45]) Let $s \mathrel{-\!\!\circ\!\!\rightarrow} t$ by simultaneously contracting redexes at positions in P. Repeated contraction of residuals of redexes in P starting from s terminates and ends in t.*

The second lemma is a close relative of [13, Lem. 5.1] and establishes the first ingredient above. It fails for P_3RSs as witnessed by the first item of Example 3.

Lemma 4 (Parallel Moves). *Let $\varrho : l \to r$ and $\vartheta : g \to d$ be PRS rules, with ϑ second-order. If $s' \leftarrow_{\vartheta} s \multimap\!\!\to_{\varrho} t$ is a fork such that g is in the context-part of the non-erasing simultaneous step, then the fork is joinable into $s' \multimap\!\!\to_{\varrho} t' \leftarrow_{\vartheta} t$, with the simultaneous step non-erasing.*

Proof. Joinability follows by FD. It remains to show non-erasingness. ϑ being of order 2, each free variable Z occurs in g as $Z(x_1, \ldots, x_n)$ with $x_i : o$ and $Z : o \to \ldots \to o \to o$ and in d as $Z(t_1, \ldots, t_n)$ with $t_i : o$. Hence, the residuals in s' of redexes of $s \multimap\!\!\to t$ are first-order substitution instances of them. Then, to show preservation of non-erasingness it suffices to show that $\mathit{Var}(s) \subseteq \mathit{Var}(s^{\sigma})$ for any first-order substitution σ, which follows by induction on s. ☐

Left-linearity and fully-extendedness are sufficient conditions for STD to hold.

Theorem 7 (STD). *Any rewrite sequence in a P_2RS can be transformed into a standard one. The transformation preserves infiniteness.*

Proof. The proof of the second part of the theorem is as for TRSs. For a proof of the first part for left-linear fully-extended (orthogonal) CRSs see [18, Sect. 7.7.3] ([26]). By the correspondence between CRSs and P_2RSs this suffices for our purposes. (STD even holds for PRSs [22, Cor. 1.5].) ☐

Proof. (of Theorem 6) Replace in the proof of Theorem 2 everywhere $\twoheadrightarrow\!\!\!\!\!\!\!+\!\!\!+$ by $\multimap\!\!\to$. That the (context)-case eventually applies follows by an appeal to FD. ☐

The proofs of the results below are obtained by analogous modifications.

Theorem 8. *Non-erasing rewrite steps are perpetual in biclosed P_2RSs.*

F_2TP can be strengthened in various ways. Unlike for TRSs, a critical step in a P_2RS need not erase a term in ∞ as witnessed by $e(f(x))\langle x{:=}a\rangle \to c\langle x{:=}a\rangle$ in the PRS $\{M(x)\langle x := N\rangle \to M(N), e(Z) \to c, f(a) \to f(a)\}$. Note that $f(x) \in \mathsf{SN}$, but by contracting the $\_\langle\_ := \_\rangle$-redex a is substituted for x and $f(a) \in \infty$.

Definition 12. *An occurrence of (the head symbol of) a subterm is* potentially infinite *if some descendant [20] of it along some reduction is in ∞. A step is ∞-erasing if it erases all potentially infinite subterms in its arguments.*

For TRSs this notion of ∞-erasingness coincides with the one of Def. 9.

Corollary 8. *Non-∞-erasing rewrite steps are perpetual in biclosed P_2RSs.*

Many variations of this result are possible. We mention two. First, the motivation for this paper originates with [13, Sect. 6.4], where we failed to obtain:

Theorem 9. *([5]) λ-δ_K-calculus is uniformly normalising.*

Proof. By Cor. 8, since λ-δ_K-calculus is weakly orthogonal. ☐

Second, we show that non-fully-extended P_2RSs may have uniform normalisation.

Theorem 10. *Non-∞-erasing steps are perpetual in $\lambda\beta\eta$-calculus [24, Prop. 27].*

Proof. It suffices to remark that η-steps can be postponed after β-steps in a standard sequence [2, Cor. 15.1.6]. Since η is terminating, an infinite standard sequence must contain infinitely many β-steps, hence may be assumed to consist of β's only and the proof of F_2TP goes through unchanged. □

By the same method, P_2RSs where non-fully-extended steps are terminating and postponable have uniform normalisation.

5 λx^-

In this section the *fundamental theorem of perpetuality for* λx^- is established:

Theorem 11 (F_xTP). *Non-erasing steps are perpetual in λx^-.*

Familiarity with the nameful λ-calculus with explicit substitutions λx^- of [4] is assumed. We define it as a *P_2RS*.

Definition 13. *The alphabet of λx^- [4] consists of the function symbols @ : $o{\to}o{\to}o$, $\lambda : (o{\to}o){\to}o$ and $\_\langle\_ := \_\rangle : (o{\leftarrow}o){\to}o{\to}o$. As above, we make @ an implicit infix operator associating to the left. The rules of λx^- are (for $x \neq y$):*

$$\begin{array}{rcl}(\lambda x.M(x))N & \to_{\mathrm{Beta}} & M(x)\langle x := N\rangle \\ x\langle x := N\rangle & \to_{=} & N \\ y\langle x := N\rangle & \to_{\neq} & y \\ (\lambda y.M(y,x))\langle x := N\rangle & \to_{\lambda} & \lambda y.M(y,x)\langle x := N\rangle \\ (M(x)L(x))\langle x := N\rangle & \to_{@} & M(x)\langle x := N\rangle L(x)\langle x := N\rangle\end{array}$$

The last four rules generate the explicit substitution *relation* $\to_x$.

$\to_x$ is a terminating and orthogonal *P_2RS*, hence the normal form of a term s exists uniquely and is denoted by $s{\downarrow_x}$. Note that $s{\downarrow_x}$ is a *pure* λ-term, i.e. it does not contain *closures* ($\_\langle\_ := \_\rangle$-symbols). λx^- implements (only) substitution [4]:

Lemma 5. *1. If $s =_x t$, then $s{\downarrow_x} = t{\downarrow_x}$.*
2. If $s \to_{\mathrm{Beta}} t$, then $s{\downarrow_x} \twoheadrightarrow_\beta t{\downarrow_x}$.
3. If s is pure and $s \to_\beta t$, then $s \to_{\mathrm{Beta}} \cdot \to_x^+ t$.

Remark that in the second item the number of β-steps might be zero, but is always positive when the Beta-steps is not inside a closure. We call λx^--sequences without steps inside closures *pretty.* λx^- *preserves strong normalisation* in the sense that any pure term which is β-terminating is λx^--terminating.

Lemma 6 (PSN). *[4, Thm. 4.19] If s is pure and β-SN, then s is λx^--SN.*

Proof. Suppose $s \in \infty$. Since $\lambda\mathrm{x}^-$ is a fully-extended left-linear sub-$\mathrm{P_2RS}$[5], we may by STD assume an infinite standard sequence $S : s_0 \to s_1 \to \ldots$ from $s = s_0$. We show that we may choose S to be pretty decent, where a sequence is *decent* [4, Def. 4.16] if for every closure $\langle x := t \rangle$ in any term, $t \in \mathsf{SN}$.

(init) s is decent since it is pure.

(step) Suppose $s_i \in \infty$ and s_i is decent. From the shape of the rules we have that 'brackets are king' [19][6]: if any step takes place in t inside some closure $\langle x := t \rangle$ in a standard sequence, then no step above the closure can be performed later in the sequence. This entails that if t is terminating, S need not perform any step inside t. Hence assume $s_i \to s_{i+1}$ is pretty.

- (Beta) Suppose $s_i \to_{\mathrm{Beta}} s_{i+1}$ contracting $(\lambda x.M(x))N$ to $M(x)\langle x := N \rangle$. We may assume that N is terminating since otherwise we could instead perform an infinite reduction on N itself, hence the reduct is decent.
- (x) Otherwise, decency is preserved, since x-steps do not create closures.

Since x is terminating S must contain infinitely many Beta-steps. Since S is pretty $S{\downarrow_{\mathrm{x}}}$ is an infinite β-sequence from s by (the remark after) Lemma 5. □

Our method relates to closure-tracking [3] as preventing to curing. It would be interesting to see whether the method can be applied to show PSN of other translations, say a translation of λ-calculus into the π-calculus or into linear logic. Our first try to do so, led to the following counterexample against [4, Conj. 6.45], stating that explicification of *redex preserving* CRSs is PSN.

Example 4. Consider the term $s = (\tilde{\lambda}(x.b))a$ in the $\mathrm{P_2RS}$ $\{(\tilde{\lambda}x.M(x))N \to M(g(N,N)), a \to b, g(a,b) \to g(a,b)\}$. The only step possible from s is $s \to b[x{:=}g(a,a)] = b$. Hence s is terminating, but since $g(a,a) \to g(a,b) \to g(a,b)$ explicifying $\mathcal{R}$ will clearly make s infinite. The PRS is redex preserving in the sense of [4, Def. 6.44] since any redex in the argument $g(N,N)$ to M occurs in N already. So s is a term for which PSN does not hold.

We expect the conjecture to hold for orthogonal CRSs. For our purpose, uniform normalisation, we will need the following corollary to Lemma 6, on preservation of infinity. It is useful in situations where terms are only the same up to the Substitution Lemma [2, Lem. 2.1.16]: $M(x,y)\langle x := N(y)\rangle\langle y := L\rangle{\downarrow_{\mathrm{x}}} = M(x,y)\langle y := P\rangle\langle x := N\langle y := L\rangle\rangle{\downarrow_{\mathrm{x}}}$.

Corollary 9. *If s is decent and $s{\downarrow_{\mathrm{x}}} = t{\downarrow_{\mathrm{x}}}$, then $s \in \infty$ implies $t \in \infty$.*

How should non-erasingness be defined for $\lambda\mathrm{x}^-$? The naïve attempt falters.

Example 5. From the term $s = ((\lambda x.z)(y\omega))\langle y := \omega\rangle$, where $\omega = \lambda x.xx$, we have a unique terminating sequence starting with a 'non-erasing' Beta-step:

$$s \to_{\mathrm{Beta}} z\langle x := y\omega\rangle\langle y := \omega\rangle \to_{\mathrm{x}} z\langle y := \omega\rangle \to_{\mathrm{x}} z$$

On the other hand, developing $\langle y := \omega\rangle$ yields the term $\omega\omega \in \infty$.

[5] It only is a *sub*-$\mathrm{P_2RS}$ since the y in the $\to_{\neq}$-rule ranges over variables not over terms.

[6] Thinking of terms as trees representing hierarchies of people, creating a redex above (overruling) someone (the ruler) from below (the people) is a revolution. For closures/brackets this is not possible, whence these are king.

Translating the example into $\lambda\beta$-calculus shows that the culprit is the 'non-erasing' Beta-step, which translates into an erasing β-step. Therefore:

Definition 14. *A* λx^-*-step contracting redex* s *to* t *is* erasing *if* $s \to t$ *is*

$$\begin{aligned}(\lambda x.M(x))N &\to_{\text{Beta}} M, \text{ with } x \notin Var(M(x){\downarrow_{\text{x}}}), \text{ or}\\ y\langle x := N\rangle &\to_{\neq} y\end{aligned}$$

Proof. (of Theorem 11) Since λx^- is a sub-P$_2$RS, it suffices by the proof of F$_2$TP to consider perpetuality of a step $s \to_{p,\varrho} t$, for some infinite standard sequence $S : s_0 \to_{q_0,\vartheta_0} s_1 \to_{q_1,\vartheta_1} \dots$ starting from $s = s_0$ such that $s_1 \leftarrow s \to t$ is an overlapping fork (case (‡) on p. 7). λx^- has only one non-trivial critical pair. It arises by @ and Beta from $s' = ((\lambda x.M(x,y))N(y))\langle y := P\rangle$, so let $s = C[s']$.

(Beta,@) In case $s \to_{\text{Beta}} C[M(x,y)\langle x := N(y)\rangle\langle y := P\rangle] = s_1$, we note that

$$\begin{aligned}s &\to_{p,@} C[(\lambda x.M(x,y))\langle y := P\rangle N(y)\langle y := P\rangle] = t\\ &\to_{\lambda} C[(\lambda x.M(x,y)\langle y := P\rangle)N(y)\langle y := P\rangle]\\ &\to_{\text{Beta}} C[M(x,y)\langle y := P\rangle\langle x := N(y)\langle y := P\rangle\rangle] = t_1\end{aligned}$$

Consider a minimal closure in s_1 (or s_1 itself) which is decent and ∞, say at position o. If o is parallel or properly below p, i.e. inside one of $M(x,y)$, $N(y)$ or P, then obviously $t_1 \in \infty$. Otherwise, o is above p and $t_1 \in \infty$ follows from Corollary 9, since $s_{1|o}{\downarrow_{\text{x}}} = t_{1|o}{\downarrow_{\text{x}}}$.

(@,Beta) The case $s \to_{p,\text{Beta}} C[M(x,y)\langle x := N(y)\rangle\langle y := P\rangle] = t$ is more involved. Construct a maximal sequence T as follows. Let $t_0 = t$ be the first term of T and set $o_0 = p$.

- Suppose $s_i \to_{q_i,\vartheta_i} s_{i+1}$ does not contract a redex below o_i. As an invariant we will use that o_i traces the position of @ (initially at p) along S. If q_i is parallel to o_i, then we set $t_i \to_{q_i,\vartheta_i} t_{i+1}$. Otherwise $q_i < o_i$ and by standardness this is only possibly in case of an @-step distributing closures over the @ at o_i. Then we set $t_{i+1} = t_i$ and $o_{i+1} = q_i$.

If this process continues, then T is infinite since in case no steps are generated $o_{i+1} < o_i$, hence eventually a step must be generated. If the process stops, say at n, then by construction $s_n = D[u]_{o_n}$ and $t_n = D[v]_{o_n}$, with $u = (\lambda x.M(x,y))\langle \boldsymbol{y} := \boldsymbol{P}\rangle N(y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle$, $v = M(x,y)\langle x := N(y)\rangle\langle \boldsymbol{y} := \boldsymbol{P}\rangle$ and $\langle \boldsymbol{y} := \boldsymbol{P}\rangle$ abbreviates a sequence of closures the first of which is $\langle y := P\rangle$. Per construction, $o_n \leq q_n$ for the step $s_n \to_{q_n} s_{n+1}$ and we are in the 'non-replicating' case: by standardness the @ cannot be replicated along S and it can only be eliminated as part of a Beta-step. Consider a maximal part of S not contracting o_n. Remark that if any of $M(x,y)$, $N(y)$ and $\boldsymbol{P}$ is infinite, then $t_n \in \infty$, so we assume them terminating.

(context) If infinitely many steps parallel to o_i take place, then $D \in \infty$, hence $t_n = D[v] \in \infty$.

(left) Suppose infinitely many steps are in $(\lambda x.M(x,y))\langle \boldsymbol{y} := \boldsymbol{P}\rangle$. This implies $M(x,y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle \in \infty$, hence $M(x,y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle\langle x := N(y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle\rangle \in \infty$, which by Corollary 9 implies $t_n \in \infty$.

(right) Suppose infinitely many steps are in $N(y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle$. By non-erasingness of $s \rightarrow_{\text{Beta}} t$, $x \in M(x,y)\downarrow_{\text{x}}$ hence

$$\begin{aligned} v &\twoheadrightarrow_{\text{x}} M(x,y)\downarrow_{\text{x}}\langle x := N(y)\rangle\langle \boldsymbol{y} := \boldsymbol{P}\rangle \\ &= E[x,\ldots,x]\langle x := N(y)\rangle\langle \boldsymbol{y} := \boldsymbol{P}\rangle \\ &\twoheadrightarrow_{\text{x}} E^*[x\langle x := N(y)\rangle\langle \boldsymbol{y} := \boldsymbol{P}\rangle,\ldots,x\langle x := N(y)\rangle\langle \boldsymbol{y} := \boldsymbol{P}\rangle] \\ &\twoheadrightarrow_{=} E^*[N(y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle,\ldots,N(y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle] \in \infty \end{aligned}$$

where E^* arises by pushing $\langle x := N(y)\rangle\langle \boldsymbol{y} := \boldsymbol{P}\rangle$ through E, and $E[\,,\ldots,\,]$ is a pure λ-calculus context with at least one hole. Hence $t = D[v] \in \infty$.

(Beta) Suppose o_n is Beta-reduced sometime in S. By standardness steps before Beta can be neither in occurrences of the closures $\langle \boldsymbol{y} := \boldsymbol{P}\rangle$ nor in $M(x,y)$, hence we may assume S proceeds as:

$$\begin{aligned} s_n &\twoheadrightarrow_{\lambda} D[(\lambda x.M(x,y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle)N(y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle] \\ &\rightarrow_{\text{Beta}} D[M(x,y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle\langle x := N(y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle\rangle] = u' \end{aligned}$$

We proceed as in item (Beta,@), using $u'\downarrow_{\text{x}} = v\downarrow_{\text{x}}$ to conclude $v \in \infty$ by Corollary 9. The only exception to this is an infinite reduction from $N(y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle$, but such a reduction can be simulated from v by non-erasingness of the Beta-step as in item (right). ☐

The proof is structured as before, only di/polluted by explicit substitutions travelling through the pivotal Beta-redex. Again, one can vary on these results. For example, it should not be difficult to show that non-∞-erasing steps are perpetual, where $y\langle x := N\rangle \rightarrow_{\neq} y$ is ∞-erasing if $N \in \infty$ and $(\lambda x.M(x))N \rightarrow_{\text{Beta}} M$ is ∞-erasing if $x \notin \mathit{Var}(\text{x}(M(x)))$ and N contains a *potentially infinite* subterm. A comparison to proof methods of related results, e.g. [6], is left to future work.

6 Conclusion

The uniform normalisation proofs in literature are mostly based on particular *perpetual* strategies, that is, strategies performing only perpetual steps. Observing that the non-computable[7] such strategies usually yield standard sequences we have based our proof on standardisation, instead of searching for yet another 'improved' perpetual strategy. This effort was successful and resulted in a flexible proof strategy with a simple invariant easily adaptable to a λ-calculus with explicit substitutions. Nevertheless, our results are still very much orthogonality-bound: the biclosedness results arise by tweaking orthogonality and the λx^- results by interpretation in the, orthogonal, $\lambda\beta$-calculus. It would be interesting to see what can be done for truly non-orthogonal systems. The fully-extendedness and left-linearity restrictions are serious ones, e.g. in the area of process-calculi (scope extrusion) or even already for λx [4], so should be ameliorated.

Acknowledgments We would like to thank R. Bloo, E. Bonelli, D. Kesner, P.-A. Melliès, A. Visser and the members of PAM at CWI for feedback.

[7] No computable strategy exists which is both perpetual and standard, since then one could for all terms s, t decide whether $\mathsf{SN}(s) \Rightarrow \mathsf{SN}(t)$ or $\mathsf{SN}(t) \Rightarrow \mathsf{SN}(s)$.

References

[1] F. Baader and T. Nipkow. *Term Rewriting and All That*. CUP, 1998.

[2] H. Barendregt. *The Lambda Calculus, Its Syntax and Semantics*. NH, 1984.

[3] Z. Benaissa, D. Briaud, P. Lescanne, and J. Rouyer-Degli. $\lambda\upsilon$, a calculus of explicit substitutions which preserves strong normalisation. *JFP*, 6(5):699–722, 1996.

[4] R. Bloo. *Preservation of Termination for Explicit Substitution*. PhD thesis, Technische Universiteit Eindhoven, 1997.

[5] C. Böhm and B. Intrigila. The ant-lion paradigm for strong normalization. *I&C*, 114(1):30–49, 1994.

[6] E. Bonelli. Perpetuality in a named lambda calculus with explicit substitutions. MSCS, To appear.

[7] Alonzo Church. *The Calculi of Lambda-Conversion*. PUP, 1941.

[8] Georges Gonthier, Jean-Jacques Lévy, and Paul-André Melliès. An abstract standardisation theorem. In *LICS'92*, pages 72–81, 1992.

[9] Bernhard Gramlich. *Termination and Confluence Properties of Structured Rewrite Systems*. PhD thesis, Universität Kaiserslautern, 1996.

[10] Gérard Huet. Confluent reductions: Abstract properties and applications to term rewriting systems. *JACM*, 27(4):797–821, 1980.

[11] Gérard Huet and Jean-Jacques Lévy. Computations in orthogonal rewriting systems, I. In *Computational Logic: Essays in Honor of Alan Robinson*, pages 395–414. MIT Press, 1991.

[12] Z. Khasidashvili. On the longest perpetual reductions in orthogonal expression reduction systems. TCS, To appear.

[13] Z. Khasidashvili, M. Ogawa, and V. van Oostrom. Perpetuality and uniform normalization in orthogonal rewrite systems. I&C, To appear. `http://www.phil.uu.nl/~oostrom/publication/ps/pun-icv2.ps`.

[14] Jan Willem Klop. *Combinatory Reduction Systems*. PhD thesis, Rijksuniversiteit Utrecht, 1980. Mathematical Centre Tracts 127.

[15] J.W. Klop. Term rewriting systems. In *Handbook of Logic in Computer Science*, volume 2, pages 1–116. OUP, 1992.

[16] Yves Lafont. From proof-nets to interaction nets. In *Advances in Linear Logic*, pages 225–247. CUP, 1995.

[17] Richard Mayr and Tobias Nipkow. Higher-order rewrite systems and their confluence. *Theoretical Computer Science*, 192:3–29, 1998.

[18] Paul-André Melliès. *Description Abstraite des Systèmes de Réécriture*. Thèse de doctorat, Université Paris VII, 1996.

[19] Paul-André Melliès. Personal communication, 1999.

[20] Vincent van Oostrom. *Confluence for Abstract and Higher-Order Rewriting*. Academisch proefschrift, Vrije Universiteit, Amsterdam, 1994.

[21] Vincent van Oostrom. Development closed critical pairs. In *HOA'95*, volume 1074 of *LNCS*, pages 185–200. Springer, 1996.

[22] Vincent van Oostrom. Normalisation in weakly orthogonal rewriting. In *RTA'99*, volume 1631 of *LNCS*, pages 60–74. Springer, 1999.

[23] F. van Raamsdonk. *Confluence and Normalisation for Higher-Order Rewriting*. Academisch proefschrift, Vrije Universiteit, Amsterdam, 1996.

[24] M.H. Sørensen. Effective longest and infinite reduction paths in untyped lambda-calculi. In *CAAP'96*, volume 1059 of *LNCS*, pages 287–301. Springer, 1996.

[25] Yoshihito Toyama. Strong sequentiality of left-linear overlapping term rewriting systems. In *LICS'92*, pages 274–284, 1992.

[26] J.B. Wells and Robert Muller. Standardization and evaluation in combinatory reduction systems, 2000. Working paper.