A Complete Axiomatization for Prefix Iteration in Branching Bisimulation

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Abstract

This paper studies the interaction of prefix iteration μ^*x with the silent step τ in the setting of branching bisimulation. That is, we present a finite equational axiomatization for Basic Process Algebra with deadlock, empty process and the silent step, extended with prefix iteration, and prove that this axiomatization is complete with respect to rooted branching bisimulation equivalence.

1 Introduction

Kleene (1956) defined a binary operator $\_^*\_$ in the context of finite automata, called *Kleene star* or *iteration*. Intuitively, the expression p^*q yields a solution for the recursive equation $X = p \cdot X + q$. In other words, p^*q can choose to execute either p, after which it evolves into p^*q again, or q, after which it terminates. This paper considers the prefix counterpart μ^*x of iteration in process algebra, where the argument μ ranges over the set of constants. Our setting is Basic Process Algebra (BPA) from Bergstra and Klop (1984), with the deadlock δ and the empty process ϵ and the silent step τ, extended with prefix iteration.

Milner (1984) was the first to study iteration in strong bisimulation equivalence, in a process algebra equivalent to $\text{BPA}_{\delta\epsilon}$ extended with the Kleene star. Milner proposed an axiomatization for this process algebra, including a conditional axiom for iteration from Salomaa (1966), and he raised the question whether his axiomatization is complete. This question is, to our knowledge, still open. Bergstra, Bethke and Ponse (1994) considered BPA with the Kleene star, and they suggested a finite equational axiomatization for this algebra. Fokkink and Zantema (1994) proved that this axiomatization is complete with respect to strong bisimulation equivalence.

Sewell (1994) proved that there does not exist a complete finite equational axiomatization for BPA_δ with the Kleene star modulo strong bisimulation, due to the fact that $x^*\delta$ and $(x^n)^*\delta$ are strongly bisimilar for $n \geq 2$. It turned out that in order to obtain an equational axiomatization, it is sufficient to get rid of expressions $(x^n)^*\delta$ in the syntax, that is, to replace the Kleene star by prefix iteration. Complete finite equational axiomatizations for prefix iteration have been given in Fokkink (1994,1995) in the settings of basic CCS and of $\mathrm{BPA}_{\delta\epsilon}$.

In a revision of Baeten and Bergstra (1992), the axiomatization for prefix iteration is applied in an extension of BPA_δ with discrete time.

Aceto and Ingólfsdóttir (1995) study basic CCS with prefix iteration together with the silent step τ, in Milner's observation congruence. They extend the axiomatization from Fokkink (1994) with two well-known equational axioms for the silent step, and with three new equational axioms which describe the interplay between the silent step and prefix iteration. They prove that their axiomatization for basic CCS with the silent step and prefix iteration is complete for observation congruence.

This paper studies the interaction of prefix iteration and the silent step τ in the context of rooted branching bisimulation equivalence from van Glabbeek and Weijland (1989). That is, we present a complete finite equational axiomatization for $\mathrm{BPA}_{\delta\epsilon\tau}$ extended with prefix iteration, with respect to rooted branching bisimulation equivalence. It turns out that two axioms are sufficient in order to describe the relation between prefix iteration and the silent step:

$$\begin{aligned} \tau^*x &= \tau x + x \\ a \cdot a^*(\tau \cdot a^*(x+y) + x) &= a \cdot a^*(x+y) \end{aligned}$$

The first axiom is based on Koomen's Fair Abstraction Rule as formulated in Baeten and van Glabbeek (1987). The second axiom is, to our knowledge, new.

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2 Preliminaries

2.1 The syntax and semantics

Assume an alphabet A of atomic actions, and three special constants δ and ϵ and τ, which represent deadlock and empty process and the silent step respectively. In the sequel we use the following notations for constants:

- a, b range over A,
- α ranges over $A \cup \{\tau\}$,
- μ ranges over $A \cup \{\delta, \epsilon, \tau\}$.

The signature of the process algebra $\mathrm{BPA}^{p*}_{\delta\epsilon\tau}(A)$ is built from constants μ, alternative composition $x + y$, sequential composition $x \cdot y$, and prefix iteration μ^*x.

Table 1 presents an operational semantics for $\mathrm{BPA}_{\delta\epsilon\tau}^{p*}(A)$ in the style of Plotkin (1981). The binary transition relation $p \xrightarrow{\alpha} q$ denotes that the process p can evolve into q after executing action α, and the unary relation $p\downarrow$ expresses that the process p can terminate successfully.

$$\epsilon\downarrow \qquad\qquad \alpha \xrightarrow{\alpha} \epsilon$$

$$\frac{x\downarrow}{x+y\downarrow \quad y+x\downarrow} \qquad \frac{x \xrightarrow{\alpha} x'}{x+y \xrightarrow{\alpha} x' \quad y+x \xrightarrow{\alpha} x'}$$

$$\frac{x\downarrow \quad y\downarrow}{x\cdot y\downarrow} \qquad \frac{x\downarrow \quad y \xrightarrow{a} y'}{x\cdot y \xrightarrow{a} y'} \qquad \frac{x \xrightarrow{a} x'}{x\cdot y \xrightarrow{a} x'\cdot y}$$

$$\alpha^* x \xrightarrow{\alpha} \alpha^* x \qquad \frac{x\downarrow}{\mu^* x\downarrow} \qquad \frac{x \xrightarrow{\alpha} x'}{\mu^* x \xrightarrow{\alpha} x'}$$

Table 1: Action rules for $\mathrm{BPA}_{\delta\epsilon\tau}^{p*}(A)$

Our model for $\mathrm{BPA}_{\delta\epsilon\tau}^{p*}(A)$ consists of all the closed terms that can be constructed from the constants in $A \cup \{\delta, \epsilon, \tau\}$ together with the three operators. That is, the BNF grammar for the collection of process terms is as follows:

$$p \ ::= \ \mu \mid p+p \mid p\cdot p \mid \mu^* p.$$

As binding convention, * binds stronger than $\cdot$, which binds stronger than $+$. Often, $p\cdot q$ will be abbreviated to pq.

The notation $p \Rightarrow p'$ in the following definition of branching bisimulation expresses that there exists a sequence $p \xrightarrow{\tau} \cdots \xrightarrow{\tau} p'$ of (zero or more) τ-transitions.

Definition 2.1 *Two processes p and q are* branching bisimilar*, denoted by $p \leftrightarrow_b q$, if there exists a symmetric binary relation $\mathcal{B}$ on processes which relates p and q, such that*

1. *if $r \xrightarrow{\alpha} r'$ and $r\mathcal{B}s$, then either*
   - *$\alpha = \tau$ and $r'\mathcal{B}s$,*
   - *or $s \Rightarrow s' \xrightarrow{\alpha} s''$ with $r\mathcal{B}s'$ and $r'\mathcal{B}s''$,*
2. *if $r\downarrow$ and $r\mathcal{B}s$, then $s \Rightarrow s'\downarrow$ with $r\mathcal{B}s'$.*

Branching bisimulation is reflexive, symmetric and transitive. The following lemma is a standard result for branching bisimulation equivalence.

Lemma 2.2 *(Stuttering Lemma.) If* $p_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} p_n$*, and if* $p_n \;\underline{\leftrightarrow}_b\; p_0$*, then* $p_i \;\underline{\leftrightarrow}_b\; p_0$ *for* $i = 1, ..., n-1$*.*

Proof. See van Glabbeek and Weijland (1989).

Branching bisimulation equivalence is not a congruence. That is, equivalences $p \;\underline{\leftrightarrow}_b\; p'$ and $q \;\underline{\leftrightarrow}_b\; q'$ do not always imply that $p + q \;\underline{\leftrightarrow}_b\; p' + q'$. For example, $\tau a \;\underline{\leftrightarrow}_b\; a$, but $\tau a + b \;\not\underline{\leftrightarrow}_b\; a + b$. In order to turn branching bisimulation into a congruence, we need a rootedness condition.

Definition 2.3 *Two processes* p *and* q *are* rooted branching bisimilar*, denoted by* $p \;\underline{\leftrightarrow}_{rb}\; q$*, if*

1. $p \xrightarrow{\alpha} p'$ *if and only if* $q \xrightarrow{\alpha} q'$ *with* $p' \;\underline{\leftrightarrow}_b\; q'$*,*
2. $p\downarrow$ *if and only if* $q\downarrow$*.*

Process terms will be considered modulo rooted branching bisimulation equivalence.

The action rules in Table 1 are in a congruence format for TSSs of Bloom (1993), called 'RBB cool'. (This format does not allow predicates such as $p\downarrow$, but by a simple coding trick this predicate can be transformed into a binary transition relation.) Hence, rooted branching bisimulation equivalence is a congruence with respect to the operators, i.e. if $p \;\underline{\leftrightarrow}_{rb}\; p'$ and $q \;\underline{\leftrightarrow}_{rb}\; q'$, then $p + q \;\underline{\leftrightarrow}_{rb}\; p' + q'$ and $p \cdot q \;\underline{\leftrightarrow}_{rb}\; p' \cdot q'$ and $\mu^* p \;\underline{\leftrightarrow}_{rb}\; \mu^* p'$. See Bloom (1993) for the definition of RBB cool, and for a proof of this congruence result.

The action rules for $\mathrm{BPA}_{\delta\epsilon\tau}(A)$ are 'pure' and 'well-founded', which are syntactic criteria from Groote and Vaandrager (1992). Moreover, the action rules for prefix iteration incorporate the Kleene star in the left-hand side of their conclusions. Hence, $\mathrm{BPA}^{p*}_{\delta\epsilon\tau}(A)$ is an operationally conservative extension of $\mathrm{BPA}_{\delta\epsilon\tau}(A)$, i.e. the action rules for prefix iteration do not influence the transition systems of $\mathrm{BPA}_{\delta\epsilon\tau}(A)$ terms. See Verhoef (1994) for a proof of this conservativity result.

2.2 The axioms

Table 2 contains the ten standard axioms for $\mathrm{BPA}_{\delta\epsilon\tau}(A)$, together with seven axioms for prefix iteration. The axioms MI1-5 stem from Fokkink (1994,1995), and the axiom MI6 from Bergstra, Bethke and Ponse (1994). The axiom MI7 is, to our knowledge, new.

In the sequel, $p = q$ will mean that the equality can be derived from the axioms in Table 2. The axiomatization is sound with respect to rooted branching bisimulation equivalence, i.e. if $p = q$ then $p \;\underline{\leftrightarrow}_{rb}\; q$. Since rooted branching bisimulation is a congruence, this can be verified by checking soundness for each axiom separately, which is left to the reader. In this paper it is proved that the axiomatization is complete with respect to bisimulation, i.e. if $p \underline{\leftrightarrow}_{rb}\ q$ then $p = q$.

A1	$x + y$	$=$	$y + x$
A2	$(x + y) + z$	$=$	$x + (y + z)$
A3	$x + x$	$=$	x
A4	$(x + y)z$	$=$	$xz + yz$
A5	$(xy)z$	$=$	$x(yz)$
A6	$x + \delta$	$=$	x
A7	δx	$=$	δ
A8	$x\epsilon$	$=$	x
A9	ϵx	$=$	x
BE2	$\alpha(\tau(x + y) + x)$	$=$	$\alpha(x + y)$
MI1	$a \cdot a^*x + x$	$=$	a^*x
MI2	$(a^*x)y$	$=$	$a^*(xy)$
MI3	$a^*(a^*x)$	$=$	a^*x
MI4	δ^*x	$=$	x
MI5	ϵ^*x	$=$	x
MI6	τ^*x	$=$	$\tau x + x$
MI7	$a \cdot a^*(\tau \cdot a^*(x + y) + x)$	$=$	$a \cdot a^*(x + y)$

Table 2: Axioms for $\mathrm{BPA}^{p*}_{\delta\epsilon\tau}(A)$

The following lemma will be used in the completeness proof.

Lemma 2.4 *For each process term p, the collection of transitions from p is finite, say $\{p \xrightarrow{\alpha_i} p_i \mid i = 1, ..., n\}$, and $p = \sum_{i=1}^n \alpha_i p_i + t(\delta, \epsilon)$, where $t(\delta, \epsilon)$ is either ϵ if $p \downarrow$, or δ otherwise.*

Proof sketch. A straightforward exercise by induction on the size of p, i.e. on the number of function symbols in p, learns that the collection of transitions from p is finite, say $\{p \xrightarrow{\alpha_i} p_i \mid i = 1, ..., n\}$.

By induction on the size of p, it follows that $p = \sum_{i=1}^n \alpha_i p_i + t(\delta, \epsilon)$, where $t(\delta, \epsilon)$ is either ϵ if $p \downarrow$, or δ otherwise. This deduction uses the axioms A1-9+MI1,4,5 and the equation $\tau \cdot \tau^*x + x = \tau^*x$. This last equation can be derived from the axioms as follows.

$$\tau \cdot \tau^*x + x \overset{\mathrm{MI6}}{=} \tau(\tau x + x) + x \overset{\mathrm{BE2}}{=} \tau x + x \overset{\mathrm{MI6}}{=} \tau^*x. \quad \square$$

3 Completeness of the Axioms

In this section we present the completeness proof.

3.1 Strong bisimulation

First, we present some definitions and results that involve strong bisimulation equivalence from Park (1981). In this section, we do not consider τ as the special constant 'silent step', but as a regular action in the alphabet.

Definition 3.1 *Two processes p and q are* strongly bisimilar*, denoted by $p \underline{\leftrightarrow} q$, if there exists a symmetric binary relation $\mathcal{B}$ on processes which relates p and q, such that*

1. *if $r \xrightarrow{\alpha} r'$ and $r\mathcal{B}s$, then there is a transition $s \xrightarrow{\alpha} s'$ with $r'\mathcal{B}s'$,*

2. *if $r\downarrow$ and $r\mathcal{B}s$, then $s\downarrow$.*

The completeness result from Fokkink (1995) says that if $p \underline{\leftrightarrow} q$, then p and q are provably equal by the axioms A1-9+MI1-5, with a replaced by α in MI1-3.

The equations $\tau \cdot \tau^*x + x = \tau^*x$ and $(\tau^*x)y = \tau^*(xy)$ and $\tau^*(\tau^*x) = \tau^*x$ can be deduced from A1-5+BE2+MI6. (For the derivation of the first equation, see the proof sketch of Lemma 2.4.) Hence, the completeness result from Fokkink (1995) induces the following proposition.

Proposition 3.2 *If $p \underline{\leftrightarrow} q$, then $p = q$.*

Note that the converse is not true, i.e. our axiomatization for $\mathrm{BPA}^{p*}_{\delta\epsilon\tau}(A)$ is not sound with respect to strong bisimulation, due to the axioms BE2 and MI6,7 for the silent step.

The following definition stems from van Glabbeek and Weijland (1989).

Definition 3.3 *A transition $p \xrightarrow{\tau} p'$ is* inert *if $p \underline{\leftrightarrow}_b p'$.*

The next lemma follows easily from the Stuttering Lemma, together with the definition of branching bisimulation. The proof is left to the reader.

Lemma 3.4 *If $p \underline{\leftrightarrow}_b q$, and if the transition systems of p' and q' do not contain any inert steps, then $p \underline{\leftrightarrow} q$.*

$$\begin{array}{rcl}(x+y)z & \longrightarrow & xz+yz\\ (xy)z & \longrightarrow & x(yz)\\ \delta x & \longrightarrow & \delta\\ \epsilon x & \longrightarrow & x\\ (a^*x)y & \longrightarrow & a^*(xy)\\ \delta^* x & \longrightarrow & x\\ \epsilon^* x & \longrightarrow & x\\ \tau^* x & \longrightarrow & \tau x + x\end{array}$$

Table 3: A term rewriting system

3.2 Construction of basic terms

In this section we define a class of *basic terms*, and we show that each process term is provably equal to a basic term.

Table 3 contains a Term Rewriting System (TRS) which reduces sequential composition to its prefix counterpart and which eliminates expressions δ^*x and ϵ^*x and τ^*x. Define a weight function on terms as follows:

$$\begin{array}{rcl}w(\mu) & = & 2\\ w(p+q) & = & w(p)+w(q)\\ w(pq) & = & w(p)^2w(q)\\ w(\mu^*p) & = & 6w(p).\end{array}$$

It is easy to see that the weight of terms always strictly decreases under application of the rewrite rules. Hence, the TRS in Table 3 is terminating, i.e. there do not exist any infinite reductions. So each process term reduces to a normal form, which cannot be reduced by the TRS. The class of normal forms is defined by

$$p \ ::= \ \mu \mid p+p \mid \alpha p \mid a^*p.$$

Definition 3.5 *The class of* basic terms *is defined by*

$$p \ ::= \ \delta \mid \epsilon \mid a \mid p+p \mid ap \mid \tau p + p \mid a\cdot a^*p.$$

Lemma 3.6 *For each term p there is a basic term p' such that $p=p'$.*

Proof. The rewrite rules in the TRS are all axioms, so each process term is provably equal to its normal forms. We prove by induction on size that each normal form is provably equal to a basic term.

First, we deal with the case of normal forms of size 1. Normal forms δ and ϵ and a are basic. Furthermore, $\tau \stackrel{\mathrm{A6,8}}{=} \tau\epsilon+\delta$ is basic.

Next, suppose that we have proved the case for normal forms of size $\leq n$.

1. Consider a normal form $p+q$ of size $n+1$. By induction, $p=p'$ and $q=q'$ where p' and q' are basic. Hence, $p+q=p'+q'$ is basic.

2. Consider a normal form ap of size $n+1$. By induction, $p=p'$ where p' is basic. Hence, $ap=ap'$ is basic.

3. Consider a normal form τp of size $n+1$. By induction, $p=p'$ where p' is basic. Hence, $\tau p=\tau p' \stackrel{\text{A6}}{=} \tau p'+\delta$ is basic.

4. Consider a normal form a^*p of size $n+1$. By induction, $p=p'$ where p' is basic. Hence, $a^*p=a^*p' \stackrel{\text{MI1}}{=} a\cdot a^*p'+p'$ is basic. □

The next lemma will be applied in the completeness theorem. It follows by induction on the size of the context $C[]$.

Lemma 3.7 *If $C[\tau r+s]$ is basic, then r and s and $C[r+s]$ are basic.*

3.3 The completeness theorem

From now on, process terms are considered modulo AC, that is, modulo associativity and commutativity of the $+$, and this equivalence is denoted by $p =_{\text{AC}} q$.

Theorem 3.8 *The axiomatization* A1-9+BE2+MI1-7 *for* $\text{BPA}^{p*}_{\delta\epsilon\tau}(A)$ *is complete with respect to rooted branching bisimulation equivalence.*

Proof. First, we prove the following two statements in parallel, by induction on size.

A. For each basic term p there is a term p' with $\tau p=\tau p'$, such that the transition system of p' does not contain any inert steps.

B. If p and q are basic terms with $p \leftrightarrow_b q$, then $\tau p=\tau q$.

Let A_n denote statement A for basic terms p of size $\leq n$, and let B_n denote statement B for basic terms p and q of sizes $\leq n$.

Suppose that we have already proved A_{n-1} and B_{n-1} for some $n\geq 0$; we prove A_n and B_n. First we prove A_n, so assume a basic term p of size n. Suppose that the transition system of p contains an inert step, for else we are done.

Recall that the basic term p is a sum of terms δ, ϵ, a, $\alpha p'$, and $a(a^*p')$, with p' a basic term, where summands $\tau p'$ always occur as an argument of alternative composition. Hence, the inert step in the transition system of p is caused by a subterm $\tau r+s$ of p which occurs in p in one of the following three forms.

- either $p =_{\text{AC}} \tau r+s$, with $r \leftrightarrow_b \tau r+s$,
- or p has a subterm $\alpha(\tau r+s)$, with $r \leftrightarrow_b \tau r+s$,

- or p has a subterm $a \cdot a^*(\tau r + s)$, with $r \underline{\leftrightarrow}_b a^*(\tau r + s)$.

In the first two cases, $r \underline{\leftrightarrow}_b \tau r+s$ yields $r \underline{\leftrightarrow}_b r+s$, and in the last case $r \underline{\leftrightarrow}_b a^*(\tau r+s)$ yields $r \underline{\leftrightarrow}_b a^*(r+s)$. We consider the three cases separately.

1. $p =_{\mathrm{AC}} \tau r + s$ and $r \underline{\leftrightarrow}_b r + s$.

   According to Lemma 3.7, r and s are basic, so $r + s$ is basic. Furthermore, the size of $r+s$ is smaller than the size of $\tau r+s$, which means that the size of $r+s$ is smaller than n. So B_{n-1} together with $r \underline{\leftrightarrow}_b r+s$ yield $\tau r = \tau(r+s)$. Hence, $\tau p =_{\mathrm{AC}} \tau(\tau r + s) = \tau(\tau(r + s) + s) \stackrel{\mathrm{BE2}}{=} \tau(r + s)$.

   Since $r + s$ is a basic term of size $< n$, A_{n-1} yields that there is a term p' with $\tau(r + s) = \tau p'$, such that the transition system of p' does not contain any inert steps. Since $\tau p = \tau(r + s) = \tau p'$, we are done.

2. $p =_{\mathrm{AC}} C[\alpha(\tau r + s)]$ and $r \underline{\leftrightarrow}_b r + s$.

   Again, B_{n-1} yields $\tau r = \tau(r + s)$. Hence, $p =_{\mathrm{AC}} C[\alpha(\tau r + s)] = C[\alpha(\tau(r + s) + s)] \stackrel{\mathrm{BE2}}{=} C[\alpha(r + s)]$.

   According to Lemma 3.7 $C[\alpha(r + s)]$ is basic, and its size is smaller than n, so A_{n-1} yields that there is a term p' with $\tau C[\alpha(r + s)] = \tau p'$, such that the transition system of p' does not contain any inert steps. Since $\tau p = \tau C[\alpha(r + s)] = \tau p'$, we are done.

3. $p =_{\mathrm{AC}} C[a \cdot a^*(\tau r + s)]$ with $r \underline{\leftrightarrow}_b a^*(r + s)$.

   $a^*(r+s)$ is a basic term of size $< n$, so B_{n-1} together with $r \underline{\leftrightarrow}_b a^*(r+s)$ yield $\tau r = \tau \cdot a^*(r+s)$. Hence, $p =_{\mathrm{AC}} C[a \cdot a^*(\tau r+s)] = C[a \cdot a^*(\tau \cdot a^*(r+s)+s)] \stackrel{\mathrm{MI7}}{=} C[a \cdot a^*(r + s)]$.

   $C[a \cdot a^*(r+s)]$ is a basic term of size $< n$, so A_{n-1} yields that there is a term p' with $\tau C[a \cdot a^*(r+s)] = \tau p'$, such that the transition system of p' does not contain any inert steps. Since $\tau p = \tau C[a \cdot a^*(r + s)] = \tau p'$, we are done.

Next, we prove B_n. Assume basic terms p and q of sizes $\leq n$ with $p \underline{\leftrightarrow}_b q$. A_n yields that there exist terms p' and q' of which the transition systems do not contain any inert steps, such that $\tau p = \tau p'$ and $\tau q = \tau q'$. Soundness of the axioms yields $p \underline{\leftrightarrow}_b \tau p \underline{\leftrightarrow}_{rb} \tau p' \underline{\leftrightarrow}_b p'$ and $q \underline{\leftrightarrow}_b \tau q \underline{\leftrightarrow}_{rb} \tau q' \underline{\leftrightarrow}_b q'$, so $p' \underline{\leftrightarrow}_b p \underline{\leftrightarrow}_b q \underline{\leftrightarrow}_b q'$. Since the transition systems of p' and q' do not contain any inert steps, Lemma 3.4 yields $p' \underline{\leftrightarrow} q'$. Hence, Proposition 3.2 yields $p' = q'$, so $\tau p = \tau p' = \tau q' = \tau q$.

Finally, we show that statement B implies the desired completeness result. First, we deduce an equation from the axioms.

$$\alpha\tau \;=\; \alpha. \tag{1}$$

It follows by substituting δ for x and ϵ for y in axiom BE2, and applying axioms A6,8.

Let p and q be process terms with $p \underline{\leftrightarrow}_{rb}\ q$. According to the definition of rooted branching bisimulation, the sets of possible transitions from p and q are $\{p \xrightarrow{\alpha_i} p_i \mid i = 1, ..., n\}$ and $\{q \xrightarrow{\alpha_i} q_i \mid i = 1, ..., n\}$ with $p_i \underline{\leftrightarrow}_b\ q_i$ for $i = 1, ..., n$, and $p \downarrow$ if and only if $q \downarrow$. Hence, Lemma 2.4 yields $p = \sum_{i=1}^{n} \alpha_i p_i + t(\delta, \epsilon)$ and $q = \sum_{i=1}^{n} \alpha_i q_i + t(\delta, \epsilon)$, where $t(\delta, \epsilon)$ is either δ or ϵ. According to Lemma 3.6, there exist basic terms p_i' and q_i' such that $p_i = p_i'$ and $q_i = q_i'$. Since $p_i \underline{\leftrightarrow}_b\ q_i$, and since the axioms are sound, we find $p_i' \underline{\leftrightarrow}_{rb}\ p_i \underline{\leftrightarrow}_b\ q_i \underline{\leftrightarrow}_{rb}\ q_i'$, so $p_i' \underline{\leftrightarrow}_b\ q_i'$. Hence, B yields $\tau p_i' = \tau q_i'$ for $i = 1, ..., n$, so

$$\begin{aligned} p \ &= \ \textstyle\sum_{i=1}^{n} \alpha_i p_i + t(\delta, \epsilon) \ = \ \sum_{i=1}^{n} \alpha_i p_i' + t(\delta, \epsilon) \ \stackrel{(1)}{=} \ \sum_{i=1}^{n} \alpha_i \tau p_i' + t(\delta, \epsilon) \\ &= \ \textstyle\sum_{i=1}^{n} \alpha_i \tau q_i' + t(\delta, \epsilon) \ \stackrel{(1)}{=} \ \sum_{i=1}^{n} \alpha_i q_i' + t(\delta, \epsilon) \ = \ \sum_{i=1}^{n} \alpha_i q_i + t(\delta, \epsilon) \ = \ q. \ \Box \end{aligned}$$

3.4 Complete axiomatizations for subalgebras of $\mathbf{BPA}_{\delta\epsilon\tau}^{p*}(A)$

In the setting without ϵ, Equality 1 can no longer be deduced from axiom BE2, so it has to be added to the axiom system. Since ϵ is absent, the more general equality $x\tau = x$ is sound. Furthermore, we can replace axiom BE2 by axiom B2, in order to get rid of the variable α in the axioms.

B1	$x\tau$	$=$	x
B2	$\tau(\tau(x+y)+x)$	$=$	$\tau(x+y)$

With a similar proof scheme as has been used for the completeness proof for $\mathrm{BPA}_{\delta\epsilon\tau}^{p*}(A)$, with the cases for ϵ omitted, we obtained the following result.

Theorem 3.9 *The axiomatization* A1-7+B1,2+MI1-4,6,7 *for* $\mathrm{BPA}_{\delta\tau}^{p*}(A)$ *is complete with respect to rooted branching bisimulation equivalence.*

In the setting of $\mathrm{BPA}_{\epsilon\tau}^{p*}(A)$ too, Equation 1 cannot be deduced from the axioms. Axiom B1 is not sound, due to the presence of the empty process, so we add Equation 1 to the axiom system, under the name BE1. Furthermore, without deadlock the following equality can no longer be deduced from axiom MI7, so it has to be added to the axiom system.

MI8	$a \cdot a^*(\tau \cdot a^* x) \ = \ a \cdot a^* x$

With a similar proof scheme as has been used for the completeness proof for $\mathrm{BPA}_{\delta\epsilon\tau}^{p*}(A)$, we obtained the following result.

Theorem 3.10 *The axiomatization* A1-5,8,9+BE1+B2+MI1-3,5-8 *for* $\mathrm{BPA}_{\epsilon\tau}^{p*}(A)$ *is complete with respect to rooted branching bisimulation equivalence.*

Finally, with a similar proof scheme we deduced the following result.

Theorem 3.11 *The axiomatization* A1-5+B1,2+MI1-3,6-8 *for* $\mathrm{BPA}_{\tau}^{p*}(A)$ *is complete with respect to rooted branching bisimulation equivalence.*

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