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Labelled Deduction in the Composition of Form and Meaning

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In the late Fifties, Jim Lambek has started a line of investigation that accounts for the composition of form and meaning in natural language in *deductive* terms: formal grammar is presented as a logic — a system for reasoning about the basic form/meaning units of language and the ways they can be put together into wellformed structured configurations. The reception of the categorial grammar logics in linguistic circles has always been somewhat mixed: the mathematical elegance of the original system ([Lambek 58]) is counterbalanced by clear descriptive limitations, as Lambek has been the first to emphasize on a variety of occasions.

As a result of the deepened understanding of the options for 'substructural' styles of reasoning, the categorial architecture has been redesigned in recent work, in ways that suggest that mathematical elegance may indeed be compatible with linguistic sophistication. A careful separation of the logical and the structural components of the categorial inference engine leads to the identification of *constants* of grammatical reasoning. At the level of the basic rules of use and proof for these constants one finds an explanation for the *uniformities* in the composition of form and meaning across languages. Cross-linguistic *variation* in the realization of the form-meaning correspondence is captured in terms of *structural* inference packages, acting as plug-ins with respect to the base logic of the grammatical constants. Structural inferences are under the explicit control of lexically anchored licensing features. These features have the status of logical constants in their own right: they control the structural aspects of grammatical resource management in a way analogous to what the 'exponentials' (or 'modalities') of linear logic do for resource multiplicity. The reader is referred to [Moortgat 97] for a systematic presentation of these developments.

The categorial set-up sketched above presents a new challenge for the 'Parsing-as-Deduction' approach to natural language processing. Consider the question whether a string of words $w_1 \cdots w_n$ constitutes a wellformed expression of type B. Under the deductive view on grammatical computation, this question is reformulated as the problem displayed in (1): given A_i as the logical 'parts-of-speech' for the words w_i, does the grammar logic allow the derivation of the conclusion B? In the original Lambek calculus, linear order is the only structural factor that affects derivability: assumptions can be faithfully represented as one-dimensional lists $A_1, \ldots, A_n$. But in the refined setting assumed here, the assumptions A_i are configured into a *structured database*: the precise configuration of the assumptions will determine which structural inferences are applicable and whether the goal formula B is derivable or not. Clearly, we cannot take the structure of the database as *given* in the statement of the parsing problem. Rather, we want to find a way of efficiently *computing* this structure in the process of grammatical deduction.

(1)
$$\begin{array}{cccccc} \text{PARSING} & w_1 & \cdots & w_n & & \\ \textit{as} & \vdots & & \vdots & & \\ \text{DEDUCTION} & \underbrace{A_1 \;\; \cdots \;\; A_n}_{\Gamma} & & & \vdash & B \end{array}$$

In §3 we provide a uniform algorithmic proof theory for the structure-sensitive style of grammatical reasoning in terms of Dov Gabbay's framework of Labelled Deduction. We'll see that this framework has exactly the right properties for dealing with the logical and the structural aspects of grammatical reasoning in a modular way. We start with a brief overview of the grammatical architecture assumed in this paper in §1. In §2 we give a linguistic illustration on the basis of a labelled Natural Deduction format that is useful for *displaying* proofs once they have been found, but that does not have the right properties for algorithmic proof *search* (parsing).

1 Grammatical Composition: Logic, Structure, and Control

In this paper, we consider a language of type formulas freely generated from a small number of atomic types $\mathcal{A}$ by means of the unary and binary connectives in (2). The binary $/, \bullet, \backslash$ are the familiar categorial product and slash connectives. The unary $\Diamond, \Box$ are the new control devices.

(2) $$\mathcal{F} ::= \mathcal{A} \mid \Diamond\mathcal{F} \mid \Box\mathcal{F} \mid \mathcal{F}/\mathcal{F} \mid \mathcal{F} \bullet \mathcal{F} \mid \mathcal{F}\backslash\mathcal{F}$$

The categorial formula language is used to talk about the form-meaning units of language: 'signs', or 'grammatical resources', as we will call them here. An appropriate framework for reasoning about structured configurations of grammatical resources is *modal logic*: we base the models for the grammar logic on frames $F = \langle W, R^2, R^3 \rangle$. The domain W, in the case at hand, is the set of linguistic resources, and for each family of n-place connectives, we have an $n+1$-place 'accessibility relation' modelling the decomposition of a grammatical compound into its constituent part(s). This type of semantics has its ancestry in the Kripke models for relevant logics introduced in the Seventies by Routley and Meyer as pointed out in [Došen 92]. Frame based semantics for the extended type languages we consider in this paper is investigated in depth in [Kurtonina 95] — see Van Benthem (this volume) for discussion.

As remarked in the introduction, we want to keep *logical* and *structural* aspects of the meaning of the constants apart. The categorial base logic is 'structurally uncommitted' in that it interprets $\Diamond$ and $\bullet$ as existential modal operators with respect to arbitrary binary and ternary composition relations R^2 and R^3. The constants $\Box$ and $/, \backslash$ are interpreted as the residuation duals of these existential modalities. See the interpretation clauses in (3).

(3) $$\begin{array}{rcl} V(\Diamond A) & = & \{x \mid \exists y(R^2xy \ \& \ y \in V(A)\} \\ V(\Box A) & = & \{y \mid \forall x(R^2xy \ \Rightarrow x \in V(A)\} \\ \\ V(A \bullet B) & = & \{x \mid \exists y \exists z[R^3xyz \ \& \ y \in V(A) \ \& \ z \in V(B)]\} \\ V(C/B) & = & \{y \mid \forall x \forall z[(R^3xyz \ \& \ z \in V(B)) \Rightarrow x \in V(C)]\} \\ V(A\backslash C) & = & \{z \mid \forall x \forall y[(R^3xyz \ \& \ y \in V(A)) \Rightarrow x \in V(C)]\} \end{array}$$

The residuation laws of (4) capture the properties of $\Diamond, \Box$ and $/, \bullet, \backslash$ with respect to derivability. The residuation inferences, together with the reflexivity and transitivity of derivability, give the essential completeness result in the sense that $A \rightarrow B$ is provable iff $V(A) \subseteq V(B)$ for every valuation V on every frame F. Restricting our attention to the binary connectives, we have the completeness result of [Došen 92] for the calculus **NL** of [Lambek 61]. For the language extended with unary connectives, see [Moortgat 96, Kurtonina 95].

(4) $$\Diamond A \rightarrow B \quad \text{iff} \quad A \rightarrow \Box B$$

$$A \rightarrow C/B \quad \text{iff} \quad A \bullet B \rightarrow C \quad \text{iff} \quad B \rightarrow A\backslash C$$

The laws of the base logic hold universally, in the sense that they do not depend on structural properties of the composition relation. Cross-linguistic variation is obtained by adding to the base logic postulate packages regulating structural aspects of the grammatical resource management regime. Semantically, these postulates 'have a price': they introduce constraints on the interpretation of the grammatical composition relations R^2, R^3, see again [Došen 92, Kurtonina 95] for thorough discussion. An illustrative sample of structural postulates is displayed in (5) below.

(5)

$$\begin{array}{rlrl} C: & A \bullet B \to B \bullet A & K1: & \Diamond(A \bullet B) \to \Diamond A \bullet B \\ A: & (A \bullet B) \bullet C \leftrightarrow A \bullet (B \bullet C) & K: & \Diamond(A \bullet B) \to \Diamond A \bullet \Diamond B \\ C_\Diamond: & A \bullet \Diamond B \to \Diamond B \bullet A & MA: & (A \bullet_j B) \bullet_i C \to A \bullet_j (B \bullet_i C) \\ A_\Diamond: & (A \bullet B) \bullet \Diamond C \leftrightarrow A \bullet (B \bullet \Diamond C) & MC: & A \bullet_i (B \bullet_j C) \to B \bullet_j (A \bullet_i C) \end{array}$$

The postulates C and A on the left impose commutativity or associativity constraints on the interpretation of the composition relation R^3. Adding A to the base residuation logic produces the familiar associative Lambek calculus **L** of [Lambek 58]; adding both the A and C postulates gives the Lambek-Van Benthem calculus **LP**. The postulates C and A change the resource management regime in a *global* fashion. In the presence of the unary modalities, one can consider refined options such as $C_\Diamond$ or $A_\Diamond$, where reordering or restructuring are not globally available, but have to be explicitly *licensed* by a structural control operator $\Diamond$. On the right, we have interaction postulates regulating the *communication* between the unary and binary multiplicatives (the weak and strong distributivity principles $K1$ and K), or between distinct binary multiplicatives (such as the weak distributivity principles of Mixed Associativity and Mixed Commutativity). These latter cases require a straightforward *multimodal* generalization of the architecture, with frames $F = \langle W, \{R^2_i\}_{i\in I}, \{R^3_j\}_{j\in J}\rangle$, where the indices keep *composition modes* apart. In §2.3, the reader will find an illustration of grammatical analysis in terms of modal control and interaction postulates like the above.

2 Labelling proofs: form and meaning

We now present two systems of *labelled* deduction for the display of derivations in the extended categorial logics. As we have seen above, fine-tuning of categorial inference is obtained by considering mixed logics where interacting regimes of structural resource management are put together. Labelled presentations of the proof systems are particularly useful here: in line with the slogan of 'bringing the semantics into the syntax' the labelling systems allow explicit reference to the grammatical resources and the logical and structural aspects of their composition. On the meaning side, we have labelling in the sense of the 'formulas-as-types' program, producing 'semantic recipes' for categorial derivations. On the structural side, labelling can capture the configuration of linguistic resources in the form dimension, and the allowable structural manipulations of these configurations in the process of grammatical reasoning.

2.1 Categorical Combinators

The first system of labelled categorial deduction we consider is the categorical presentation of [Lambek 88]. In the categorical presentation, deductions take the form of 'arrows' $f : A \to B$, where the proof label f codes a process of deducing B from A, i.e. a proof of the semantic inclusion $v(A) \subseteq v(B)$. For every type formula A, we have an identity arrow 1_A, capturing the reflexivity of derivability, and we have a rule of inference which from given proofs f and g produces a new proof $g \circ f$ for their sequential composition, thus capturing the transitivity of derivability. The pure residuation logic is then obtained by imposing the additional rules of inference of Def 2.1, which establish the residuation laws for $\Diamond, \Box$ and $/, \bullet, \backslash$. One can now study equality of proofs in terms of appropriate categorical equations for the labelling system, cf. [Lambek 93], and [Troelstra 92] for discussion in the context of combinatorial linear logic.

Definition 2.1 *The pure logic of residuation: combinator proof terms ([Lambek 88]).*

$$1_A : A \to A \qquad \frac{f : A \to B \quad g : B \to C}{g \circ f : A \to C}$$

$$\frac{f : \Diamond A \to B}{\mu_{A,B}(f) : A \to \Box B} \qquad \frac{g : A \to \Box B}{\mu^{-1}_{A,B}(g) : \Diamond A \to B}$$

$$\frac{f : A \bullet B \to C}{\beta_{A,B,C}(f) : A \to C/B} \qquad \frac{f : A \bullet B \to C}{\gamma_{A,B,C}(f) : B \to A\backslash C}$$

$$\frac{g : A \to C/B}{\beta^{-1}_{A,B,C}(g) : A \bullet B \to C} \qquad \frac{g : B \to A\backslash C}{\gamma^{-1}_{A,B,C}(g) : A \bullet B \to C}$$

One easily derives the arrows for the laws of left and right functional application, and their unary counterpart. See (6), where we write $\mathbf{app}^{\backslash}$ for the proof $\gamma^{-1}(1_{B\backslash A})$, $\mathbf{app}^{/}$ for $\beta^{-1}(1_{A/B})$, and **co-unit** for $\mu^{-1}(1_{\Box A})$.

(6) $$\mathbf{co\text{-}unit} : \Diamond\Box A \to A \quad \mathbf{app}^{/} : A/B \bullet B \to A \quad \mathbf{app}^{\backslash} : B \bullet B\backslash A \to A$$

As examples of derived rules of inference, we have the Isotonicity laws for $\Diamond$ and $\bullet$. The $f \cdot g$ law is known as 'parallel composition' in the categorical setting, as contrasted with the 'sequential composition' of arrows $g \circ f$.

(7) $$\frac{f : A \to B}{(f)^{\diamond} : \Diamond A \to \Diamond B} \qquad \frac{f : A \to B \quad g : C \to D}{f \cdot g : A \bullet C \to B \bullet D}$$

In (8), we give the derivation of $(f)^{\diamond}$ as $\mu^{-1}(\mu(1_{\Diamond B}) \circ f)$. For sequential composition, see [Lambek 58].

(8) $$\cfrac{\cfrac{f : A \to B \qquad \cfrac{1_{\Diamond B} : \Diamond B \to \Diamond B}{\mu(1_{\Diamond B}) : B \to \Box\Diamond B}}{\mu(1_{\Diamond B}) \circ f : A \to \Box\Diamond B}}{\mu^{-1}(\mu(1_{\Diamond B}) \circ f) : \Diamond A \to \Diamond B}$$

Variation in grammatical resource management is obtained by adding to the pure residuation logic the required structural postulates, cf. (5) above. Each of these postulates, as an extra axiom schema, is labelled with its own primitive structural combinator.

The categorical labelling fully encodes a proof, both in its logical and in its structural aspects. As an example, we derive an implicational form of the Mixed Commutativity postulate MC from (5), dropping formula subscripts and composition mode indices for legibility. (f here stands for $\gamma^{-1}(\gamma(\beta^{-1}(1_{A/B})) \circ \gamma^{-1}(1_{C\backslash B}))$, as the reader will no doubt want to check.) Notice that the proof term is composed of a logical part (the residuation combinators γ, β and their inverses) and a structural component (the combinator **mc**).

(9) $$\cfrac{\cfrac{\cfrac{\mathbf{mc} : C \bullet_i (A/_j B \bullet_j C\backslash_i B) \to A/_j B \bullet_j (C \bullet_i C\backslash_i B) \qquad f : A/_j B \bullet_j (C \bullet_i C\backslash_i B) \to A}{f \circ \mathbf{mc} : C \bullet_i (A/_j B \bullet_j C\backslash_i B) \to A}}{\gamma(f \circ \mathbf{mc}) : A/_j B \bullet_j C\backslash_i B \to C\backslash_i A}}{\beta(\gamma(f \circ \mathbf{mc})) : A/_j B \to (C\backslash_i A)/_j(C\backslash_i B)}$$

2.2 Natural Deduction and Curry-Howard Labelling

In order to relate the categorical proof terms to the Curry-Howard-de Bruyn formulas-as-types interpretation, we now move to a Natural Deduction presentation, which we first consider in its unlabelled form. The arrows $f : A \to B$ are replaced by statements $\Gamma \vdash B$ representing a deduction of a formula B from a *structured database* of assumptions Γ. The structural 'packaging' of the resources is what distinguishes the categorial systems from linear logic — in the latter, the database can be seen as a *multiset* of assumptions: the occurrence aspect of the formulas matters, but not their further structuring.

To build a structured database of antecedent formulae, we need a language of structural connectives matching the language of logical connectives. This strategy goes back essentially to Belnap's Display Logic — see [Goré 98] for an up-to-date discussion of the substructural connections. We write (unary) $\langle\cdot\rangle$ for the structural counterpart of $\Diamond$, and (binary) $(\cdot \circ \cdot)$ for the structural connective corresponding to $\bullet$.

(10) $$\mathcal{S} ::= \mathcal{F} \mid \langle \mathcal{S} \rangle \mid \mathcal{S} \circ \mathcal{S}$$

Definition 2.2 *The pure residuation logic: Natural Deduction presentation. Introduction and Elimination rules for the constants. (Notation: $\Gamma[\Delta]$ for a structure Γ with a distinguished occurrence of a substructure Δ.)*

$$\frac{}{A \vdash A}\ (Ax)$$

$$\frac{\Gamma \vdash \Box A}{\langle \Gamma \rangle \vdash A}\ (\Box E) \qquad \frac{\langle \Gamma \rangle \vdash A}{\Gamma \vdash \Box A}\ (\Box I)$$

$$\frac{\Gamma \vdash A}{\langle \Gamma \rangle \vdash \Diamond A}\ (\Diamond I) \qquad \frac{\Delta \vdash \Diamond A \quad \Gamma[\langle A \rangle] \vdash B}{\Gamma[\Delta] \vdash B}\ (\Diamond E)$$

$$\frac{\Delta \vdash A \quad \Gamma \vdash A\backslash B}{\Delta \circ \Gamma \vdash B}\ (\backslash E) \qquad \frac{A \circ \Gamma \vdash B}{\Gamma \vdash A\backslash B}\ (\backslash I)$$

$$\frac{\Gamma \vdash B/A \quad \Delta \vdash A}{\Gamma \circ \Delta \vdash B}\ (/E) \qquad \frac{\Gamma \circ A \vdash B}{\Gamma \vdash B/A}\ (/I)$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma \circ \Delta \vdash A \bullet B}\ (\bullet I) \qquad \frac{\Delta \vdash A \bullet B \quad \Gamma[A \circ B] \vdash C}{\Gamma[\Delta] \vdash C}\ (\bullet E)$$

It is not difficult to derive the Natural Deduction rules from the categorical formulation. Let us write $\Gamma^\natural$ for the formula that results from replacing the structural connectives $\circ$ and $\langle\cdot\rangle$ in Γ by their logical counterparts $\bullet$ and $\Diamond$. The Axiom case coincides in the two presentations. $(\Box I)$ and $(\Box E)$ become the residuation inferences μ and μ^{-1}, respectively. $(/I)$ and $(\backslash I)$ become the β and γ half of residuation. $(\Diamond I)$ and $(\bullet I)$ are the Monotonicity rules of inference (7) — derived rules of inference, as we saw above. For $(\backslash E)$, we have the derivation in (11) which composes Monotonicity with Application. The $(/E)$ case is similar.

(11) $$\frac{\dfrac{f : \Delta^\natural \to A \quad g : \Gamma^\natural \to A\backslash B}{f \cdot g : (\Delta \circ \Gamma)^\natural \to A \bullet A\backslash B} \quad \mathbf{app}^\backslash : A \bullet A\backslash B \to B}{\mathbf{app}^\backslash \circ (f \cdot g) : (\Delta \circ \Gamma)^\natural \to B}$$

For $(\bullet E)$, we have (12). We write $\pi(f)$ for the sequence of μ, β, γ inferences that isolate the target formula $(A \circ B)^\natural$ on the left hand side of the arrow, moving the context to the right hand side (notation $C \mid \Gamma^\natural$). At that point, we compose with the major premise g, and put the context back in place on the left hand side via π^{-1}. The $(\Diamond E)$ case is similar.

$$
(12) \qquad \cfrac{g : \Delta^\natural \to A \bullet B \quad \cfrac{\cfrac{f : (\Gamma[A \circ B])^\natural \to C}{\vdots}}{\pi(f) : (A \circ B)^\natural \to C \mid \Gamma^\natural}}{\cfrac{\cfrac{\pi(f) \circ g : \Delta^\natural \to C \mid \Gamma^\natural}{\vdots}}{\pi^{-1}(\pi(f) \circ g) : (\Gamma[\Delta])^\natural \to C}}
$$

Structural rules S, in the Natural Deduction presentation, take the form of inferences

$$
(13) \qquad \frac{\Gamma[\Delta'] \vdash A}{\Gamma[\Delta] \vdash A}\ S
$$

where the formula equivalents $\Delta^\natural$ and $\Delta'^\natural$ of the structures Δ and Δ' match the left and right hand sides of a structural postulate $\sigma : A \to B$. Their derivation from the categorical presentation, then, follows the lines of (12), with the structural combinator axiom σ taking the place of the open premise g. As an illustration, consider (14), the Natural Deduction rule corresponding to the distributivity postulate $\Diamond(A \bullet B) \to \Diamond A \bullet \Diamond B$.

$$
(14) \qquad \frac{\Gamma[\langle \Delta_1 \rangle \circ \langle \Delta_2 \rangle] \vdash A}{\Gamma[\langle \Delta_1 \circ \Delta_2 \rangle] \vdash A}\ K
$$

Let us turn now to the more familiar decoration of Natural Deduction derivations with λ term annotation for the Curry-Howard-de Bruyn 'formulas-as-types' interpretation. Instead of the formula A, we take the labelled formula $t : A$ as the 'basic declarative unit'. Rules of inference manipulate both the formula and its label, and we build a recipe t for the construction of the meaning of the goal formula B out of a structured configuration of labelled assumptions $x_i : A_i$. The term decoration rules for the pure logic of residuation are given in (2.3). Introduction and Elimination of the implications correspond to functional abstraction and application, respectively. (We can collapse $/$ and $\backslash$ in the meaning dimension, coding the order requirements of these operators in the antecedent term structure.) Introduction and Elimination rules for $\bullet$ are associated with the pairing and projection operations. In an entirely analogous way, we have 'cap' and 'cup' operations for the Introduction and Elimination rules for $\Diamond$ and $\Box$. Substructural versions of the 'formulas-as-types' program, and of the relevant term equations, are studied in depth in [Gabbay & de Queiroz, Wansing 92a].

Definition 2.3 *Natural deduction. Proof terms.*

$$
\frac{}{x : A \vdash x : A}\ (Ax)
$$

$$
\frac{\Delta \vdash u : A \quad \Gamma \vdash t : A\backslash B}{\Delta \circ \Gamma \vdash (t\ u) : B}\ (\backslash E) \qquad \frac{x : A \circ \Gamma \vdash t : B}{\Gamma \vdash \lambda x.t : A\backslash B}\ (\backslash I)
$$

$$
\frac{\Gamma \vdash t : B/A \quad \Delta \vdash u : A}{\Gamma \circ \Delta \vdash (t\ u) : B}\ (/E) \qquad \frac{\Gamma \circ x : A \vdash t : B}{\Gamma \vdash \lambda x.t : B/A}\ (/I)
$$

$$
\frac{\Gamma \vdash t : A \quad \Delta \vdash u : B}{\Gamma \circ \Delta \vdash \langle t, u \rangle : A \bullet B}\ (\bullet I) \qquad \frac{\Delta \vdash u : A \bullet B \quad \Gamma[x : A \circ y : B] \vdash t : C}{\Gamma[\Delta] \vdash t[(u)_0/x, (u)_1/y] : C}\ (\bullet E)
$$

$$\frac{\Gamma \vdash t : \Box A}{\langle\Gamma\rangle \vdash {}^{\vee}t : A}\ (\Box E) \qquad \frac{\langle\Gamma\rangle \vdash t : A}{\Gamma \vdash {}^{\wedge}t : \Box A}\ (\Box I)$$

$$\frac{\Gamma \vdash t : A}{\langle\Gamma\rangle \vdash {}^{\cap}t : \Diamond A}\ (\Diamond I) \qquad \frac{\Delta \vdash u : \Diamond A \quad \Gamma[\langle x : A\rangle] \vdash t : B}{\Gamma[\Delta] \vdash t[{}^{\cup}u/x] : B}\ (\Diamond E)$$

The Curry-Howard term decoration records the application of the *logical rules* of inference — the Elimination and Introduction rules for the connectives. The *structural rules* of resource management are not reflected in the Curry-Howard labelling: structural rules, schematically, manipulate a structural subterm of the antecedent, leaving the succedent formula annotation unaffected.

(15)
$$\frac{\Gamma[\Delta'] \vdash t : A}{\Gamma[\Delta] \vdash t : A}\ S$$

If we restrict the attention exclusively to the formula labels, we see a loss of information with respect to the categorical proof terms that faithfully encoded both the logical and structural aspects of a derivation. But of course, in the 'sequent style' Natural Deduction presentation, the antecedent has a term structure of its own, given by the structural operations $\circ$ and $\langle\cdot\rangle$, and structural rules manipulate this term structure.

2.3 Illustration: Crossed Dependencies

The components of the grammatical architecture proposed in the previous section are summarized below.

Logic. The core notions of 'grammatical composition' are characterized in terms of universal laws, independent of the structural properties of the composition relation. The operations of the base logic (introduction/elimination of the grammatical constants) provide the interface to a derivational theory of meaning via the Curry-Howard interpretation of proofs.

Structure. Packages of resource-management postulates function as 'plug-in' modules with respect to the base logic. They offer a logical perspective on structural variation, within languages and cross-linguistically.

Control. A vocabulary of control operators provides explicit means to fine-tune grammatical resource management, by imposing structural constraints or by licensing structural relaxation.

In order to illustrate the increased expressive power of the multimodal style of reasoning, we take a brief look at crossed dependencies in Dutch. As is well known, a proper analysis of the syntactic and semantic aspects of crossed dependencies is beyond the reach of strictly context-free grammar formalisms — beyond the reach of the standard Lambek calculus **L**, in the categorial case. The relevant phenomena are displayed in (16) below. As the examples (a) to (c) show, Dutch is a verb-final language: in their canonical position (the embedded clause), verbs look for their arguments to the left. Crossed dependencies arise in the presence of modal auxiliaries such as 'kunnen' ('can'), 'willen' ('want'). These auxiliaries select for an infinitival complement, but rather than consuming this complement in its entirety, they are prefixed to the clause-final infinitival *head* of their complement, 'skipping over' the arguments of the infinitival, if any. By connecting the infinitive 'plagen' to its direct object 'Alice' and the auxiliary 'wil' to its subject 'Tweedledum' in (e), one can see where the dependencies cross.

(16) *a* als Alice slaapt (slaapt: $np\backslash s$)
if Alice sleeps
b als Tweedledum Alice plaagt (plaagt: $np\backslash(np\backslash s)$)
if Tweedledum Alice teases ('if T teases A')
c of Alice Tweedledum gek vindt (vindt: $ap\backslash(np\backslash(np\backslash s))$)
whether Alice Tweedledum crazy considers ('whether A considers T crazy')
d als Alice wil slapen (wil: vp/inf, slapen: inf)
if Alice wants sleep ('if A wants to sleep')
e als Tweedledum Alice wil plagen (plagen: $np\backslash inf$, wil: ??)
if Tweedledum Alice wants tease ('if T wants to tease A')
f *if Tweedledum wil Alice plagen

Consider the provisional type assignments in (16), where we write vp for $np\backslash s$. On the basis of (d), one could assign 'wil' the type vp/inf, so that it prefixes itself to its infinitival complement. But to obtain the combination 'wil plagen' with the transitive infinitive in (e), one would need a directionally crossed or mixed form of composition (schematically, $A/B, C\backslash B \Rightarrow C\backslash A$) which is invalid in **L**, as it violates the order sensitivity of the types involved. The grammatical example (e), in other words, is underivable given the **L** type assignments above. The sequence 'wil Alice plagen' in (f), on the contrary, *is* derivable, but it is *ungrammatical* in the embedded clausal context we are considering.

We can overcome these problems of overgeneration and undergeneration by moving to a multimodal setting, as shown in [Moortgat & Oehrle 94], and by exploiting the structural control devices $\Diamond, \Box$. The structural package in (17) makes a distinction between two binary modes. The regular combination of heads with their phrasal complements is realized by $\bullet_1$: subcategorizational requirements of the verbs in (16a-c), and to the transitive infinitive in (e), will be expressed in terms of the $\backslash_1$ implication. The head adjunction operation that gives rise to crossed dependencies is realized by $\bullet_0$: the type assignment for 'wil' in (16d-f) selects the infinitival complement in terms of $/_0$.

(17)

$$
\begin{array}{ll}
P1 & \Diamond_1(A \bullet_1 B) \rightarrow A \bullet_1 \Diamond_1 B \\
P2 & \Diamond_1 A \rightarrow \Diamond_0 A \\
P3 & \Diamond_0(A \bullet_0 B) \rightarrow \Diamond_0 A \bullet_0 \Diamond_0 B \\
P4 & A \bullet_1 (\Diamond_0 B \bullet_0 C) \rightarrow \Diamond_0 B \bullet_0 (A \bullet_1 C)
\end{array}
$$

Consider next the control component, expressed in terms of modalities $\Diamond_0$ and $\Diamond_1$, together with their residuals. In order to lexically anchor the structural control, we use a 'key-and-lock' strategy. Verbal elements (tensed and infinitival) are lexically typed with $\Box_0$ as their main connective — the transitive infinitive 'plagen', for example, would get the type assignment $\Box_0(np\backslash_1 inf)$. As we have seen above, subcategorizational requirements are expressed in terms of implications with respect to the composition modes $\bullet_1$ and $\bullet_0$. But before these implications can be used in a derivation, the $\Box_0$ lock has to be removed, by means of the basic reduction law $\Diamond_0\Box_0 A \rightarrow A$.

The role of the control devices $\Diamond_1$ and $\Diamond_0$ in (17) is to check whether the clause final verbal structure is indeed wellformed. We assign a complementizer like 'als' ('if') the type $sbar/_1\Box_1 s$, i.e. the goal type for an embedded clause is $\Box_1 s$. To prove that a structure Γ is of type $\Box_1 s$ amounts to proving $\langle\Gamma\rangle^1 \vdash s$ (via Box Introduction). Here our postulate package can start its work. $P1$ recursively inspects phrasal structure, and looks for the verbal head at the end. At the point where there is no more phrasal $\bullet_1$ structure to traverse, $P2$ switches the control to inspection of the verbal head constituent itself. This can either be a simple verb (which can then be directly unlocked by means of $\Diamond_0\Box_0 A \rightarrow A$), or it can be a compound verbal cluster, constructed by means of the head adjunction operation $\bullet_0$. In the latter case, $P3$ recursively checks whether the components of a verbal cluster are indeed verbs. Postulate $P4$ — a modally controlled version of mixed commutativity — undoes the crossed dependencies and makes sure that the phrasal complements that were skipped over can be consumed by means of $\backslash_1$ Elimination.

A Natural Deduction derivation for the verb phrase 'Alice wil plagen' of (16e) is given below. In (18), we focus on *structural* composition, dropping type formulae in the antecedent, and semantic labels in the succedent. Notice that the upper part of the derivation proceeds in 'bottom up' fashion from lexical type assignments purely in terms of *logical* inferences — Elimination rules for the implication and box connectives — producing the structure

$$\langle \text{wil} \rangle_0 \circ_0 (\text{Alice} \circ_1 \langle \text{plagen} \rangle_0)$$

The structural inferences $P1$–$P4$ mediate between this structure and the structure

$$\langle \text{Alice} \circ_1 (\text{wil} \circ_0 \text{plagen}) \rangle_1$$

that is obtained in 'top down' fashion from the final conclusion by means of the logical $\Box_1$ Introduction inference.

(18)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{wil} \vdash \Box_0(vp/_0 inf)}{\langle \text{wil} \rangle_0 \vdash vp/_0 inf}\,\Box_0 E \qquad \dfrac{\text{Alice} \vdash np \qquad \dfrac{\text{plagen} \vdash \Box_0(np\backslash_1 inf)}{\langle \text{plagen} \rangle_0 \vdash np\backslash_1 inf}\,\Box_0 E}{\text{Alice} \circ_1 \langle \text{plagen} \rangle_0 \vdash inf}\,\backslash_1 E}{\langle \text{wil} \rangle_0 \circ_0 (\text{Alice} \circ_1 \langle \text{plagen} \rangle_0) \vdash vp}\,/_0 E}{\text{Alice} \circ_1 (\langle \text{wil} \rangle_0 \circ_0 \langle \text{plagen} \rangle_0) \vdash vp}\,P4}{\text{Alice} \circ_1 \langle \text{wil} \circ_0 \text{plagen} \rangle_0 \vdash vp}\,P3}{\text{Alice} \circ_1 \langle \text{wil} \circ_0 \text{plagen} \rangle_1 \vdash vp}\,P2}{\langle \text{Alice} \circ_1 (\text{wil} \circ_0 \text{plagen}) \rangle_1 \vdash vp}\,P1}{\text{Alice} \circ_1 (\text{wil} \circ_0 \text{plagen}) \vdash \Box_1 vp}\,\Box_1 I$$

In (19), we concentrate on the composition of *meaning*: we drop the structured antecedent database, and present just the succedent formulae with their Curry-Howard term labels. Observe that the modal auxiliary 'wil' has the required scope over the combination of the infinitive 'plagen' and its direct object 'Alice'.

(19)
$$\dfrac{\dfrac{\dfrac{\text{wil} : \Box_0(vp/_0 inf)}{{}^\vee\text{wil} : vp/_0 inf}\,\Box_0 E \qquad \dfrac{\text{alice} : np \qquad \dfrac{\text{plagen} : \Box_0(np\backslash_1 inf)}{{}^\vee\text{plagen} : np\backslash_1 inf}\,\Box_0 E}{({}^\vee\text{plagen alice}) : inf}\,\backslash_1 E}{({}^\vee\text{wil } ({}^\vee\text{plagen alice})) : vp}\,/_0 E}{{}^\wedge({}^\vee\text{wil } ({}^\vee\text{plagen alice})) : \Box_1 vp}\,\Box_1 I$$

Notice also that the distinction between $\Diamond_1$ and $\Diamond_0$ effectively imposes the constraint that the (standard Dutch) verb-raising cluster cannot contain phrasal compositions. Because we have $\Diamond_1 A \rightarrow \Diamond_0 A$, but not the other way around, the following attempt at deriving the ungrammatical (16f) '(als Tweedledum) wil Alice plagen' fails: $\Diamond_0$ does not distribute through a phrasal $\bullet_1$ configuration. This then solves the 'overgeneration' part of the problems with (16).

(20)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{wil} \vdash \Box_0(vp/_0 inf)}{\langle \text{wil} \rangle_0 \vdash vp/_0 inf}\,\Box_0 E \qquad \dfrac{\text{FAILS}}{\langle \text{Alice} \circ_1 \text{plagen} \rangle_0 \vdash inf}}{\langle \text{wil} \rangle_0 \circ_0 \langle \text{Alice} \circ_1 \text{plagen} \rangle_0 \vdash vp}\,/_0 E}{\langle \text{wil} \circ_0 (\text{Alice} \circ_1 \text{plagen}) \rangle_0 \vdash vp}\,P3}{\langle \text{wil} \circ_0 (\text{Alice} \circ_1 \text{plagen}) \rangle_1 \vdash vp}\,P2}{\text{wil} \circ_0 (\text{Alice} \circ_1 \text{plagen}) \vdash \Box_1 vp}\,\Box_1 I$$

3 Proof Search and Labelling

The categorical and Natural Deduction formats are handy for *presenting* proofs once they have been found. But they do not provide an appropriate basis for automated proof *search*. In this section we consider two proof formats which *do* have an algorithmic interpretation: Gentzen sequent calculus and proof nets. Labelled presentations here make it possible to give a uniform presentation of the Curry-Howard 'derivational meaning' at the level of **LP**, i.e. at a level where multiplicity of assumptions matters but where one can abstract from the structural aspects of composition. Instead, these aspects are controlled via an appropriate structural labelling regime. A labelled Gentzen presentation can be seen as a first step towards a modular treatment of 'logic' and 'structure'. But the Gentzen format still suffers from spurious non-determinism, which can be effectively removed as soon as we move to a (labelled) proof net approach.

3.1 Labelled Gentzen Calculus

The relations between Natural Deduction and sequent calculus for resource logics are well-understood, syntactically and on the level of the Curry-Howard interpretation, see for example [Gabbay & de Queiroz, Girard e.a. 89, Wansing 92a]. The move from Natural Deduction to Gentzen sequent presentation requires that we reformulate all logical rules of inference in such a way that a connective is introduced in the *conclusion*, either in the antecedent (rules of use, left rules) or in the succedent (rules of proof, right rules). In the presence of a Cut Elimination result to the effect that the Cut rule in (21) does not increase the set of derivable theorems (or semantic recipes, modulo logical equivalence), one immediately obtains a procedure for decidable proof search based on systematic removal of connectives from conclusion to premises.

(21)
$$\frac{\Delta \Rightarrow u : A \quad \Gamma[x : A] \Rightarrow t : B}{\Gamma[\Delta] \Rightarrow t[u/x] : B} \text{ Cut}$$

The Introduction rules in the Natural Deduction presentation have the required form — they can be taken over unchanged as rules of proof in the sequent calculus. The rules of use for $\Diamond$ and $\bullet$ are obtained from the Natural Deduction Elimination rules for these connectives by instantiating the major premise as the identity axiom.

(22)
$$\frac{\overline{A \bullet B \vdash A \bullet B}\ (Ax) \quad \Gamma[A \circ B] \vdash C}{\Gamma[A \bullet B] \vdash C}\ (\bullet E) \quad \rightsquigarrow \quad \frac{\Gamma[A \circ B] \Rightarrow C}{\Gamma[A \bullet B] \Rightarrow C}\ (\bullet L)$$

In the rules of use for $\Box$ and $/, \backslash$, we recognize compiled Cuts, on $\Diamond\Box A \Rightarrow A$, and on Application.

(23)
$$\frac{\Gamma[A] \Rightarrow B}{\Gamma[\langle \Box A \rangle] \Rightarrow B}\ \Box L \qquad \frac{\Delta \Rightarrow A \quad \Gamma[B] \Rightarrow C}{\Gamma[\Delta \circ A\backslash B] \Rightarrow C}\ \backslash L \qquad \frac{\Delta \Rightarrow A \quad \Gamma[B] \Rightarrow C}{\Gamma[A/B \circ \Delta] \Rightarrow C}\ /L$$

Considering proof search from a 'parsing as deduction' point of view, one notices an important difference between the case of the *associative* system **L**, and the generalized multimodal categorial logics that form the subject of this paper. Because of the global availability of Associativity in **L**, one can say that strong and weak generative capacity for this system coincide. Parsing a string $w_1 \cdots w_n$ as an expression of type B comes down to proving the sequent $A_1, \ldots, A_n \Rightarrow B$, where the A_i are types assigned to the lexical resources w_i, and where the antecedent is a 'flat' sequence of assumptions without hierarchical structure. In the general multimodal case, we need to know the structural configuration of the antecedent assumptions in terms of the structural connectives $\circ$ and $\langle\cdot\rangle$ with their mode indications. As remarked in the introduction to this paper, one cannot take the antecedent structuring as 'given' without trivializing the parsing problem. Rather, we have to find a proof format where the structure of the antecedent database is gradually 'discovered' in the proof process.

In the categorial literature, a variety of labelled sequent formulations have been proposed for this purpose — see, among others, [Moortgat 92, Oehrle 95, Morrill 94, Hepple 95]. One considers labelled sequents $x_1 : A_1, \ldots, x_n : A_n \Rightarrow t : B$ where the antecedent is simply a multiset of labelled formulae, representing the lexical assumptions (seen as occurrences, i.e. with the x_i distinct), and where t is a structure label built from these x_i by means of the structural operations of the multimodal system one is dealing with. Below we present the structural labelling of [Kurtonina 95], which is complete for the general multimodal architecture ($\Diamond$ and $\bullet$ plus their residuals, and structural rule packages relativized to composition modes). The syntax of the labelling system is given in (24). Definition 3.1 presents the labelled sequent rules. (We have slightly adapted the notation of [Kurtonina 95]. Of course, one can also label the formulas with Curry-Howard terms for semantic interpretation — but we concentrate on the structural aspects here.)

(24)

ϕ, ψ	$\longrightarrow$	x	atomic labels
		$\mathbf{un}(x, \phi)$	unary tree
		$\mathbf{bin}(x, \phi, \psi)$	binary tree

Definition 3.1 *Labelled Gentzen calculus: [Kurtonina 95]. A structure label is called* proper *if all its atomic subterms are distinct. Notation: x, y, z for atomic structure terms, t, u, v for proper structure terms. Γ, Δ finite multisets of formulas decorated with atomic structure labels. $t[u \rightsquigarrow v]$ denotes the substitution of u for v in t.*

$$\frac{}{x : A \Rightarrow x : A}\ Ax$$

$$\frac{x : A, \Gamma \Rightarrow t : B}{y : \Box A, \Gamma \Rightarrow t[\mathbf{un}(x, y) \rightsquigarrow x] : B}\ \Box L \qquad \frac{\Gamma \Rightarrow \mathbf{un}(x, t) : A}{\Gamma \Rightarrow t : \Box A}\ \Box R$$

$$\frac{y : A, \Gamma \Rightarrow t : B}{x : \Diamond A, \Gamma \Rightarrow t[x \rightsquigarrow \mathbf{un}(x, y)] : B}\ \Diamond L \qquad \frac{\Gamma \Rightarrow t : A}{\Gamma \Rightarrow \mathbf{un}(x, t) : \Diamond A}\ \Diamond R$$

$$\frac{\Gamma \Rightarrow u : B \quad x : A, \Delta \Rightarrow t : C}{y : A/B, \Gamma, \Delta \Rightarrow t[\mathbf{bin}(x, y, u) \rightsquigarrow x] : C}\ /L \qquad \frac{z : B, \Gamma \Rightarrow \mathbf{bin}(x, t, z) : A}{\Gamma \Rightarrow t : A/B}\ /R$$

$$\frac{\Gamma \Rightarrow u : B \quad x : A, \Delta \Rightarrow t : C}{z : B\backslash A, \Gamma, \Delta \Rightarrow t[\mathbf{bin}(x, u, z) \rightsquigarrow x] : C}\ \backslash L \qquad \frac{y : B, \Gamma \Rightarrow \mathbf{bin}(x, y, t) : A}{\Gamma \Rightarrow t : B\backslash A}\ \backslash R$$

$$\frac{y : A, z : B, \Gamma \Rightarrow t : C}{x : A \bullet B, \Gamma \Rightarrow t[x \rightsquigarrow \mathbf{bin}(x, y, z)] : C}\ \bullet L \qquad \frac{\Gamma \Rightarrow t : A \quad \Delta \Rightarrow u : B}{\Gamma, \Delta \Rightarrow \mathbf{bin}(x, t, u) : A \bullet B}\ \bullet R$$

The above rules of labelled deduction represent the pure residuation logic. Recovery of the configuration of the antecedent in terms of unary $\langle\cdot\rangle$ and binary $\circ$ structural operations, and the underlying pairs and triples for the composition relations R^2 and R^3 in the semantics, is straightforward. Structural rules, in this presentation, translate into labelling rules

(25)
$$\frac{\Gamma \Rightarrow t[u'] : A}{\Gamma \Rightarrow t[u] : A}$$

replacing a subterm u by a structural alternative u', where u and u' are the labelling versions of the left- and righthand sides of a structural postulate $A \rightarrow B$. Below the distributivity principle K as an illustration.

$$(K) \quad \Diamond(A \bullet B) \rightarrow \Diamond A \bullet \Diamond B$$

$$u : \mathbf{un}(x, \mathbf{bin}(y, t', t'')) \qquad u' : \mathbf{bin}(x, \mathbf{un}(y', t'), \mathbf{un}(y'', t''))$$

Let us evaluate the labelled Gentzen presentation from the perspective of algorithmic proof search. On the formula level, the format allows 'backward chaining' search (elimination of connectives) on the basis of a goal formula and an antecedent multiset of lexical resources. But the flow of information on the level of the structure labels is at odds with the backward chaining regime, and requires destructive term manipulations in the rules that have $t[u \rightsquigarrow v]$ in the conclusion. This problem makes the labelled Gentzen format suboptimal for the purposes of 'parsing-as-deduction' for the general multimodal categorial framework. (Of course, there is also the problem of spurious non-determinism in rule application order characteristic for naive sequent proof search. But this problem can be tackled by adding 'procedural control', as shown in the categorial literature by [König 91, Hepple 90, Hendriks 93], and by [Hodas & Miller 94, Andreoli 92, Cervesato e.a.], among others, in the context of 'linear' refinements of Logic Programming.)

3.2 Labelled Proof Nets

In this section, we consider labelled versions of the 'proof nets' of Linear Logic as an optimalization of sequent proof search. Proof nets can be decorated with Curry-Howard λ-term labelling in a straightforward way, as shown in [Roorda 91, de Groote & Retoré 96]. In order to capture the syntactic fine-structure of systems more discriminating than **LP**, and multimodal architectures with structural inference packages, we complement the semantic labeling with *structural* labeling.

The construction of a proof net corresponding to a sequent $\Gamma \Rightarrow B$ can be presented as a three stage process. The first stage is deterministic and consists in unfolding the formula decomposition tree for the A_i antecedent terminal formulae of Γ and for the goal formula B. The unfolding has to keep track of the antecedent/succedent occurrences of subformulae: we work with signed formulae, and distinguish $(\cdot)^\bullet$ (antecedent) from $(\cdot)^\circ$ (succedent) unfolding, corresponding to the sequent rules of use and proof for the connectives. We also distinguish two types of decomposition steps: $\exists$-type decomposition for the $\Diamond$L,$\bullet$L, /R,\R rules, and $\forall$-type decomposition corresponding to the $\Box$L,/L,\L,$\bullet$R rules. (For the binary connectives, these are one-premise and two-premise inferences, respectively.)

Definition 3.2 *Formula decomposition.*

$$\frac{(A)^\bullet \quad (B)^\circ}{(A/B)^\bullet}\,\forall \quad \frac{(B)^\bullet \quad (A)^\circ}{(A/B)^\circ}\,\exists \qquad \frac{(B)^\circ \quad (A)^\bullet}{(B\backslash A)^\bullet}\,\forall \quad \frac{(A)^\circ \quad (B)^\bullet}{(B\backslash A)^\circ}\,\exists$$

$$\frac{(A)^\bullet \quad (B)^\bullet}{(A \bullet B)^\bullet}\,\exists \quad \frac{(B)^\circ \quad (A)^\circ}{(A \bullet B)^\circ}\,\forall$$

$$\frac{(A)^\bullet}{(\Diamond A)^\bullet}\,\exists \qquad \frac{(A)^\circ}{(\Diamond A)^\circ}\,\forall \qquad \frac{(A)^\bullet}{(\Box A)^\bullet}\,\forall \qquad \frac{(A)^\circ}{(\Box A)^\circ}\,\exists$$

We call the result of the unfolding a *proof frame*. The second stage, corresponding to the Axiom case in the Gentzen presentation, consists in linking the signed atomic formulae (literals) with opposite polarity marking. We call an arbitrary linking connecting the leaves of the proof frame a *proof structure*. Not every proof structure corresponds to a sequent derivation. The final stage is to perform a wellformedness check on the proof structure graph in order to identify it as a *proof net*, i.e. a structure which effectively corresponds to a sequent derivation. For the checking of the well-formedness conditions, there are various alternatives for Girard's original 'long trip' condition, which (in the case of the binary connectives) checks the graph for connectedness and acyclicity. We do not discuss these checking procedures here, but move on to labelled versions of the proof net format.

The proof net version of Curry-Howard labelling is presented in Def 3.3.

Definition 3.3 *Formula decomposition with Curry-Howard terms for* **LP** *meaning composition. We use x, y, z (t, u, v) for object-level variables (terms), M, N for meta-level (search) variables. The search variables are instantiated in establishing the axiom links. Newly introduced object-level variables and metavariables in the rules below are chosen fresh.*

$$\text{Axiom links}\quad \overline{t:(A)^{\bullet}\quad M:(A)^{\circ}}\quad \overline{M:(A)^{\circ}\quad t:(A)^{\bullet}}\quad \text{with } M:=t$$

$$\frac{(t\ M):(A)^{\bullet}\quad M:(B)^{\circ}}{t:(A/B)^{\bullet}}\ \forall \qquad \frac{x:(B)^{\bullet}\quad N:(A)^{\circ}}{\lambda x.N:(A/B)^{\circ}}\ \exists$$

$$\frac{M:(B)^{\circ}\quad (t\ M):(A)^{\bullet}}{t:(B\backslash A)^{\bullet}}\ \forall \qquad \frac{N:(A)^{\circ}\quad x:(B)^{\bullet}}{\lambda x.N:(B\backslash A)^{\circ}}\ \exists$$

$$\frac{(t)_0:(A)^{\bullet}\quad (t)_1:(B)^{\bullet}}{t:(A\bullet B)^{\bullet}}\ \exists \qquad \frac{N:(B)^{\circ}\quad M:(A)^{\circ}}{\langle M,N\rangle:(A\bullet B)^{\circ}}\ \forall$$

$$\frac{{}^{\cup}t:(A)^{\bullet}}{t:(\Diamond A)^{\bullet}}\ \exists \qquad \frac{M:(A)^{\circ}}{{}^{\cap}M:(\Diamond A)^{\circ}}\ \forall \qquad \frac{{}^{\vee}t:(A)^{\bullet}}{t:(\Box A)^{\bullet}}\ \forall \qquad \frac{M:(A)^{\circ}}{{}^{\wedge}M:(\Box A)^{\circ}}\ \exists$$

Proof nets, as we have considered them so far, give a geometric representation for the occurrence-sensitivity of **LP** derivability, but they ignore structural aspects of well-formedness. Roorda shows that one can impose a further geometric criterion of planarity on the axiom linkings to capture the order-sensitivity of the **L** refinement. It is not clear, however, how such a geometric approach would generalize to the general multimodal architectures we have been studying in this paper, where typically a base logic is combined with variable packages of structural postulates.

In order to obtain a general algorithmic proof theory for the multimodal systems, we now complement the Curry-Howard labelling with a system of *structure* labeling, that serves the same purpose as the antecedent structuring in the (labelled) Natural Deduction presentation of Def 2.3. The labeling regime of Def 3.4 is related to proposals in [Hepple 95, Morrill 95, Oehrle 95], but makes adjustments to accommodate the multimodal architecture in its full generality.

Definition 3.4 *Structure labels: syntax. The labeling system uses atomic formula labels x and structure labels $\langle\sigma\rangle, (\sigma\circ\tau)$, for the $\forall$ formula decomposition nodes. For the $\exists$ nodes, we use* auxiliary *labels: expressions that must be rewritten to structure/formula labels under the residuation reductions of Def 3.5.*

σ,τ	$\longrightarrow$	x	*(atoms)*	$(\sigma\circ\tau)$	*(constructor $\bullet$)*
		$\langle\sigma\rangle$	*(constructor $\Diamond$)*	${}^{\triangleleft}(\sigma)$	*(left-destructor $\bullet$)*
		$\lfloor\sigma\rfloor$	*(destructor $\Diamond$)*	$(\sigma)^{\triangleright}$	*(right-destructor $\bullet$)*
		$\lceil\sigma\rceil$	*(goal $\Box$)*	$x\backslash\sigma$	*(goal $\backslash$)*
				σ/x	*(goal $/$)*

Definition 3.5 *Labelled formula decomposition: structure labels and residuation term reductions (boxed)* REDEX $\succ$ CONTRACTUM. *We use x, y, z (t, u, v) for object-level formula (structure) labels, Γ, Δ for meta-level search variables. Newly introduced formula labels and metavariables in the rules below are chosen fresh.*

$$\frac{(t \circ \Delta) : (A)^{\bullet} \quad \Delta : (B)^{\circ}}{t : (A/B)^{\bullet}}\ \forall \qquad \frac{x : (B)^{\bullet} \quad \Gamma : (A)^{\circ}}{\Gamma/x : (A/B)^{\circ}}\ \exists$$

$$\boxed{(t \circ x)/x \succ t}$$

$$\frac{\Delta : (B)^{\circ} \quad (\Delta \circ t) : (A)^{\bullet}}{t : (B\backslash A)^{\bullet}}\ \forall \qquad \frac{\Gamma : (A)^{\circ} \quad x : (B)^{\bullet}}{x\backslash\Gamma : (B\backslash A)^{\circ}}\ \exists$$

$$\boxed{x\backslash(x \circ t) \succ t}$$

$$\frac{{}^{\triangleleft}(t) : (A)^{\bullet} \quad (t)^{\triangleright} : (B)^{\bullet}}{t : (A \bullet B)^{\bullet}}\ \exists \qquad \frac{\Delta : (B)^{\circ} \quad \Gamma : (A)^{\circ}}{(\Gamma \circ \Delta) : (A \bullet B)^{\circ}}\ \forall$$

$$\boxed{({}^{\triangleleft}(t) \circ (t)^{\triangleright}) \succ t}$$

$$\frac{\lfloor t \rfloor : (A)^{\bullet}}{t : (\Diamond A)^{\bullet}}\ \exists \quad \frac{\Gamma : (A)^{\circ}}{\langle \Gamma \rangle : (\Diamond A)^{\circ}}\ \forall \qquad \frac{\langle t \rangle : (A)^{\bullet}}{t : (\Box A)^{\bullet}}\ \forall \quad \frac{\Gamma : (A)^{\circ}}{\lceil \Gamma \rceil : (\Box A)^{\circ}}\ \exists$$

$$\boxed{\langle \lfloor t \rfloor \rangle \succ t} \qquad \boxed{\lceil \langle t \rangle \rceil \succ t}$$

The basic residuation reductions in Def 3.5 are dictated by the identities for complex formulae $\Diamond A, \Box A, A \bullet B, A/B, B\backslash A$. Structural postulates $A \to B$ translate to reductions $\sigma(B) \succ \sigma(A)$, where $\sigma(\cdot)$ is the structure label translation of a formula. The reduction for the distributivity postulate K is given as an illustration in (26). Notice that both residuation reductions and structural postulate reductions are asymmetric, capturing the asymmetry of the derivability relation.

(26) $$\Diamond(A \bullet B) \to \Diamond A \bullet \Diamond B \quad \overset{\sigma(\cdot)}{\rightsquigarrow} \quad (\langle t \rangle \circ \langle u \rangle) \succ \langle (t \circ u) \rangle$$

The structural labelling is used in parsing in the following way. To determine whether a string $x_1 \cdots x_n$ can be assigned the goal type B on the basis of a multiset of lexical assumptions $\Gamma = x_1 : A_1, \ldots, x_n : A_n$, one considers the multiset of signed labelled literals resulting from the formula decomposition of the $x_i : (A_i)^{\bullet}$ in Γ and of the goal formula $\Delta : (B)^{\circ}$. One resolves literals with opposite signature, unifying the search variable decorating $(p)^{\circ}$ against the label decorating $(p)^{\bullet}$, making sure in the matching that the proof net conditions of acyclicity and connectedness are respected. Notice that the labelling regime is set up in such a way that the $(p)^{\circ}$ literals are always decorated with unstructured search variables: unification at the axiom linkings, in other words, is simple one-sided matching. Labels can be rewritten under the residuation and/or structural postulate rewritings. We require the label assigned to the goal type B to be *normal*, in the sense that all residuation redexes must be reduced. We say that the input string $x_1 \cdots x_n$ is parsed if it is the yield of the normalized structure label assigned to the goal type B.

Illustration

For an illustration of the proof net labelled deduction system, we now return to the multimodal head adjunction analysis of §2.3. The relevant structural postulates are repeated below with the corresponding term rewriting rules for the structural labelling system.

$$
\begin{array}{llll}
(27) & P1 & \Diamond_1(A \bullet_1 B) \rightarrow A \bullet_1 \Diamond_1 B & t \circ_1 \langle u \rangle_1 \succ \langle t \circ_1 u \rangle_1 \\
 & P2 & \Diamond_1 A \rightarrow \Diamond_0 A & \langle t \rangle_0 \succ \langle t \rangle_1 \\
 & P3 & \Diamond_0(A \bullet_0 B) \rightarrow \Diamond_0 A \bullet_0 \Diamond_0 B & \langle t \rangle_0 \circ_0 \langle u \rangle_0 \succ \langle t \circ_0 u \rangle_0 \\
 & P4 & A \bullet_1 (\Diamond_0 B \bullet_0 C) \rightarrow \Diamond_0 B \bullet_0 (A \bullet_1 C) & \langle t \rangle_0 \circ_0 (u \circ_1 v) \succ u \circ_1 (\langle t \rangle_0 \circ_0 v)
\end{array}
$$

We try to derive the type $\Box_1(np\backslash_1 s)$ for the string 'Alice wil plagen' on the basis of the lexical type assignments in (28). (We have expanded the atomic formula inf of our earlier assignments to $np\backslash_1 is$ in order to illustrate the unfolding of higher-order types — one can read is as 'infinitival clause'.)

(28)

$$
\begin{array}{ll}
\text{Alice} & := np \\
\text{plagen} & := \Box_0(np\backslash_1(np\backslash_1 is)) \\
\text{wil} & := \Box_0((np\backslash_1 s)/_0(np\backslash_1 is))
\end{array}
$$

The labelled literals resulting from the formula unfolding for the lexical resources and for the goal formula $\Box_1(np\backslash_1 s)$ is given in (29). We use upper case italic for the unknowns associated with succedent (goal) literals; upper case Roman for the fresh atomic formula labels for hypothetical assumptions.

(29)

$$
\begin{array}{lll}
 & (np)^\bullet & \text{Alice} \\
C & (is)^\circ & \\
 & (np)^\bullet & \mathrm{P} \\
H & (np)^\circ & \\
 & (s)^\bullet & (H \circ_1 (\langle \text{wil} \rangle_0 \circ_0 \mathrm{P}\backslash_1 C)) \\
K & (np)^\circ & \\
N & (np)^\circ & \\
 & (is)^\bullet & (N \circ_1 (K \circ_1 \langle \text{plagen} \rangle_0)) \\
P & (s)^\circ & \\
 & (np)^\bullet & \mathrm{Q}
\end{array}
$$

The step-by-step decomposition tree for the higher-order type assignment for the verb 'wil' is presented in (30).

(30)

$$
\dfrac{\dfrac{\dfrac{H : (np)^\circ \quad H \circ_1 (\langle \text{wil} \rangle_0 \circ_0 \mathrm{P}\backslash_1 C) : (s)^\bullet}{\langle \text{wil} \rangle_0 \circ_0 \mathrm{P}\backslash_1 C : (np\backslash_1 s)^\bullet}\ \forall \quad \dfrac{\mathrm{P} : (np)^\bullet \quad C : (is)^\circ}{\mathrm{P}\backslash_1 C : (np\backslash_1 is)^\circ}\ \exists}{\langle \text{wil} \rangle_0 : ((np\backslash_1 s)/_0(np\backslash_1 is))^\bullet}\ \forall}{\text{wil} : (\Box_0((np\backslash_1 s)/_0(np\backslash_1 is)))^\bullet}\ \forall
$$

The resolution steps (axiom linkings) and the term rewritings leading up to the identification of the goal label are presented below.

$$
\begin{array}{rcl}
K^{np} & = & \text{Alice} \\
N^{np} & = & \text{P} \\
C^{is} & = & (\text{P} \circ_1 (\text{Alice} \circ_1 \langle \text{plagen} \rangle_0)) \\
H^{np} & = & \text{Q} \\
P^{s} & = & (\text{Q} \circ_1 (\langle \text{wil} \rangle_0 \circ_0 \text{P} \backslash_1 (\text{P} \circ_1 (\text{Alice} \circ_1 \langle \text{plagen} \rangle_0)))) \\
\text{Goal} & = & \lceil \text{Q} \backslash_1 (\text{Q} \circ_1 (\langle \text{wil} \rangle_0 \circ_0 \text{P} \backslash_1 (\text{P} \circ_1 (\text{Alice} \circ_1 \langle \text{plagen} \rangle_0)))) \rceil_1 \\
& & \rightsquigarrow \quad (\text{Res}\backslash), (\text{Res}\backslash) \\
& & \lceil \langle \text{wil} \rangle_0 \circ_0 (\text{Alice} \circ_1 \langle \text{plagen} \rangle_0)) \rceil_1 \rightsquigarrow \quad (P4) \\
& & \lceil \text{Alice} \circ_1 (\langle \text{wil} \rangle_0 \circ_0 \langle \text{plagen} \rangle_0)) \rceil_1 \rightsquigarrow \quad (P3) \\
& & \lceil \text{Alice} \circ_1 \langle \text{wil} \circ_0 \text{plagen} \rangle_0 \rceil_1 \rightsquigarrow \quad (P2) \\
& & \lceil \text{Alice} \circ_1 \langle \text{wil} \circ_0 \text{plagen} \rangle_1 \rceil_1 \rightsquigarrow \quad (P1) \\
& & \lceil \langle \text{Alice} \circ_1 (\text{wil} \circ_0 \text{plagen}) \rangle_1 \rceil_1 \rightsquigarrow \quad (\text{Res}\Box) \\
& & (\text{Alice} \circ_1 (\text{wil} \circ_0 \text{plagen}))
\end{array} \tag{31}
$$

As was the case with the labelled sequent presentation of §3.1, from the labelling information one can reconstruct the structural configuration of the antecedent database, and a sequent or Natural Deduction representation of the proof from the axiom linkings in (31).

The labelled proof net format discussed here is the basis for Richard Moot's theorem prover Grail, a general grammar development environment for multimodal categorial systems. We refer the interested reader to [Moot 98a, Moot 98b] for information on the design of the Grail system — specifically, for discussion of efficient strategies to combine structural label rewriting with checking of proof net conditions.

4 Conclusion

The quinquagenarian who is the subject of this Festschrift has edited a volume ([Gabbay (ed)]) raising the question 'What is a logical system?' An answer is suggested in an interview he has given for the Computational Linguistics Magazine TA! The interview appears under the title '*I* am a logic' ([Gabbay 95]). Following up on this suggestion, we have compared a number of proof formats for grammatical reasoning from a 'Labelled Deduction' point of view. Labelled presentations of the categorial proof theory turn out to be attractive: they naturally accommodate the required modular treatment of logical and structural aspects of grammatical resource management.

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