AN INSIDE VIEW OF EXP
or
The closed fragment of the provability of $I\Delta_0+\Omega_1$ with a propositional constant for EXP

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1 Introduction

Paris & Wilkie, in their paper *On the scheme of induction for bounded arithmetic formulas* (Paris & Wilkie[87]), paint a gripping picture of the interrelations between $I\Delta_0+\Omega_1$ and $I\Delta_0$+EXP. Two of their most memorable results are their Corollary 8.14: $I\Delta_0+\text{EXP}\nvdash\text{Con}(I\Delta_0+\Omega_1)$, and their Theorem 8.19: $I\Delta_0+\text{EXP}+\text{Con}(I\Delta_0+\Omega_1)\nvdash\text{Con}(I\Delta_0+\text{EXP})$. In this paper I give a generalization of theorems in this style. Consider the closed modal language generated by $\bot,\top$, the propositional connectives and $\Box$, with an additional logical constant EXP. We interpret the propositional constants as themselves, $\Box$ as provability in $I\Delta_0+\Omega_1$ and EXP as the arithmetical axiom EXP. In this language Paris and Wilkie's results can be reformulated as $I\Delta_0+\Omega_1\nvdash(\text{EXP}\rightarrow\Diamond\top)$ [as usual $\Diamond$ abbreviates $\neg\Box\neg$] and $I\Delta_0+\Omega_1\nvdash((\text{EXP}\wedge\Diamond\top)\rightarrow\Diamond\text{EXP})$. In this paper I characterize all principles of the closed modal language under the given interpretation that are provable in $I\Delta_0+\Omega_1$. One special case of our result of a distinctly different flavour than the theorems of Paris and Wilkie discussed above is: $I\Delta_0+\Omega_1\vdash(\Diamond\Diamond\top\rightarrow\Diamond\text{EXP})$.

Our result can be described as a solution of a variant for a special case of Friedman's 35th problem. Friedman original problem is to give a characterization of the formulas of the closed fragment of the language of modal propositional logic which are provable under the standard provability interpretation in reasonable arithmetical theories like PA. Friedman's problem was solved independently by van Benthem, Boolos (see Boolos[76]) and Magari (see Magari[75]). Their result works (modulo a slight refinement in case a theory proves its own n-iterated inconsistency for some n) for all Δ_1^b-axiomatized theories containing a sufficiently large fragment of $I\Delta_0+\Omega_1$ or even better Buss's S_2^1. The reason that the result goes through so easily in weak theories is that it doesn't require Rosser style arguments: to formalize Rosser style arguments one seems to need EXP. In contrast Solovay's proof of his arithmetical completeness theorem for Provability Logic doesn't work in $I\Delta_0+\Omega_1$. (For an elaboration of this theme see Verbrugge[88].) A solution of Friedmans problem for the case of Heyting's Arithmetic was given in Visser[85].

Hájek and Svejdar in Hajék & Svejdar[198?] generalize Friedman's problem by adding a binary operator $\rhd$ for relative interpretability to the language. If the theory we consider is T $A\rhd B$ means: T+B is relatively interpretable in T+A. Hájek and Svejdar solve the generalized problem for all Δ_1^b-axiomatized extensions of $I\Delta_0+\Omega_1$ (again modulo a slight refinement in case T proves its own n-iterated inconsistency). In section 6 of this paper I prove a similar generalization of our main result.

The contents of the paper are as follows: in section 3 the necessary conventions and elementary facts are introduced. Section 4 contains our main technical lemma. The lemma is a variant of the main lemma of Visser[88]. It is the result of formalizing a model theoretical argument due to Paris and Wilkie. In Section 5 our main result is proved and section 6 gives the generalization to the language also involving interpretability.

2 Prerequisites

We presuppose some knowledge of either Boolos[79] or Smoryński[85], and of either Buss[85] or Paris & Wilkie[87]. At a few places results from Pudlák[85],[86] and from Visser[87b] are used.

The reader who is not familiar with Buss[85] or Paris & Wilkie[87] and who is interested in the modal material could try to understand the statement of lemma 4.1 and then proceed immediately to section 5.

3 Facts, notions and conventions

3.1 Theories and Provability

We will assume that the axiom-set of a theory T is given by a $\Delta_1{}^b$-predicate (see Buss[1985]). We take this predicate to be part of the identity conditions of the theory. Proof_T is the $\Delta_1{}^b$ proof predicate based on the predicate defining T's axiom set.

We write par abus de langage '$\mathrm{Proof}_T(u, \phi(\underline{x}_1,...,\underline{x}_n))$' for: $\mathrm{Proof}_T(u, \ulcorner\phi(\dot{x}_1,...,\dot{x}_n)\urcorner)$, here:

i) all free variables of ϕ are among those shown.

ii) $\ulcorner\phi(\dot{x}_1,...,\dot{x}_n)\urcorner$ is the "Gödelterm" for $\phi(x_1,...,x_n)$ as defined in Smoryński[85], p43. Here we use instead of the usual numerals the efficient numerals of Paris & Wilkie[87], so that: $I\Delta_0+\Omega_1\vdash\forall x_1,...,x_n\exists y\ \ulcorner\phi(\dot{x}_1,...,\dot{x}_n)\urcorner = y$.

$\Box_T\phi(x_1,...,x_n)$ will stand for: $\mathrm{Prov}_T(\ulcorner\phi(\dot{x}_1,...,\dot{x}_n)\urcorner)$.

Occurrences of terms inside $\Box_T$ should be treated with some care. Is $\Box_T(\phi[t/x])$ intended $(\Box_T\phi(x))[t/x]$? We will always use the first, i.e. the small scope reading. In cases where: U prov that t is total and $U\vdash t=x\rightarrow\Box_V t=\underline{x}$, the scope distinction may be ignored within U w.r.t. $\Box_V$. W have: $U\vdash (\Box_V\phi(x))[t/x] \leftrightarrow \Box_V(\phi[t/x])$.

We will use the same convention for occurrences of variables inside the interpretability predicate. For some uses in section 4 our conventions are not sufficient. Rather than introducing a heavier notational apparatus I prefer to explain what is going on there in words.

The theory that we will be looking at in this paper is $I\Delta_0+\Omega_1$: this theory is explained in Paris & Wilkie[87]. It is (modulo some translation work) the same as Buss's theory S_2 (see Buss[85]). Sometimes, especially in subscripts, we will call $I\Delta_0+\Omega_1$ simply Ω. We will also be looking at $I\Delta_0$+EXP, which we will call sometimes -if no confusion is possible- simply EXP.

3.1.1 Cuts & a strengthened Löb's Principle

We follow the discussion of cuts of Paris & Wilkie[87]. For reasons of convenience we use a slightly idiosyncratic notion of cut: a cut I is given by an arithmetical predicate, is downwards closed w.r.t. the standard ordering of the natural numbers, is closed under successor, addition, multiplication and ω_1 (i.e. $x^{\log(x)}$). The attentive reader of Paris & Wilkie[87] will easily see that our restricted notion is not really restrictive. We will say that I is a T-cut if T proves the arithmetization of "I is a cut".

In section 5 we will use a strengthened Löb's principle: this is a direct adaptation of Pudlák's strengthening of Gödels Second Incompleteness Theorem in Pudlák[85]. Let's say that a T-cut I is T-reasonable if according to T we have enough instances of Δ_0-induction in I to verify the various metamathematical principles formalized by Paris and Wilkie in $I\Delta_0+\Omega_1$. It is well known that every T-cut can be shortened to a T-reasonable T-cut. Moreover if T proves 'enough' instances of $I\Delta_0$ then automatically every T-cut is T-reasonable (by downwards preservation of Π_1-sentences). Let T extend Q. We have:

Strengthened Löb's Principle (SLP)
$I\Delta_0+\Omega_1 \vdash$ for all T-reasonable T-cuts I $\Box_T(\Box_T{}^I A\rightarrow A) \rightarrow \Box_T A$

Proof: Reason in $I\Delta_0+\Omega_1$: Let I be a T-reasonable T-cut and suppose $\Box_T(\Box_T{}^I A\rightarrow A)$. By the Diagonalization Lemma we can find a sentence λ such that $\Box_T(\lambda \leftrightarrow (\Box_T{}^I\lambda\rightarrow A))$. We also have $\Box_T\Box_T{}^I(\lambda\leftrightarrow(\Box_T{}^I\lambda\rightarrow A))$ and hence: $\Box_T(\Box_T{}^I\lambda \rightarrow \Box_T{}^I(\Box_T{}^I\lambda\rightarrow A))$ (because in I we have 'enough axioms of $I\Delta_0+\Omega_1$). Moreover: $\Box_T(\Box_T{}^I\lambda\rightarrow\Box_T{}^I\Box_T{}^I\lambda)$. Ergo $\Box_T(\Box_T{}^I\lambda\rightarrow\Box_T{}^I A)$ and hence $\Box_T(\Box_T{}^I\lambda\rightarrow A)$. Conclude: $\Box_T\lambda$. It follows that for some x $\Box_T\mathrm{Proof}_T(x,\lambda)$. By a result of both Pudlák and Paris & Wilkie: $\Box_T x\in I$, hence $\Box_T\Box_T{}^I\lambda$ and so: $\Box_T A$. $\Box$

3.2 Interpretability

Interpretations are in this paper: one dimensional global relative interpretations without parameters. For a discussion see Pudlák[83] or Visser[88b]. We say that: U is *interpretable* via interpretation M in V if for every theorem C of U there is a proof in V of C^M. Here C^M is the translation of C under M. In the definition I assumed that the theorems are sentences; if we allow formulas D(x,...) as theorems we should take: $(\delta(x)\wedge...)\rightarrow D(x,...)^M$, where δ is the formula giving the domain of the interpretation M.

Warning: our definition of interpretability speaks of theorems not axioms. In strong theories a definition involving axioms is equivalent to ours. As far as I can see the proof of equivalence needs Σ-induction. 'Theorems-interpretability' seems to be what is needed for applications.

We write: $M{:}U \rhd V$, for the arithmetization of: V is interpretable in U via M. We can arrange it so that M occurs in the arithmetization as a number, so it is possible to quantify over M in the theory. Define:

$$
\begin{array}{ll}
U \rhd V & :\Leftrightarrow \exists M\, M{:}\, U \rhd V \\
M{:}A \rhd_U B & :\Leftrightarrow M{:}(U+A) \rhd (U+B) \\
A \rhd_U B & :\Leftrightarrow (U+A) \rhd (U+B) \\
U \equiv V & :\Leftrightarrow U \rhd V \wedge V \rhd U \\
A \equiv_U B & :\Leftrightarrow (U+A) \equiv (U+B)
\end{array}
$$

In Visser[88b] It is shown that the following principles are valid in any sequential theory extending $I\Delta_0+\Omega_1$.

L1	$\vdash A \Rightarrow \vdash \Box A$
L2	$\vdash \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
L3	$\vdash \Box A \rightarrow \Box\Box A$
L4	$\vdash \Box(\Box A \rightarrow A) \rightarrow \Box A$
J1	$\vdash \Box(A \rightarrow B) \rightarrow A \rhd B$
J2	$\vdash (A \rhd B \wedge B \rhd C) \rightarrow A \rhd C$
J3	$\vdash (A \rhd C \wedge B \rhd C) \rightarrow (A \vee B) \rhd C$
J4	$\vdash A \rhd B \rightarrow (\Diamond A \rightarrow \Diamond B)$
J5	$\vdash \Diamond A \rhd A$
W	$\vdash A \rhd B \rightarrow A \rhd (B \wedge \Box \neg A)$

The principles L1-J5 make up the theory IL. IL+W=:ILW. I conjecture that the principles of ILW are precisely the principles valid in every sequential theory extending $I\Delta_0+\Omega_1$. To be precise the conjecture is:

$$\mathrm{ILW} \vdash A \Leftrightarrow \text{for all sequential T extending } I\Delta_0+\Omega_1 \text{ for all T-interpretations } (.)^* \; T \vdash A^*$$

4 Doing some simple model theory in $I\Delta_0+\Omega_1$

In this section we formalize a model theoretic argument from Paris and Wilkie[87]. The result will be our main technical tool in sections 5 and 6.

4.1 Main Lemma

For every $A(x,y) \in \Delta_0$ with only x,y free:

$$I\Delta_0+\Omega_1 \vdash (\forall\ I\Delta_0+\Omega_1\text{-cut } I \Diamond_\Omega \exists x \in I\ \forall y\ A(x,y)) \equiv_\Omega (EXP \wedge \exists x \forall y\ A(x,y)).$$

Proof: Some details of the proof not given here are presented in Visser[88b].

"$\rhd$" We reason in $I\Delta_0+\Omega_1$. Let J be a (standard) $I\Delta_0+\Omega_1$-cut such that $\Box_\Omega(\forall x \in J\ exp(x)$ exists). Let itexp(x,0):=x, itexp(x,y+1):=exp(itexp(x,y)). One can find a Δ_0-formula representing the graph of itexp, such that the recursive clauses of the definition are verifiable.

Reason in $I\Delta_0+\Omega_1$ (so this is really in $I\Delta_0+\Omega_1$ in $I\Delta_0+\Omega_1$):

Suppose that for every $I\Delta_0+\Omega_1$-cut I: $\Diamond_\Omega \exists x \in I \forall y\ A(x,y)$. By a result of Pudlák (see Pudlák[86], the proof of Lemma 4.2): $\forall u \in J\ \exists\ I\Delta_0+\Omega_1$-cut I $\Box_\Omega(\forall v \in I$ itexp(v,u) exists). It follows that: $\forall u \in J\ \Diamond_\Omega \exists x$ (itexp(x,u) exists $\wedge\ \forall y\ A(x,y)$). Let c be a new constant and let $V := I\Delta_0+\Omega_1+\forall y\ A(c,y)+\{\text{itexp}(c,\underline{u}) \text{ exists} \mid u \in J\}$. As is easily seen V is consistent.

We want to formalize the following more or less trivial model theoretical argument (keeping in mind that model≈interpretation). For the moment read 'ω' for J. Pick a model K of V. Say D is the domain of K. Let $D^* = \{d \in D \mid \text{for some } n \in \omega\ K \models d \leq \text{itexp}(c,\underline{n})\}$. Let K* be the restriction of K to D*. Clearly $K^* \models EXP$. Because the $I\Delta_0$-axioms are Π_1: $K^* \models I\Delta_0$; similarly $K^* \models \forall y\ A(c,y)$. Conclude that $K^* \models I\Delta_0+EXP+\exists x \forall y\ A(x,y)$.

We formalize the Henkin construction to produce an internal model K of V.

We proceed as follows: first define the usual Henkin tree for formulas in the language extended with Henkin constants. The formula treated at depth x will be precisely the formula with code x. Some care should be taken to make the Henkin constants not too big. We pick the leftmost path π in the tree. We cannot prove that our path is infinite in the usual sense, but we can produce an $I\Delta_0+\Omega_1$-cut I_0 such that for each x in I_0 there is a sequence in π with length x. Without loss of generality we may assume that $I_0 \subseteq J$. Let K be the set of formulas given by elements of π with length in I_0; clearly $K \subseteq I_0$. Let D be the set of Henkin constants in I_0. It can be arranged that if (the code of) $\exists x B(x)$ is in K and b is the Henkin constant of $\exists x B(x)$, then b is in D. We can show: $\forall x \in I_0\ \text{Prov}_V(x) \rightarrow K(x)$.

We use d,d', e,.. to range over D. We write e.g. K(B(d,d')) for K(b(d,d')), where b(d,d') is a term for: the code of the sentence obtained by substituting the Henkin constants coded by d and d' for u and v in B(u,v). We write for x in I_0 e.g. $K(C(\underline{x}))$ for K(c(x)), where c(x) is a term for: the code of the sentence obtained by substituting the efficient numeral of x for u in C(u).

K is one form of appearance of the 'model K' we are looking for. Its other form of appearence is as an interpretation $(.)^K$. The domain of this interpretation is going to be D. Let R be a relation of the language of V, we have: $R^K(d,...) :\leftrightarrow K(R(d,...))$. For arbitrary formulas $B(d,...)$ $B^K(d,...)$ is defined as usual. For vividness we will write $K\models B(d,...)$ for $B^K(d,...)$.

As usual we can show $\forall x\, K(\mathrm{conj}(x,y))\leftrightarrow(K(x)\wedge K(y))$, etc. . By an *external* induction we can show:

*   For d,... in D: $K(B(d,...)) \leftrightarrow K\models B(d,...)$.

More on the meaning of * and its proof below: see the discussion on **.

Finally we can define a homomorphism f from I_0 to the natural numbers of the 'internal model' K. Consider x in I_0, f(x) will be the code of the Henkin constant of $\ulcorner\exists u\ u=\underline{x}\urcorner$. We will have: $K(f(x)=\underline{x})$. We can arrange it so (by shortening I_0 if necessary) that the range of f is downwards closed in K.

Let c* be the Henkin constant of $\exists x\ x=c$. We have $K(c^*=c)$. Moreover: $\forall x\in I_0 \Box_V(\mathrm{itexp}(c,\underline{x})$ exists), ergo $\forall x\in I_0\, K(\mathrm{itexp}(c,\underline{x})$ exists), so $\forall x\in I_0\, K(\mathrm{itexp}(c^*,f(x))$ exists). Conclude: $\forall x\in I_0\, K\models(\mathrm{itexp}(c^*,f(x))$ exists). Let $D^*:=\{d\in D \mid \exists x\in I_0\, K\models d\leq \mathrm{itexp}(c^*,f(x))\}$. Clearly: $c^*\in D^*$ and $\forall d\in D^*\exists e\in D^*\, K\models \exp(d)=e$.

Let $(.)^{K^*}$ be like $(.)^K$ exept that we use D* instead of D. We write for d,... in D*: $K^*\models B(d,...)$ for $B^{K^*}(d,...)$. Because the graph of exp is Δ_0 it follows by a simple argument that $K^*\models \mathrm{EXP}$. Moreover $K\models\forall y\, A(c^*,y)$, A is Δ_0, hence $K^*\models\forall y\, A(c^*,y)$ and thus $K^*\models\exists x\forall y\, A(x,y)$.

Finally we have for all codes z of instances Z of Δ_0-induction: $\Box_\Omega \underline{z}\in I_0$ and $\Box_\Omega \mathrm{Prov}_V(\underline{z})$, hence $\Box_\Omega K(\underline{z})$, so $\Box_\Omega(K\models Z)$. Because these Z have Π_1 form we may conclude: $\Box_\Omega(K^*\models Z)$.

Let's look at these last four steps a bit more carefully. As is well known (see e.g. Paris & Wilkie [87]) the proofs of $\ulcorner\underline{z}\in I_0\urcorner$ and $\ulcorner\mathrm{Prov}_V(\underline{z})\urcorner$ can be explicitily bounded by terms in z involving the usual arithmetical operations and ω_1 (ω_1-terms for short). (A moment's reflection shows that I_0 is given by a standard formula.) Hence the proof of $\ulcorner K(\underline{z})\urcorner$ can be bounded by an ω_1-term in z.

Next we move to $\Box_\Omega(K\models D)$ using (momentarily confusing formulas and their codes):

** $\forall C\Box_\Omega(\ \forall d,...\in D\ (K(C(d,...)) \leftrightarrow K\models C(d,...))\ )$.

Let's call the statement following $\Box_\Omega$ in **: E{C}. To prove ** we use $\Delta_0(\omega_1)$-induction, which is available in $I\Delta_0+\Omega_1$. To do this we must bound the $I\Delta_0+\Omega_1$-proofs of E{C} with ω_1-terms in C; in other words the lengths (=number of symbols) of these proofs should be bounded by a polynomial in |C|, i.e. the length of C. Let's call the length of the proof of E{C}: $\lambda(C)$. I consider a specific example: say $C=(F\wedge G)$ and suppose we have proofs of E{F} and E{G}. To construct a proof of E{C} we give proofs of: $C=\mathrm{conj}(F,G)$, and $\forall x\, K(\mathrm{conj}(x,y))\leftrightarrow(K(x)\wedge K(y))$. The length of the first proof is polynomially bounded in |C| and the length of the second one is standard. Now the proofs of E{F}, E{G}, C=conj(F,G), and $\forall x\ K(conj(x,y)) \leftrightarrow (K(x) \wedge K(y))$ can be combined to a proof of E{C} of length bounded by: $\lambda(F)+\lambda(G)+P(|C|)$, where P is a suitable polynomial. For each connective we find such a polynomial. Let Q be a polynomial that majorizes all polonomials corresponding to the connectives. Noting that $|F|+|G|<|C|$ it is now easy to show that: $\lambda(C) \leq |C|.Q(|C|)$, e.g. in the case considered we have e.g:

$$\lambda(C) \leq \lambda(F)+\lambda(G)+Q(|C|) \leq |F|Q(|F|)+|G|Q(|G|)+Q(|C|) \leq (|F|+|G|+1)Q(|C|) \leq |C|Q(|C|).$$

Finally we move to $\Box_\Omega(K^* \models Z)$. Here we use:

*** $\quad \forall C \Box_\Omega(\forall d,... \in D^*\ (K^* \models C(d,...) \leftrightarrow K \models C(d,...))\ )$.

The proof shares many features with the proof of **. Again the lengths of the proofs will be polynomially bounded in $|C|$. Let t range over ω_1-terms. An important lemma is:

+ $\quad \forall t \Box_\Omega(\forall d,... \in D^* \forall e \in D\ (\ (K \models e=t(d,...)) \rightarrow e \in D^*\ )\ )$.

The lemma is proved by induction on t using a bound on the lengths of the proofs that is polynomial in $|t|$.

Concluding: let AX be the set of axioms of $I\Delta_0$+EXP+$\exists x \forall y\ A(x,y)$. We have for a suitable ω_1-term t: $\forall C \in AX\ \exists p<t(C)\ Proof_\Omega(p, \ulcorner K^* \models C \urcorner)$. By induction we find for a suitable ω_1-term u:

$$\forall x \forall C<x\ (Proof_{AX}(x,C) \rightarrow \exists z<u(x)\ Proof_\Omega(z, \ulcorner K^* \models C \urcorner). \qquad \Box$$

"$\lhd$" Let $\Im$ be an $I\Delta_0$+EXP-cut such that $I\Delta_0+EXP \vdash \forall u \in \Im \forall v$ itexp(v,u) exists. We first show for B in Δ_0 having only x,y free:

$$I\Delta_0+EXP \vdash \forall I \in \Im(\ \Box_\Omega^{\Im}\text{"I is a cut"} \rightarrow ((\exists z \in \Im\ \Box_{\Omega,z} \forall x \in I\ \exists y\ B(x,y)) \rightarrow \forall x \exists y\ \ B(x,y))\ ).$$

Reason in $I\Delta_0$+EXP: Suppose $I \in \Im$, $\Box_\Omega^{\Im}$"I is a cut", $z \in \Im$ and $\Box_{\Omega,z} \forall x \in I\ \exists y\ B(x,y)$. For some $u \in \Im$ and for all v $\Box_{\Omega,u} \underline{v} \in I$. It follows that for some $w \in \Im$: $\forall x \Box_{\Omega,w} \exists y\ B(\underline{x},y)$. Using the estimate on cut-elimination in Paris & Wilkie[87], p293 we may conclude: $\forall x \Delta_\Omega \exists y\ B(\underline{x},y)$. Paris and Wilkie also show reflection for tableaux provability in $I\Delta_0$+EXP. w.r.t. Π_2-formulas, hence: $\forall x \exists y\ B(x,y)$.

From the above we have by Σ-completeness, contraposition and by weakening the statement a bit: for A in Δ_0 having only x,y free:

$$I\Delta_0+\Omega_1 \vdash \Box_{EXP}(\ \exists x \forall y A(x,y) \rightarrow\ \ (\forall\ I\Delta_0+\Omega_1\text{-cut } I \Diamond_\Omega \exists x \in I\ \forall y\ A(x,y))^{\Im}\ ).$$

From this the result we're looking for is immediate using $\Im$ as our interpretation. $\Box$

4.2 Corollary

For any Σ_2-sentence B: $I\Delta_0+\Omega_1 \vdash B \rhd_\Omega (B \wedge \neg EXP)$.

Proof: from 4.1 we have: $I\Delta_0+\Omega_1 \vdash (B \wedge EXP) \rhd_\Omega \Diamond_\Omega B$, hence by principle W: $I\Delta_0+\Omega_1 \vdash (B \wedge EXP) \rhd_\Omega ((\Diamond_\Omega B) \wedge \Box_\Omega(B \rightarrow \neg EXP))$, so $I\Delta_0+\Omega_1 \vdash (B \wedge EXP) \rhd_\Omega \Diamond_\Omega (B \wedge \neg EXP)$. Conclude by J5: $I\Delta_0+\Omega_1 \vdash (B \wedge EXP) \rhd_\Omega (B \wedge \neg EXP)$. Also $I\Delta_0+\Omega_1 \vdash (B \wedge \neg EXP) \rhd_\Omega (B \wedge \neg EXP)$, hence by J3:

$I\Delta_0+\Omega_1\vdash B\rhd_\Omega(B\wedge\neg EXP)$. □

4.3 Corollary

i) Suppose A is Δ_0 having only x,y free, then:
$I\Delta_0+\Omega_1\vdash \Box_{EXP}\forall x\exists y\, A(x,y)\leftrightarrow \Box_\Omega\exists\, I\Delta_0+\Omega_1\text{-cut } I\Box_\Omega\forall x\in I\,\exists y\, A(x,y)$.

ii) Suppose B is a Σ_2-sentence, then $I\Delta_0+\Omega_1\vdash \Box_\Omega(B\rightarrow EXP)\rightarrow\Box_\Omega\neg B$.

Proof: (i) is immediate from 4.1 and (ii) is immediate from 4.2. □

4.4 Corollary

Suppose A is a Σ_1-sentence, then:

i) $I\Delta_0+\Omega_1\vdash \Box_{EXP}A\leftrightarrow\Box_\Omega\Box_\Omega A$

ii) $I\Delta_0+\Omega_1\vdash \Box_{EXP}(\Box_\Omega A\rightarrow A)\rightarrow\Box_{EXP}A$

Proof: (i) is immediate from 4.3(i). For (ii) we have:

$$\begin{aligned} I\Delta_0+\Omega_1\vdash \Box_{EXP}(\Box_\Omega A\rightarrow A) &\rightarrow \Box_\Omega\exists\, I\Delta_0+\Omega_1\text{-cut } I\Box_\Omega(\Box_\Omega^I A\rightarrow A) && (4.3(i))\\ &\rightarrow \Box_\Omega\Box_\Omega A && (SLP)\\ &\rightarrow \Box_{EXP}A && (4.3(i)) \end{aligned}$$

□

5 The closed fragment of the provability logic of $I\Delta_0+\Omega_1$ with a constant for EXP

Λ is the closed language of provability logic, i.e. Λ is the smallest set containing $\bot,\top$, which is closed under $\neg,\wedge,\vee,\rightarrow$ and $\Box$. If a logical constants c,c',... are added to Λ we write: $\Lambda[c,c',...]$. $\Diamond$ abbreviates $\neg\Box\neg$.

The degrees of falsity DF are defined as follows: $\Box^0\bot:=\bot$, $\Box^{n+1}\bot:=\Box\Box^n\bot$, $\Box^\omega\bot:=\top$. Dually the degrees of truth are defined by: $\Diamond^0\top:=\top$, $\Diamond^{n+1}\top:=\Diamond\Diamond^n\top$, $\Diamond^\omega\top:=\bot$. If X is a set of formulas we write Boole(X) for the set of Boolean combinations of elements of X.

We will only consider a fixed interpretation of our languages: the propositional connectives are interpreted as themselves, $\Box$ is interpreted as $\Box_\Omega$, EXP is interpreted as the arithmetical axiom EXP. The fact that our interpretation is constant makes that we can conveniently confuse modal formulas and their arithmetical counterparts. From now on we will do so.

The system LC[EXP] in Λ[EXP] is given by the following principles:

L1 $\vdash A\Rightarrow\vdash\Box A$

L2 $\quad \vdash \Box(A\to B) \to (\Box A \to \Box B)$

L3 $\quad \vdash \Box A \to \Box\Box A$

L4 $\quad \vdash \Box(\Box A \to A) \to \Box A$

C1 $\quad \vdash \Box(EXP\to B) \leftrightarrow \Box\Box B$, for $B\in$ Boole(DF)

C2 $\quad \vdash \Box(\neg EXP\to B) \leftrightarrow \Box B$, for $B\in$ Boole(DF)

We verify the validity of LC[EXP] for interpretations in $I\Delta_0+\Omega_1$. C2 is immediate from 4.2(ii).

In our verification of C1 we will use the "some finite subset" notation: $\{A \| P(A)\}$ means approximately: some finite (possibly empty) subset of $\{A \mid P(A)\}$. When the notation is repeatedly used however it will function in an anaphoric way: so sometimes it means: the finite subset we were talking about; or even: the finite subset connected in the evident way with the finite subset we were talking about.

Verification of C1 in $I\Delta_0+\Omega_1$: Consider B in Boole(DF). Clearly B is equivalent to a sentence of the form $\bigwedge\{\Box^{\alpha}\bot \to \Box^{k}\bot \| k<\alpha\}$. (Here: α ranges over $\omega+1$.) By 4.2(i) we have that:

$$I\Delta_0+\Omega_1 \vdash \Box(EXP\to B) \leftrightarrow \Box\exists\ I\Delta_0+\Omega_1\text{-cut } I\Box\bigwedge\{\Box^{\alpha,I}\bot \to \Box^{k}\bot \| k<\alpha\}.$$

On the other hand:

$$\begin{array}{lll}
I\Delta_0+\Omega_1 \vdash \exists\ I\Delta_0+\Omega_1\text{-cut } I\Box\bigwedge\{\Box^{\alpha,I}\bot \to \Box^{k}\bot \| k<\alpha\} & \to & \\
\qquad \exists I\Delta_0+\Omega_1\text{-cut} I \bigwedge\{\Box(\Box^{\alpha,I}\bot \to \Box^{k}\bot) \| k<\alpha\} & \to & \\
\qquad \exists I\Delta_0+\Omega_1\text{-cut} I \bigwedge\{\Box(\Box^{k+1,I}\bot \to \Box^{k}\bot) \| k<\alpha\} & \to & \text{(SLP)} \\
\qquad \bigwedge\{\Box^{k+1}\bot \| k\in\omega\} & \to & (\alpha^{*}=\inf\{k \| k\in\omega\}) \\
\qquad \Box^{1+\alpha^{*}}\bot & \to & \\
\qquad \Box\bigwedge\{\Box^{\alpha}\bot \to \Box^{k}\bot \| k<\alpha\} & \to & \\
\qquad \exists\ I\Delta_0+\Omega_1\text{-cut } I\Box\bigwedge\{\Box^{\alpha,I}\bot \to \Box^{k}\bot \| k<\alpha\}. & &
\end{array}$$

Ergo $I\Delta_0+\Omega_1 \vdash \Box(EXP\to B) \leftrightarrow \Box\Box B$. □

5.1 Theorem

i) For every $A\in\Lambda$[EXP]: LC[EXP]$\vdash \Box A \leftrightarrow \Box^{\alpha}\bot$, for some $\alpha\in\omega+1$.

ii) For every $A\in\Lambda$[EXP] there is a $B\in$ Boole(DF$\cup\{$EXP$\}$): LC[EXP]$\vdash A \leftrightarrow B$.

iii) For every $A\in\Lambda$[EXP]: LC[EXP]$\vdash \Box A \Rightarrow$ LC[EXP]$\vdash A$.

Proof: for (i) and (ii) it is sufficient to show that for $B\in$ Boole(DF$\cup\{$EXP$\}$): LC[EXP]$\vdash \Box B \leftrightarrow \Box^{\alpha}\bot$, for some $\alpha\in\omega+1$. The rest of the argument is a simple induction. As is easily seen there are C,D in Boole(DF) such that LC[EXP]$\vdash B \leftrightarrow ((EXP\to C)\wedge(\neg EXP\to D))$, hence LC[EXP]$\vdash \Box B \leftrightarrow (\Box(EXP\to C)\wedge\Box(\neg EXP\to D))$, so by C1, C2: LC[EXP]$\vdash \Box B \leftrightarrow (\Box\Box C\wedge\Box D)$. So by the usual reasoning the desired result follows.

To prove (iii) suppose LC[EXP]$\vdash \Box A$. We note that by (ii): A is LC[EXP]-equivalent to:

$(EXP \rightarrow \bigwedge\{\Box^{\alpha}\bot \rightarrow \Box^{k}\bot \mid k<\alpha\}) \wedge (\neg EXP \rightarrow \bigwedge\{\Box^{\beta}\bot \rightarrow \Box^{n}\bot \mid n<\beta\})$. If both conjunctions are empty we are done. If not it follows that for some m LC[EXP]$\vdash \Box^{m}\bot$ and hence $I\Delta_0+\Omega_1 \vdash \Box^{m}\bot$, quod non. □

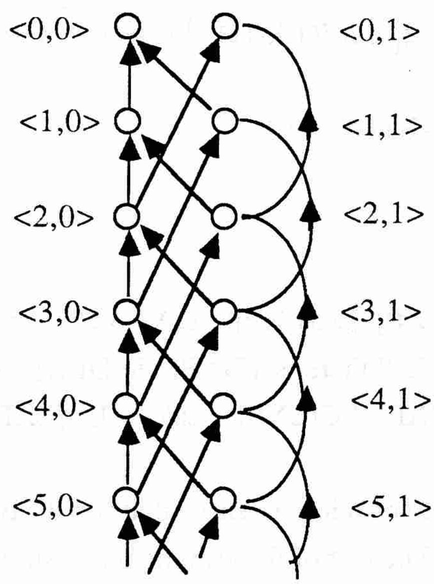
Consider two Kripke models $K=\langle W,R,\Vdash\rangle$ and $K'=\langle W',R',\Vdash'\rangle$. A *Λ-bisimulation* β between K and K' is a relation between W and W' such that: (i) for every k in W there is a k' in W' with kβk'; (ii) for every k' in W' there is a k in W with kβk'; (iii) if kβk' and kRs, then there is an s' with k'R's' and sβs'; (iv) if kβk' and k'R's', then there is an s with kRs and sβs'. As is easily seen: if β is a Λ-bisimulation between K and K' and kβk', then for $A \in \Lambda$: $k \Vdash A \Leftrightarrow k' \Vdash' A$.

5.2 Theorem

$$LC[EXP] \vdash A \Leftrightarrow I\Delta_0+\Omega_1 \vdash A.$$

Proof: "⇒" has already been checked. For "⇐" suppose $I\Delta_0+\Omega_1 \vdash A$. Suppose that LC[EXP] does not prove A, then LC[EXP] does not prove $\Box A$, so $\Box A$ must be LC[EXP]-equivalent to $\Box^{k}\bot$ for some k. We find $I\Delta_0+\Omega_1 \vdash \Box A$, hence $I\Delta_0+\Omega_1 \vdash \Box^{k}\bot$. Quod non. □

We define a Kripke model M as follows: the domain of M is $\{\langle n,i\rangle \mid n\in\omega, i\in\{0,1\}\}$; M has an accessibility relation given by: $\langle n,i\rangle R \langle m,j\rangle :\Leftrightarrow n>m+j$. We stipulate $\langle n,i\rangle \Vdash EXP :\Leftrightarrow i=1$. The forcing relation is extended to the whole language in the usual way. We show that LC[EXP] is valid in M. As is easily seen R is transitive and upwards wellfounded. Hence the principles L1-L4 are valid on M.



The model M

Let N be the model with domain ω and accessibility relation R* given by: $nR^{*}m :\Leftrightarrow n>m$. Define a relation β between nodes of N and nodes of M by $n\beta\langle m,i\rangle :\Leftrightarrow n=m$. It is easily seen that β is a

Λ-bisimulation between N and M. Conclude that for A in Λ: $<n,0>\Vdash A \Leftrightarrow <n,1>\Vdash A$.

Verification of C1 in M: suppose B is a Boolean combination of degrees of falsity.

First suppose $<n,i>\Vdash \Box\Box B$ and $<n,i>R<m,j>$ and $<m,j>\Vdash EXP$, i.e. $j=1$. We have: $n>m+1$, so $<n,i>R<m+1,0>R<m,0>$. Hence $<m,0>\Vdash B$. B is in Λ, so $<m,1>\Vdash B$. Conclude: $<n,i>\Vdash \Box(EXP\to B)$.

Suppose for the converse: $<n,i>\Vdash \Box(EXP\to B)$ and $<n,i>R<m,j>R<p,k>$. Clearly $n>m+j>p+k$, so $n>p+1$ and thus $<n,i>R<p,1>$. $<p,1>\Vdash EXP$ and so $<p,1>\Vdash B$. B is in Λ so we may conclude: $<p,k>\Vdash B$. Ergo $<n,i>\Vdash \Box\Box B$ $\Box$

Verification of C2 in M: suppose B is a Boolean combination of degrees of falsity.

One direction is trivial. Suppose: $<n,i>\Vdash \Box(\neg EXP\to B)$ and $<n,i>R<m,j>$. Clearly $<n,i>R<m,0>$, so $<m,0>\Vdash B$. B is in Λ so we may conclude: $<m,j>\Vdash B$. Ergo $<n,i>\Vdash \Box B$. $\Box$

5.3 Theorem

$LC[EXP]\vdash A \Leftrightarrow M\Vdash A.$

Proof: entirely analogous to the proof of 5.2. $\Box$

6 The closed fragment of the interpretability logic of $I\Delta_0+\Omega_1$ with a constant for EXP

The system ILC[EXP] is given by the following principles:

L1	$\vdash A \Rightarrow \vdash \Box A$
L2	$\vdash \Box(A\to B)\to(\Box A\to\Box B)$
L3	$\vdash \Box A\to\Box\Box A$
L4	$\vdash \Box(\Box A\to A)\to\Box A$
J1	$\vdash \Box(A\to B)\to A\rhd B$
J2	$\vdash (A\rhd B\wedge B\rhd C)\to A\rhd C$
J3	$\vdash (A\rhd C\wedge B\rhd C)\to(A\vee B)\rhd C$
J4	$\vdash A\rhd B\to(\Diamond A\to\Diamond B)$
J5	$\vdash \Diamond A\rhd A$
W	$\vdash A\rhd B\to A\rhd(B\wedge\Box\neg A)$
C	$\vdash (EXP\wedge B)\equiv\Diamond B$, where $B\in Boole(DF)$

We verify the validity of ILC[EXP] for interpretations in $I\Delta_0+\Omega_1$.

Verification of C in $I\Delta_0+\Omega_1$:

Suppose $B\in$ Boole(DF). Clearly B is equivalent to a sentence of the form $\bigvee\{\Diamond^k\top\wedge\Box^\alpha\bot \| k<\alpha\}$, where α ranges over $\omega+1$. By 4.1 we have that:

$$I\Delta_0+\Omega_1\vdash (EXP\wedge B)\equiv(\forall\ I\Delta_0+\Omega_1\text{-cuts } I\Diamond\bigvee\{\Diamond^k\top\wedge\Box^{\alpha,I}\bot \| k<\}).$$

By contraposition of the reasoning concerning the verification of C1:

$$I\Delta_0+\Omega_1\vdash (\forall\ I\Delta_0+\Omega_1\text{-cuts } I\Diamond\bigvee\{\Diamond^k\top\wedge\Box^{\alpha,I}\bot \| k<\alpha\}) \leftrightarrow \Diamond B.$$

Conclude: $I\Delta_0+\Omega_1\vdash (EXP\wedge B)\equiv\Diamond B$. □

6.1 Theorem

i) For every $A\in\Lambda[\rhd,EXP]$: ILC[EXP]$\vdash \Box A \leftrightarrow \Box^\alpha\bot$, for some $\alpha\in\omega+1$.

ii) For every $A,B\in\Lambda[\rhd,EXP]$: ILC[EXP]$\vdash A\rhd B \leftrightarrow \Box^\alpha\bot$, for some $\alpha\in\omega+1$.

iii) For every $A\in\Lambda[\rhd,EXP]$ there is a $B\in$ Boole(DF$\cup$\{EXP\}): LC[EXP]$\vdash A \leftrightarrow B$.

iv) For every $A\in\Lambda[EXP]$: LC[EXP]$\vdash \Box A \Rightarrow$ LC[EXP]$\vdash A$.

Proof: for (i), (ii), (iii) it is sufficient to show that for $A,B\in$ Boole(DF$\cup$\{EXP\}): ILC[EXP]$\vdash \Box A \leftrightarrow \Box^\alpha\bot$, for some $\alpha\in\omega+1$ and ILC[EXP]$\vdash A\rhd B \leftrightarrow \Box^\alpha\bot$, for some $\alpha\in\omega+1$. The rest of the argument is a simple induction. We can restrict ourselves to the case of $\rhd$ noting that $\Box A$ is equivalent to $A\rhd\bot$.

First consider C in Boole(DF). We show: ILC[EXP]$\vdash (EXP\wedge C)\equiv\Diamond^\alpha\top$, for some α. We have:

$$ILC[EXP]\vdash (EXP\wedge C)\equiv\Diamond C\equiv\Diamond^\alpha\top.$$

Next we show: ILC[EXP]$\vdash (\neg EXP\wedge C)\equiv\Diamond^\beta\top$, for some β. First note:

$$\begin{aligned} ILC[EXP]\vdash (EXP\wedge C) &\rhd \Diamond C \\ &\rhd (\Diamond C\wedge\Box(C\to\neg EXP)) \\ &\rhd \Diamond(\neg EXP\wedge C) \\ &\rhd (\neg EXP\wedge C) \end{aligned}$$

Also: ILC[EXP]$\vdash (\neg EXP\wedge C)\rhd(\neg EXP\wedge C)$, hence ILC[EXP]$\vdash C\rhd(\neg EXP\wedge C)$. We find:

$$\begin{aligned} ILC[EXP]\vdash (\neg EXP\wedge C) &\equiv C \\ &\equiv (C\vee\Diamond C) \\ &\equiv \Diamond^\beta\top \end{aligned}$$

Consider A in Boole(DF$\cup$\{EXP\}). Clearly A is equivalent to $(EXP\wedge C)\vee(\neg EXP\wedge D)$ for some C and D in Boole(DF). By the above: ILC[EXP]$\vdash (EXP\wedge C)\equiv\Diamond^\alpha\top$, for some α and ILC[EXP]$\vdash (\neg EXP\wedge D)\equiv\Diamond^\beta\top$, for some β. Hence ILC[EXP]$\vdash A\equiv(\Diamond^\alpha\top\vee\Diamond^\beta\top)\equiv\Diamond^\gamma\top$, for some γ. Conclude for A,B in Boole(DF$\cup$\{EXP\}): ILC[EXP]$\vdash A\rhd B \leftrightarrow \Diamond^\gamma\top\rhd\Diamond^\delta\top$ for some γ,δ. If $\gamma\geq\delta$: ILC[EXP]$\vdash A\rhd B \leftrightarrow \top$, and we are done. If $\gamma<\delta$:

$$
\begin{aligned}
\mathrm{ILC[EXP]} \vdash A \rhd B &\leftrightarrow \Diamond^{\gamma}\top \rhd \Diamond^{\delta}\top \\
&\leftrightarrow \Diamond^{\gamma}\top \rhd (\Diamond^{\delta}\top \wedge \Box\neg\Diamond^{\gamma}\top) \\
&\leftrightarrow \Diamond^{\gamma}\top \rhd (\Diamond^{\delta}\top \wedge \Box^{\gamma+1}\bot) \\
&\leftrightarrow \Diamond^{\gamma}\top \rhd \bot \\
&\leftrightarrow \Box^{1+\gamma}\bot
\end{aligned}
$$

The proof of of (iv) is the same as the proof of 5.1(iii). □

6.2 Theorem

$$\mathrm{ILC[EXP]} \vdash A \Leftrightarrow I\Delta_0+\Omega_1 \vdash A.$$

Proof: the same as the proof of 5.2. □

We define a Kripke model M as follows: the domain of M is $\{\langle n,i\rangle \mid n\in\omega, i\in\{0,1\}\}$; M has an two accessibility relations R and S given by: $\langle n,i\rangle R\langle m,j\rangle :\Leftrightarrow n>m+j$ and $\langle n,i\rangle S\langle m,j\rangle :\Leftrightarrow n+i\geq m+j$. We stipulate $\langle n,i\rangle \Vdash \mathrm{EXP} :\Leftrightarrow i=1$. The forcing relation is extended to the whole language in the usual way using R as the accessibility relation for □ and:

$x\Vdash A\rhd B :\Leftrightarrow$ for all y: xRy and $y\Vdash A \Rightarrow$ there is a z with ySz and $z\Vdash B$.

As before R is transitive and upwards wellfounded. We have: $R\subseteq S$; S is reflexive and transitive; S satisfies property P, i.e.: $xRySz \Rightarrow xRz$.

Excursion: The property '$xRySz \Rightarrow xRz$' makes M into an ILP-model (see Visser[88a] or Visser[88b] or De Jongh & Veltman[88]). This implies that the principle: $A\rhd B \rightarrow \Box(A\rhd B)$ is valid on M. There are a priori reasons, given the fact that M fully characterizes what is and what is not provable in the restricted language and seeing the methods we used, that this should be so. For suppose M would provide a counterexample to the principle. This shows or at least strongly suggests that $I\Delta_0+\Omega_1$ is not finitely axiomatizable. (The loophole here is that it might be the case that, yes, $I\Delta_0+\Omega_1$ is in fact finitely axiomatizable but, no, its finite axiomatizability is not verifiable in $I\Delta_0+\Omega_1$.) But the problem of finite axiomatizability of $I\Delta_0+\Omega_1$ is connected with difficult complexity theoretic problems and it seems clear that the methods used in section 4 are not 'heavy' enough to solve such problems. So a full characterization of the valid principles of $\Lambda[\mathrm{EXP},\rhd]$ in $I\Delta_0+\Omega_1$ using light methods as in section 4 cannot but satisfy principle P.

Verification of C in M: suppose B is a Boolean combination of degrees of falsity.

First suppose $\langle n,i\rangle \Vdash \Box\Box B$ and $\langle n,i\rangle R\langle m,j\rangle$ and $\langle m,j\rangle \Vdash \mathrm{EXP}$, i.e. j=1. We have: n>m+1, so $\langle n,i\rangle R\langle m+1,0\rangle R\langle m,0\rangle$. Hence $\langle m,0\rangle \Vdash B$. B is in Λ, so $\langle m,1\rangle \Vdash B$. Conclude: $\langle n,i\rangle \Vdash \Box(\mathrm{EXP}\rightarrow B)$.

Suppose for the converse: $\langle n,i\rangle \Vdash \Box(\mathrm{EXP}\rightarrow B)$ and $\langle n,i\rangle R\langle m,j\rangle R\langle p,k\rangle$. Clearly n>m+j>p+k, so n>p+1 and thus $\langle n,i\rangle R\langle p,1\rangle$. $\langle p,1\rangle \Vdash \mathrm{EXP}$ and so $\langle p,1\rangle \Vdash B$. B is in Λ so we may conclude:

$\langle p,k\rangle \Vdash B$. Ergo $\langle n,i\rangle \Vdash \Box\Box B$ □

6.3 Theorem

$$\mathrm{ILC[EXP]} \vdash A \Leftrightarrow \mathrm{M} \Vdash A.$$

Proof: entirely analogous to the proof of 5.3. □

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