NNIL, a study in intuitionistic propositional logic

Albert Visser
Johan van Benthem
Dick de Jongh
Gerard R. Renardel de Lavalette

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Abstract

In this paper we study *NNIL*, the class of formulas of the Intuitionistic Propositional Calculus *IPC* with no nestings of implications to the left. We show that the formulas of this class are precisely the formulas of the language of *IPC* that are preserved under taking submodels of Kripke models for *IPC* (for various notions of submodel). This makes *NNIL* an analogue of the purely universal formulas in Predicate Logic. We prove a number of interpolation properties for *NNIL*, and explore the extent to which these properties can be generalized to more complicated classes of formulas.

1 Introduction

In this paper we study a special class of formulas of the Intuitionistic Propositional Calculus *IPC*. This is the class of Π_1-formulas or *NNIL*-formulas[1]. These formulas are the formulas without nestings of implications to the left. Examples of *NNIL*-formulas are:

$$p,\ \top,\ (p \to (q \vee (r \to s))) \wedge ((q \wedge t) \to ((r \to p) \vee (s \to r))).$$

The usual Kripke semantics for *IPC* provides us with a translation of *IPC*-formulas to one-variable formulas of the Classical Predicate Calculus (*CQC*). The *NNIL*-formulas are seen to translate to purely universal formulas or Π_1-formulas in the sense of *CQC*. In fact our results imply that every Π_1-formula in the sense of *CQC*, under certain further appropriate general conditions shared by all *IPC*-formulas, is provably equivalent to a *NNIL*-formula.

We will prove in this paper that the *NNIL*-formulas are precisely the *IPC*-formulas preserved under taking submodels. Moreover the *NNIL*-formulas satisfy the interrelated properties of left and right approximation and (uniform) left and right interpolation. We define and discuss these properties in section 3.

[1] As we will see 'Π_1' is the more systematic name for our formula class. The name 'Π_1', however, was introduced only fairly recently. '*NNIL*' on the other hand is in use since 1984. '*NNIL*' is pronounced as 'NIL', where the first 'N' is pronounced with some slight hesitation.

1.1 Motivation

This paper is a case study concerned with the interplay of two sources of complexity in *IPC*: the number of propositional variables occurring in a formula, on the one hand, and the nesting degree of implications, on the other. We take as focus of our study the nearly lowest complexity class w.r.t., nesting degree, *NNIL*. It turns out that *NNIL* has some very good properties, justifying its separate study: it satisfies strong versions of interpolation and it is precisely the class preserved under taking submodels of Kripke models.

NNIL lies at the intersection of rather diverse research interest of the authors. We briefly mention a few of these, with pointers for further reading, as this simple class of formulas turns out to be a useful concrete testing ground for broad theoretical issues.

- In many cases classical model-theoretic properties, such as Łoś's Theorem or the Löwenheim-Skolem Theorem, can be transferred to intuitionistic or modal propositional logic. The possibility of such transfer can be systematically explored. See Andréka et al. ??, van Benthem 1991b, de Rijke 1993.
- *NNIL* can be viewed as a fragment of *IPC* in an extended sense, since it is generated like the language of *IPC*, restricting the formation rule for implications. Thus the study of its interpolation properties, has some interest for the study of interpolation in fragments of *IPC*. For information on interpolation in fragments, see Porebska 1985, Renardel de Lavalette 1981, 1986, 1989, Zucker 1978. (There is also some work on the structure of finite fragments. See de Jongh et al. 1991. The question of the structure of $NNIL(\vec{p})$ will, however, not be taken up in this paper.)
- The class *NNIL* turns up naturally in the study of the propositional admissible rules for theories, like Heyting's Arithmetic (*HA*). Using *NNIL*, one can fully characterize these rules, for the case, where one restricts oneself to substitutions of Σ-formulas. For background see e.g., de Jongh 1982, de Jongh and Visser 1993, Visser 1985 and Visser 1994.

1.2 Historical note

We briefly outline the genesis of the paper. In 1983-1984, A. Visser was studying the provability logic of Heyting's Arithmetic *HA*. As a subproblem he considered Σ_1-substitutions of propositional formulas. *NNIL* emerged from this work on *HA*. Two questions came up of a purely propositional character. The first was whether the *NNIL*-formulas are precisely the ones that are preserved under taking submodels of Kripke models. This question was answered positively by Johan van Benthem (in correpondence with Visser) using a model-theoretical argument close to the argument presented in this paper and, independently, by Visser using different methods (see Visser 1985). Van Benthem's proof appeared (for the case of temporal logic) in his van Benthem 1991b. The other question was to prove *NNIL*-interpolation. This question was posed in Visser 1985. In that paper, an indirect proof for *NNIL* left-interpolation (see 1.1.4 in Visser 1985) was given. A direct proof both for left and for right interpolation is given in Renardel de Lavalette 1986. Renardel's proofs are presented in section 4 of this paper.

1.3 Organization of the Paper

Section 2 contains the syntactical preliminaries. In remark 2.1 we formulate a translation of *IPC* into predicate logic. We will see that via this translation implicational complexity in *IPC* corresponds to quantifier complexity in predicate logic. Following the translation we will designate the classes of implicational complexity as Π_n. In section 3 we consider properties like uniform interpolation from a mildly abstract point of view. Section 4 provides a proof of *NNIL* interpolation using cut-elimination. Section 5 gives the basics of Kripke models and section 6 adds the basics of subsimulations between models. In this section we prove the promised result that the *NNIL*-formulas are precisely the ones preserved under submodels. Section 7 contains the proof of uniform *NNIL* interpolation by model-theoretical means. In section 8 we prove that (uniform) right interpolation also holds for Π_2, but that uniform left interpolation fails for Π_2 and that right interpolation fails for Π_3. Appendix A contains a characterization of *IPC* as a fragment of Predicate Logic. In appendix B we develop the notions of simulation appropriate for the model-theoretic characterization of arbitrary Π_n.

Each of the selections:

$$\langle 1,2,3,4 \rangle, \langle 1,2,3,5,6,7\rangle, \langle 1,2,3,5,6,7,8\rangle, \langle 1,2,5,6,\mathrm{A}\rangle, \langle 1,2,5,\mathrm{B}\rangle.$$

can be read as a reasonably selfcontained paper, and, of course, each of their unions can.

2 Syntax and Formula Classes

Let $\mathcal{L}$ be the language of Intuitonistic Propositional Logic *IPC*. We take as connectives: $\wedge$, $\vee$, $\rightarrow$, $\top$ and $\bot$. $\neg A$ is defined as $A \rightarrow \bot$. PV is a fixed set of propositional variables, denoted by $p, q, \ldots$. For definiteness, we stipulate that PV has cardinality $\aleph_0$. None of our results, however, depends upon this assumption. PV together with $\top$, $\bot$ is the set of *atoms*. $A, B, C, \ldots$ are formulas; Γ, Δ, $\Gamma', \ldots$ are finite (possibly empty) sets of formulas. We write Γ,Δ for the union of Γ and Δ ; Γ, A stands for $\Gamma, \{A\}$. Let $\mathcal{P}$ be a set of propositional variables. We write $\mathcal{L}(\mathcal{P})$ for $\mathcal{L}$ restricted to $\mathcal{P}$. Similar notation will be used for other classes of formulas. $\vec{p}, \vec{q}, \vec{r}, \ldots$ will range over *finite* sets of propositional variables.

The substitution operator $A[p := B]$ ("substitute B for all occurrences of p in A") is defined as usual. *PV(A)* is the set of of propositional variables occurring in A.

We define a measure of complexity ρ, which counts the left-nesting of $\rightarrow$, as follows:

- $\rho(p) := \rho(\bot) := \rho(\top) := 0$
- $\rho(A \wedge B) := \rho(A \vee B) := max(\rho(A), \rho(B))$
- $\rho(A \rightarrow B) := max(\rho(A) + 1, \rho(B))$

We define $\Pi_n := \{A \in \mathcal{L} \mid \rho(A) \leq n\}$. This name is explained by the translation presented in remark 2.1.

We will at some points confuse propositional formulas with their equivalence classes modulo *IPC*-provable equivalence. Under this confusion *IPC* becomes the Lindenbaum algebra $\mathcal{H}_{IPC}$, which is the free Heyting algebra[2] on $\aleph_0$ generators.

Remark 2.1 (connection with Predicate Logic) We consider $\mathcal{L}_{Krip}$, the language of Predicate Logic with constant b, relation symbols $=$, $\leq$ and with infinitely many unary predicate symbols P,Q,R,.... We define the Π_n- and Σ_n-formulas of our language as follows:

- $\Pi_0 := \Sigma_0 :=$ all Boolean combinations of atomic formulas
- Π_{n+1} and Σ_{n+1} are the smallest classes such that:
  - $\Sigma_n \subseteq \Pi_{n+1}$
  - $\Pi_n \subseteq \Sigma_{n+1}$
  - Π_{n+1} is closed under $\wedge$, $\vee$ and $\forall$
  - Σ_{n+1} is closed under $\wedge$, $\vee$ and $\exists$
  - If $A \in \Sigma_{n+1}$ and $B \in \Pi_{n+1}$, then $A \rightarrow B \in \Pi_{n+1}$
  - If $A \in \Pi_{n+1}$ and $B \in \Sigma_{n+1}$, then $A \rightarrow B \in \Sigma_{n+1}$

Of course, our definition coincides, *modulo provable equivalence*, with the more usual one based on the number of quantifier changes of formulas in prenex normal form. We prefer the present definition, since under it all formulas are literally —not merely modulo provable equivalence— in some formula class.

Let *Krip* be the theory in $\mathcal{L}_{Krip}$ consisting of:

- Classical Predicate Logic
- The theory of identity for $=$
- The theory of partial orders for $\leq$ with bottom element b
- Axioms expressing the persistence property: $(Px \wedge x \leq y) \rightarrow Py$, for all unary predicate symbols P

We translate formulas of *IPC* into $\mathcal{L}_{Krip}$. *I(A,x)*, the Kripke translation of A at x, is defined by:

[2]For information about Heyting algebras, see Troelstra and van Dalen 1988b or see Pitts 1992, section 4.

- $I(p,x) := Px$, $I(\bot,x) := \bot$, $I(\top,x) := \top$
- $I(A \wedge B, x) := I(A,x) \wedge I(B,x)$
- $I(A \vee B, x) := I(A,x) \vee I(B,x)$
- $I(A \to B, x) := \forall y \geq x(I(A,y) \to I(B,y))$

We have by a simple induction on A: $I(A,x) \in \Pi_{\rho(A)}$.

Clearly, every Kripke model (see section 5) can be considered as a model of *Krip* and vice versa. We have: $k \Vdash_{\mathbb{K}} A \Leftrightarrow \mathbb{K} \models I(A,k)$ where $\Vdash$ is Kripke forcing (see section ref5) and $\models$ is the satisfaction relation of Predicate Logic. One can show in analogy to a result of Johan van Benthem for modal logic, that every one-variable formula $A(x)$ of $\mathcal{L}_{Krip}$, that is (i) persistent and (ii) preserved under (partial) bisimulations, is *CQC*-provably equivalent to a formula $I(B,x)$ for some $B \in \mathcal{L}$. For a precise formulation of the result and a proof, see appendix A.

As sketched above our measure of complexity corresponds via the Kripke translation with *depth of quantifier alternations*. In modal logic there is a similar correspondence: the relevant measure of complexity is there *depth of box-diamond alternations*. A striking difference between the modal and the intuitionistic case is that the $\Pi_n(\vec{p})$ of modal logic are generally infinite modulo provable equivalence, where, as we will see, the $\Pi_n(\vec{p})$ of *IPC* are finite modulo provable equivalence.

Note, finally, that there are alternative hierarchies both for modal logic and for *IPC* corresponding simply to depth of quantifiers. In *IPC* one counts *depth of implications*; in modal logic one counts *depth of boxes*. For this notion the complexity classes restricted to $\vec{p}$ are finite modulo provable equivalence both in modal logic and in *IPC*. ❑

The subject in this paper is the class $NNIL := \Pi_1$. As is easily seen every formula in Π_{n+1} is provably equivalent to a formula resulting from substituting Π_n-formulas in a *NNIL*-formula.

Consider a *NNIL*-formula. Conjunctions and disjunctions in front of implications can be removed using:

- $\vdash ((A \vee B) \to C) \leftrightarrow ((A \to C) \wedge (B \to C))$
- $\vdash ((A \wedge B) \to C) \leftrightarrow (A \to (B \to C))$

So *NNIL* coincides modulo provable equivalence with $NNIL_0$, the smallest class containing the propositional atoms, closed under conjunction and disjunction and:

- if $A \in NNIL_0$, then $(p \to A) \in NNIL_0$

Theorem 2.2 *$NNIL(\vec{p})$ is finite (modulo provable equivalence).*

Proof. Each element of $NNIL(\vec{p})$ can be rewritten as a conjunction of disjunctions of atoms and elements of the form: $p \to A$, where A is in $NNIL(\vec{p} \backslash \{p\})$. (To arrange that p does not occur in A one uses the fact that $(p \to B(p))$ is equivalent to $(p \to B(\top))$.) So, the result follows immediately with induction on the cardinality of $\vec{p}$. ❑

By our earlier observation that every $\Pi_{n+1}(\vec{p})$-formula can be obtained by substituting $\Pi_n(\vec{p})$-formulas in a *NNIL*-formula, it follows by induction on n, that:

Theorem 2.3 $\Pi_n(\vec{p})$ *is finite (modulo provable equivalence).* ❑

$\Pi_n(\vec{p})$ is a distributive lattice under $\wedge$ and $\vee$. Since it is finite, it is also a Heyting algebra, with implication, say: $\to_{n,\vec{p}}$. Note that $\to$ need not be $\to_{n,\vec{p}}$. We *do* have (for $A, B \in \Pi_n(\vec{p})$): $\vdash (A \to_{n,\vec{p}} B) \to (A \to B)$.

We end this section by introducing the various notions of interpolation and approximation. Consider any class of formulas X. Define:

- X satisfies *left-interpolation*[3] (*IPL*) if for every $\vec{p}$, $\vec{q}$ and for every $A \in X(\vec{p})$ and $B \in \mathcal{L}(\vec{q})$ with $\vdash A \to B$, there is an I in $X(\vec{p} \cap \vec{q})$ such that $\vdash A \to I$ and $\vdash I \to B$. I is called the X *left-interpolant*.
- X satisfies *right-interpolation* (*IPR*) if for every $\vec{p}$, $\vec{q}$ and for every $A \in \mathcal{L}(\vec{p})$ and $B \in X(\vec{q})$ with $\vdash A \to B$, there is an I in $X(\vec{p} \cap \vec{q})$ such that $\vdash A \to I$ and $\vdash I \to B$. I is called the X *left-interpolant*.
- X satisfies *uniform left-interpolation* (*UIPL*) if for every B and every $\vec{p}$, there is a formula $B^*(\vec{p}) \in X(\vec{p})$, such that for all $\vec{q}$ and for all $A \in X(\vec{q})$ satisfying $\vec{q} \cap PV(B) \subseteq \vec{p}$ we have: $\vdash A \to B \Leftrightarrow \vdash A \to B^*(\vec{p})$. We call $B^*(\vec{p})$ the uniform X *left-interpolant* of B.
- X satisfies *uniform right-interpolation* (*UIPR*) if for every A, and $\vec{p}$, there is a formula $A^\circ(\vec{p}) \in X(\vec{p})$, such that for all $\vec{q}$ and for all $B \in X(\vec{q})$ satisfying $PV(A) \cap \vec{q} \subseteq \vec{p}$ we have: $\vdash A \to B \Leftrightarrow \vdash A^\circ(\vec{p}) \to B$. We call $A^\circ(\vec{p})$ the uniform X right-interpolant of A.
- X satisfies *left-approximation* (*APL*) if for every B, there is a formula $B^* \in X$, such that for all $A \in X$: $\vdash A \to B \Leftrightarrow \vdash A \to B^*$.
- X satisfies *right-approximation* (*APR*) if for every A, there is a formula $A^\circ \in X$, such that for all $B \in X$: $\vdash A \to B \Leftrightarrow \vdash A^\circ \to B$.

3 Closure and Interior Operations on the Lindenbaum algebra of *IPC*

Many important classes of formulas can be represented (modulo provable equivalence) as sets of fixed points of (or, equivalently, ranges of) closure (interior) operations on the Lindenbaum algebra of *IPC*, $\mathcal{H}_{IPC}$. This algebra is isomorphic to the free Heyting algebra on $\aleph_0$ generators. Let H be the domain of $\mathcal{H}_{IPC}$. We write $\mathcal{H}_{IPC}(\vec{p})$ for the subalgebra obtained by restricting $\mathcal{H}_{IPC}$ to $H(\vec{p})$, the set of elements generated by $\vec{p}$. $\mathcal{H}_{IPC}(\vec{p})$ is isomorphic to the free Heyting algebra generated from $\vec{p}$.

[3] Our use of *left* and *right* is determined by whether the element of X is on the lefthand side or on the righthand side in the relevant occurrence of '$\vdash A \to B$'.

Note that if A is in $H(\vec{p})$ and in $H(\vec{q})$, then it is in $H(\vec{p}\cap\vec{q})$. Thus we can coherently define $PV_0(A)$ as the minimum set of generators $\vec{p}$, such that A is in $H(\vec{p})$. In this section we will notationally confuse formulas and their equivalence classes (or their standard interpretations in $\mathcal{H}_{IPC}$). Note that $PV(A)$ (for *formula* A)) is not necessarily identical to $PV_0(A)$ (for the corresponding *equivalence class* A). E.g., $PV(p \to p) = \{p\}$ and $PV_0(p \to p) = \emptyset$.

$\Phi : \mathcal{H}_{IPC} \to \mathcal{H}_{IPC}$ is a *closure operation* if:

- $A \leq \Phi(A)$ (Φ is increasing)
- $A \leq B \Rightarrow \Phi(A) \leq \Phi(B)$ (Φ is monotonic)
- $\Phi(A) = \Phi(\Phi(A))$ (Φ is idempotent)

Ψ is a *interior operation* if Ψ is monotonic, idempotent and has the following property:

- $\Psi(A) \leq A$ (Ψ is decreasing)

Note that if $X = \Phi(H)$ for some closure operation Φ, then Φ is completely determined by X, since: $\Phi(A) = Min(\{B \in X \mid A \leq B\})$. Thus $\Phi(A)$ is the *smallest upper X-approximation* of A. Similarly, for interior operations Ψ we have: $\Psi(A) = Max(\{B \in X \mid B \leq A\})$, *the greatest lower X-approximation* of A. We will sometimes write Φ_X and X_Φ to express the dependence on X and a closure operation Φ. We will use Ψ_X for a corresponding interior operation.

The correspondence between closure (interior) operations and subsets of H can also be viewed as follows. Let $\mathcal{X}$ be the partial order given by X with the restriction to X of $\leq$, the ordering of $\mathcal{H}_{IPC}$. Let emb_X be the embedding of $\mathcal{X}$ in $\mathcal{H}_{IPC}$, considered as a partial order. emb_X is order preserving and hence can be considered as a functor between $\mathcal{X}$ and $\mathcal{H}_{IPC}$ considered as categories. A closure operation Φ, such that X is the range of Φ, corresponds precisely to a left adjoint of emb_X. This means that Φ can be viewed as the unique functor (if there is one) from $\mathcal{H}_{IPC}$ to $\mathcal{X}$, which satisfies:

$$\forall A \in H \, \forall B \in X \, (\Phi(A) \leq B \Leftrightarrow A \leq emb_X(B))$$

Similarly, an interior operation Ψ, such that X is the range of Ψ, corresponds to a right adjoint of emb_X. I.e., Ψ is the unique functor (if there is one) satsfying:

$$\forall A \in X \, \forall B \in H \, (A \leq \Psi(B) \Leftrightarrow emb_X(A) \leq B)$$

We will view $\mathcal{X}$ as a substructure of $\mathcal{H}_{IPC}$ and, thus, we will employ these equivalences, suppressing the 'emb_X'.

It is well-known that, if Φ is a closure operation, then X_Φ is closed under $\wedge$. Similarly, if Ψ is a interior operation, then X_Ψ is closed under $\vee$.

Theorem 3.1 *Suppose X is finite and closed under conjunction (disjunction), then Φ_X (Ψ_X) exists.*

Proof. Trivial. ❑

We call an operation Θ *propositional variable preserving* or, briefly, *preserving* if:

- $PV_0(\Theta(A)) \subseteq PV_0(A)$.

Equivalently Θ is preserving if for all $\vec{p}$ $\Theta(H(\vec{p})) \subseteq H(\vec{p})$.

We state a simple sufficient condition for closure (interior) operations to be preserving. It is clear that a permutation of the propositional variables induces a unique permutation of H.

Theorem 3.2 *Suppose that Θ is a closure (interior) operation and that $X := X_\Theta$ is closed under permutations of propositional variables. Then Θ is preserving.*

Proof. Suppose e.g., that Θ is a closure operation and that X is closed under permutations of the propositional variables. We have:

$$\begin{aligned} \Theta\sigma A \leq B \in X \quad & \Leftrightarrow \quad \sigma A \leq B \in X \\ & \Leftrightarrow \quad A \leq \sigma^{-1} B \in X \\ & \Leftrightarrow \quad \Theta A \leq \sigma^{-1} B \in X \\ & \Leftrightarrow \quad \sigma\Theta A \leq B \in X \end{aligned}$$

Since ΘA is in X, $\sigma\theta A$ is also in X and, hence, $\Theta\sigma A = \sigma\Theta A$. Suppose $A \in H(\vec{p})$ and $\Theta(A) \in H(\vec{p},\vec{q})$, where $\vec{q}$ is disjoint from $\vec{p}$. Let $\vec{r}$ be of the same cardinality as $\vec{q}$ and disjoint from both $\vec{p}$ and $\vec{q}$. Take σ some permutation that leaves $\vec{x}$ fixed and interchanges $\vec{q}$ and $\vec{r}$. We have:

$$\Theta(A) = \Theta\sigma A = \sigma\Theta A \in H(\vec{p},\vec{r}).$$

Hence $\Theta(A) \in H(\vec{p},\vec{q}) \cap H(\vec{p},\vec{r}) = H(\vec{p})$. ❑

X is called *PV-finite* if, for each $\vec{p}$, $X(\vec{p})$ is finite. A closure (interior) operation is called PV-finite if its image is.

A well-known example of a PV-finite, preserving closure operation is double negation $\neg\neg$. $X_{\neg\neg}$ is the class of stable formulas.

A major discovery on *IPC* is the result by A. Pitts (Pitts' Uniform Interpolation Theorem, see Pitts 1992, Theorem 1). Pitts' Theorem can be rendered in our terminology as follows:

Theorem 3.3 (Pitts' Uniform Interpolation Theorem) *For every $\vec{p}$ both $\Phi_{H(\vec{p})}$ and $\Psi_{H(\vec{p})}$ exist. Both operations are preserving.* ❑

We define: $\mathcal{E}\vec{p} := \Phi_{H(\vec{p})}$ and $\mathcal{A}\vec{p} := \Psi_{H(\vec{p})}$. Equivalently Pitts' result can be viewed as providing a preserving closure operation $\exists\vec{p} := \Phi_{H(PV\setminus\vec{p})}$ and a preserving interior operation $\forall\vec{p} := \Phi_{H(PV\setminus\vec{p})}$. Note that e.g.,

$$\exists p A \leq B \in H(PV\setminus\{p\}) \Leftrightarrow A \leq B \in H(PV\setminus\{p\}).$$

This shows that $\exists p$ has the defining property of existential quantification. To see the equivalence of our two fromulations, we may e.g., define:

- $\mathcal{E}\vec{p}A := \exists\vec{q}A$, where $\vec{q}$ is finite set of variables, disjoint from $\vec{p}$, such that $A \in H(\vec{p},\vec{q})$.

Since $H(\vec{p}) \subseteq H(PV\backslash\vec{q})$, it is immediate that:

$$\exists\vec{q}A \leq B \in H(\vec{p}) \Leftrightarrow A \leq B \in H(\vec{p}).$$

Conversely define:

- $\exists\vec{p}A := \mathcal{E}\vec{q}A$, where $\vec{q} := PV_0(A)\backslash\vec{p}$.

We have, using ordinary interpolation:

$$\begin{array}{rcl} \mathcal{E}\vec{q}A \leq B \in H(PV\backslash\vec{p}) & \Leftrightarrow & \exists I{\in}H(\vec{q})\ \mathcal{E}\vec{q}A \leq I \leq B \in H(PV\backslash\vec{p}) \\ & \Leftrightarrow & \exists I{\in}H(\vec{q})\ A \leq I \leq B \in H(PV\backslash\vec{p}) \\ & \Leftrightarrow & A \leq B \in H(PV\backslash\vec{p}) \end{array}$$

S. Ghilardi and M. Zawadowski in their Ghilardi and Zawadowski ??, give an alternative proof of Pitts' Theorem using Kripke models. The work of Ghilardi and Zawadowski provides a Kripke semantics for the propositional quantifiers.

Lemma 3.4 *Consider a closure operation Φ and an interior operation Ψ. Let $X := X_\Phi$ and $Y := Y_\Psi$. Then: X is closed under Ψ iff Y is closed under Φ.*

Proof. We prove the left-to-right direction. Right-to-left is dual. Suppose X is closed under Ψ. Let A be in Y. We have: $A \leq \Phi(A) \in X$. Hence: $A = \Psi(A) \leq \Psi(\Phi(A)) \in X$. And so: $\Phi(A) \leq \Phi(\Psi(\Phi(A))) = \Psi(\Phi(A))$. On the other hand: $\Psi(\Phi(A)) \leq \Phi(A)$. Ergo: $\Phi(A) = \Psi(\Phi(A))$, and so $\Phi(A) \in Y$. ❑

Theorem 3.5 *Let Φ be a closure operation, then Φ is preserving iff X_Φ is closed under $\mathcal{A}\vec{p}$ for all $\vec{p}$. Similarly, let Ψ be an interior operation, then Ψ is preserving iff X_Φ is closed under $\mathcal{E}\vec{p}$ for all $\vec{p}$.*

Proof. Immediate by lemma 3.4. ❑

It follows for example that $\mathcal{A}\vec{p}\neg\neg A$ is stable.

Theorem 3.6 (Glueing Theorem) *Consider any subset X of $\mathcal{H}_{IPC}$. Then X is the image of a preserving closure operation iff each $X(\vec{p})$ is the image of a preserving closure operation. Similarly for interior.*

Proof. Suppose X is the image of the preserving closure operation Φ. It is easily seen that $\mathcal{E}\vec{p}\circ\Phi$[4] is a preserving closure operation with image $X(\vec{p})$. E.g., by the fact that Φ is preserving:

$$\mathcal{E}\vec{p}\circ\Phi\circ\mathcal{E}\vec{p}\circ\Phi = \mathcal{E}\vec{p}\circ\Phi\circ\Phi = \mathcal{E}\vec{p}\circ\Phi$$

.

[4] We read composition in the order of application.

Conversely suppose that for each $\vec{p}$ $\Theta_{\vec{p}}$ is a preserving closure operation with image $X(\vec{p})$. Note that $\Theta_{\vec{q}}(A) \in X(\vec{q} \cap PV_0(A))$. Consider any A and any $\vec{q} \supseteq PV_0(A)$. We claim: $\Theta_{\vec{q}}(A) = \Theta_{PV_0(A)}(A)$. First $\Theta_{\vec{q}}(A) \in X(PV_0(A))$ and $A \leq \Theta_{\vec{q}}(A)$, hence: $\Theta_{PV_0(A)}(A) \leq \Theta_{PV_0(A)}(\Theta_{\vec{q}}(A)) = \Theta_{\vec{q}}(A)$. Conversely $PV_0(\Theta_{PV_0(A)}(A)) \subseteq PV_0(A) \subseteq \vec{q}$ and $A \leq \Theta_{PV_0(A)}(A)$, hence: $\Theta_{\vec{q}}(A) \leq \Theta_{PV_0(A)}(A)$.

Set: $\Phi(A) := \Theta_{PV_0(A)}(A)$. Consider any B with $B \in X$ and any $\vec{q}$ with $\vec{q} \supseteq PV_0(A) \cup PV_0(B)$. We find:

$$\begin{aligned} \Theta_{PV_0(A)}(A) \leq B \in H(\vec{q}) \quad &\Leftrightarrow \quad \Theta_{\vec{q}}(A) \leq B \in H(\vec{q}) \\ &\Leftrightarrow \quad A \leq B \in H(\vec{q}) \end{aligned}$$

❑

It follows that if X is PV-finite and closed under conjunction, then X is the image of a preserving closure operation if each $\Phi_{X(\vec{p})}$ is preserving.

Theorem 3.7 *Suppose that X is the image of a preserving closure operation Φ and that X is closed under $\mathcal{E}\vec{p}$, then $\Phi \circ \mathcal{E}\vec{p} = \mathcal{E}\vec{p} \circ \Phi$.*

Proof. Left to the reader. ❑

We want to connect closure and interior operations to the notions of interpolation and approximation introduced in section 2. Some care should be taken here, since we switch between the algebraic and the syntactic. It is easy to formulate algebraic variants of interpolation and approximation. E.g., the algebraic counterpart of *IPL* looks as follows. Let $X \subseteq H$. We have:

- X satisfies *left-interpolation* (*IPL*) if for every $\vec{p}$, $\vec{q}$ and for every $A \in X(\vec{p})$ and $B \in H(\vec{q})$ with $A \leq B$, there is an I in $X(\vec{p} \cap \vec{q})$ such that $A \leq I \leq B$. I is called the X *left-interpolant*.

It is easy to see that if we have one of our interpolation or approximation properties for the *syntactical* class X, then ipso facto we have the property for the corresponding algebraic class. The converse, however, does not hold. The problem is that we do not demand that the syntactic class is closed under provable equivalence. Consider for example the syntactic class $\{(p \wedge \neg p), (q \to q)\}$. This class does not satisfy *IPL*. On the other hand, its algebraic counterpart is $\{\top, \bot\}$, which *does* satify *IPL*. It is not difficult to see that it is sufficient for making the transition from an algebraic property to the corresponding syntactical one, that X is *variable-sound*. Variable soundness is defined as follows:

- X is *variable-sound* iff for all $A \in X$ there is a $B \in X$ such that A is provably equivalent with B and $PV(B) = PV_0(A)$.

If X is closed under substitution of $\top$, then X is variable-sound.

Theorem 3.8 *Consider any $X \subseteq H$. Suppose X satisfies* IPR*, X is PV-finite and X is closed under finite conjunctions, then X is —modulo provable equivalence— the image of a preserving closure operation. Similarly for IPL, disjunctions and interior.*

Proof. Consider $A \in H$ and let $\vec{p} := PV_0(A)$. Define:

$$\Phi(A) := \bigwedge\{C{\in}X(\vec{p}) \mid A \leq C\}.$$

Clearly, $PV_0(\Phi(A)) \subseteq PV_0(A))$. Moreover:

$$\begin{array}{rcl} \Phi(A) \leq B \in X & \Leftrightarrow & \exists I{\in}X(\vec{p})\ \ \Phi_X(A) \leq I \leq B \in X \\ & \Leftrightarrow & \exists I{\in}X(\vec{p})\ \ \Phi(A) \leq I \leq B \in X \\ & \Leftrightarrow & \exists I{\in}X(\vec{p})\ \ A \leq I \leq B \in X \\ & \Leftrightarrow & A \leq B \in X \end{array}$$

Hence Φ is a preserving closure operation with range X. ❑

Theorem 3.9 *X satisfies APR(L) iff X is the image of a closure (interior) operation. (In fact $(.)^*$ and $(.)^\circ$* are *the desired operations.)*

Proof. Trivial. ❑

Theorem 3.10 *The following are equivalent:*

1. *X is the image of a preserving closure (interior) operation*
2. *X satisfies UIPR(L)*
3. *X satisfies IPR(L) and APR(L)*

Proof.

(1) $\Rightarrow$ (2) Suppose e.g., X is the image of a preserving closure operation Φ. Consider any $A \in H(\vec{q})$ and any $\vec{p}$. Take $A^\circ(\vec{p}) := \Phi\mathcal{E}\vec{p}(A)$. Since both Φ and $\mathcal{E}\vec{p}$ are closure operations, we have: $A \leq A^\circ(\vec{p})$. Since Φ is preserving, we find that $A^\circ(\vec{p})$ is in $H(\vec{p})$. Consider any B and $\vec{r}$ with $B \in X(\vec{r})$ and assume that $\vec{q}{\cap}\vec{r} \subseteq \vec{p}$. Using ordinary interpolation, we obtain:

$$\begin{array}{rcl} A \leq B & \Leftrightarrow & \exists I{\in}H(\vec{p})\ \ A \leq I \leq B \\ & \Leftrightarrow & \exists I{\in}H(\vec{p})\ \ \mathcal{E}\vec{p}A \leq I \leq B \\ & \Leftrightarrow & \mathcal{E}\vec{p}A \leq B \\ & \Leftrightarrow & \Phi\mathcal{E}\vec{p}A \leq B \end{array}$$

(2) $\Rightarrow$ (3) Clearly, *UIPR(L)* implies *IPR(L)*. To get e.g., *APR* from *UIPR*, take: $A^\circ := A^\circ(PV(A))$.

(3) $\Rightarrow$ (1) Suppose e.g., *IPR* and *APR*. Define $\Phi(A) := A^\circ$. It is immediate that Φ is a closure operation. We have $A \leq A^\circ$ and $A^\circ \in X$. So by *IPR* there is a $B \in X(PV(A))$ with $A \leq B \leq A^\circ$. On the other hand, by the defining property of $(.)^\circ$, $A^\circ \leq B$ and, hence, $A^\circ = B$. We conclude that Φ is preserving. ❑

4 Interpolation, the Proof-theoretic Approach

We prove that *NNIL* satisfies both left- and right-interpolation by a proof-theoretic argument. The proof consists in constructing the interpolant I from a (cut-free) proof of $A \vdash B$ in a sequent calculus. The results of section 3 imply that *NNIL* also satisfies uniform left- and right interpolation, since *NNIL* is PV-finite and closed under disjunction and conjunction.

4.1 Basic Definitions

Definition 4.1.1 (Positive and negative occurrences) We define the set of all positively [negatively] occurring propositional variables in A, $PV^+(A)$ [$PV^-(A)$], by:

- $PV^+(\top) = PV^-(\top) = PV^+(\bot) = PV^-(\bot) = \emptyset$
- $PV^+(p) = \{p\}$, $PV^-(p) = \emptyset$
- $PV^+(A \wedge B) = PV^+(A \vee B) = PV^+(A) \cup PV^+(B)$
- $PV^-(A \wedge B) = PV^-(A \vee B) = PV^-(A) \cup PV^-(B)$
- $PV^+(A \to B) = PV^-(B \to A) = PV^-(A) \cup PV^+(B)$

Clearly, $PV(A) = PV^+(A) \cup PV^-(A)$.

Definition 4.1.2 (The derivation system) We use the sequent calculus given in table 1.

p	$\Gamma, p \vdash p$	
$\top$	$\Gamma \vdash \top$	
$\bot$	$\Gamma, \bot \vdash C$	
$\wedge$	$\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge R$	$\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C} \wedge L$
$\vee$	$\dfrac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2} \vee R, i = 1, 2$	$\dfrac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C} \vee L$
$\to$	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \to R$	$\dfrac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C} \to L$

TABLE 1 The Sequence Calculus

Related systems can be found in Schütte 1962 and Takeuti 1975. All these systems are equivalent in the sense that they yield the same class of derivable formulas and that the following rule schemes may be derived:

- *CE* cut elimination: if $\Gamma \vdash A$ and $\Gamma, A \vdash B$ then $\Gamma \vdash B$
- *W* weakening: if $\Gamma \vdash A$ then $\Gamma, \Delta \vdash A$
- *S* substitution: if $\Gamma \vdash A$ then $\Gamma[p := B] \vdash A[p := B]$
- *PS* positive substitution: if $p \notin PV^-(A)$ then $A, (p \to B) \vdash A[p := B]$

The proofs are standard.

4.2 Schütte's Interpolation Method

Schütte gives in Schütte 1962 a method to build an interpolant from a derivation of $A \vdash B$. This method yields for every derivable sequent $\Gamma, \Delta \vdash C$ an interpolant I satisfying:

$$\Gamma \vdash I,\ \Delta, I \vdash C \textit{ and } PV(I) \subseteq PV(\Gamma) \cap PV(\Delta, C).$$

Using the shorthand $\Gamma[I]\Delta \vdash C$ for ($\Gamma \vdash I$ and $\Delta, I \vdash C$), Schütte's method is presented in table 2.

$ip1$	$\Gamma[\top]\Delta, p \vdash p$	$ip2$	$\Gamma, p[p]\Delta \vdash p$
$i\top$	$\Gamma[\top]\Delta \vdash \top$		
$i\bot 1$	$\Gamma[\top]\Delta, \bot \vdash C$	$i\bot 2$	$\Gamma, \bot[\bot]\Delta \vdash C$
$i{\wedge}R$	$\dfrac{\Gamma[I_1]\Delta \vdash A \quad \Gamma[I_2]\Delta \vdash B}{\Gamma[I_1{\wedge}I_2]\Delta \vdash A{\wedge}B}$		
$i{\vee}L1$	$\dfrac{\Gamma[I_1]A, \Delta \vdash C \quad \Gamma[I_2]B, \Delta \vdash C}{\Gamma[I_1{\wedge}I_2]A{\vee}B, \Delta \vdash C}$	$i{\vee}L2$	$\dfrac{\Gamma, A[I_1]\Delta \vdash C \quad \Gamma, B[I_2]\Delta \vdash C}{\Gamma, A{\vee}B[I_1{\vee}I_2]\Delta \vdash C}$
$i{\to}L1$	$\dfrac{\Gamma[I_1]\Delta \vdash A \quad \Gamma[I_2]B, \Delta \vdash C}{\Gamma[I_1{\wedge}I_2]A{\to}B, \Delta \vdash C}$	$i{\to}L2$	$\dfrac{\Delta[I_1]\Gamma \vdash A \quad \Gamma, B[I_2]\Delta \vdash C}{\Gamma, A{\to}B[I_1{\to}I_2]\Delta \vdash C}$

TABLE 2 Schütte's method

We explain this notation with an example. $i{\wedge}R$ means:

if $\Gamma \vdash I_1$ *and* $I_1, \Delta \vdash A$ *and* $\Gamma \vdash I_2$ *and* $I_2, \Delta \vdash B$, *then* $\Gamma \vdash I_1{\wedge}I_2$ *and* $I_1{\wedge}I_2, \Delta \vdash A{\wedge}B$.

So $i{\wedge}R$ indicates how an interpolant for $\Gamma, \Delta \vdash A{\wedge}B$ can be obtained from interpolants for $\Gamma, \Delta \vdash A$ and $\Gamma, \Delta \vdash B$. For rules not mentioned here (${\wedge}L$,

$\vee R$, $\to R$, $\neg R$), the interpolant for the conclusion is the same as for the premise.

Now the Interpolation theorem is proved as follows. Assume $A \vdash B$, then there is a derivation of $A \vdash B$ in the sequent calculus defined in 4.1.2. With induction over the length of the derivation it is shown (using Schütte's method) that any partition $\Gamma, \Delta \vdash C$ of a sequent in the derivation has an interpolant I, i.e.,

$$\Gamma \vdash I, I, \Delta \vdash C \textit{ and } PV(I) \subseteq PV(\Gamma) \cap PV(\Delta, C)$$

Hence $A \vdash B$ has an interpolant.

Applying Schütte's method to derivations of $A \vdash B$ with $A \in \mathit{NNIL}$ does not always yield an $I \in \mathit{NNIL}$:

$$\cfrac{\cfrac{\cfrac{\cfrac{p[p] \vdash p \qquad q[q]p \vdash q}{p \to q[p \to q]p \vdash q}\to L \qquad p, r[r]p \to q \vdash r}{p, q \to r[(p \to q) \to r]p \to q \vdash r} \to L}{p \wedge (q \to r)[(p \to q) \to r]p \to q \vdash r} \wedge L}{p \wedge (q \to r)[(p \to q) \to r] \vdash (p \to q) \to r} \to R$$

It turns out that $(i \to L2)$, the only place where an $\to$ is added to I, has to be modified. This will be done in the next subsection.

4.3 The Proof

We first prove *NNIL* left-interpolation and then *NNIL* right-interpolation.

Lemma 4.3.1 *Assume* $\Gamma, \Delta \vdash C$ *and* $\Gamma \subseteq \mathit{NNIL}_0$. *Then there is an* I *with:*

i) $\Gamma \vdash I$ *and* $I, \Delta \vdash C$
ii) $PV(I) \subseteq PV(\Gamma) \cap PV(\Delta, C)$
iii) $I \in \mathit{NNIL}$
iv) $\{C\} \cap PV^-(I) \subseteq PV(\Delta)$ *(i.e., if* $C \in PV^-(I)$, *then* $C \in PV(\Delta)$*).*

Proof. Induction over the length of a derivation of $\Gamma, \Delta \vdash C$.

If $\Gamma, \Delta \vdash C$ is an axiom or the conclusion of a rule different from $\to L$, apply Schütte's method (4.2). (i)-(iv) follow directly by induction.

If $\Gamma, \Delta \vdash C$ is the conclusion of $\to L$. Let $A \to B$ be the "new" formula in the conclusion. We distinguish two cases: $A \to B \in \Gamma$ or $A \to B \in \Delta$.

Case 1: $A \to B \in \Gamma$. Then $A \to B \in \mathit{NNIL}_0$, so A is a propositional variable, p say. By the induction hypothesis, we have a Γ' with $\Gamma' \cup \{p \to B\} = \Gamma$ and I_1, I_2 with:

a) $\Gamma' \vdash I_1; I_1, \Delta \vdash p$; $\Gamma', B \vdash I_2$; $I_2, \Delta \vdash C$
b) $PV(I_1) \subseteq PV(\Gamma') \cap PV(\Delta, p)$; $PV(I_2) \subseteq PV(\Gamma', B) \cap PV(\Delta, C)$
c) $I_1, I_2 \in \mathit{NNIL}$
d) $p \in PV^-(I_1) \Rightarrow p \in PV(\Delta)$; $\{C\} \cap PV^-(I_2) \subseteq PV(\Delta)$

Now we must find an I and show that (i)-(iv) hold. We consider three subcases.

Subcase 1A: $C = p$. Put $I := I_1$. Ad (i): $\Gamma \vdash I$ follows from $\Gamma' \vdash I_1$ (by (a)), $\Gamma' \subseteq \Gamma$ and W. We get (ii), (iii) and (iv) directly from (b), (c) and (d).

Subcase 1B: $C \neq p$, $p \in PV(\Delta)$. Put $I := I_1 \wedge (p \to I_2)$.

Ad (i):

$$\dfrac{\Gamma', p \to B \vdash I_1 \quad (a,W) \qquad \dfrac{\dfrac{\Gamma', p \vdash p \quad \Gamma', p, B \vdash I_2 \quad (a,W)}{\Gamma', p \to B, p \vdash I_2}}{\Gamma', p \to B \vdash p \to I_2}}{\Gamma', p \to B \vdash I_1 \wedge (p \to I_2) \text{ i.e., } \Gamma \vdash I}$$

$$\dfrac{\dfrac{I_1, \Delta \vdash p \quad (a) \quad I_1, I_2, \Delta \vdash C \quad (a,W)}{I_1, p \to I_2, \Delta \vdash C}}{I_1 \wedge (p \to I_2), \Delta \vdash C \text{ i.e., } I, \Delta \vdash C}$$

Ad (ii): $PV(I_1, I_2) \subseteq PV(\Gamma) \cap PV(\Delta, C)$ is easy; as $\Gamma = \Gamma' \cup \{p \to B\}$ and $p \in PV(\Delta)$ we also have $p \in PV(\Gamma) \cap PV(\Delta, C)$, so

$$PV(I) = PV(I_1) \cup PV(I_2) \cup \{p\} \subseteq PV(\Gamma) \cap PV(\Delta, C).$$

Ad (iii): $I_1, I_2 \in NNIL$ (by (c)), so $I = I_1 \wedge (p \to I_2) \in NNIL$ by definition of *NNIL*.

Ad (iv): Assume $C \in PV^-(I)$, then $C \in PV^-(I_1) \cup PV^-(I_2)$, for $C \neq p$. Now if $C \in PV^-(I_1)$ then $C \in PV(\Delta) \cup \{p\}$ by (b), so $C \in PV(\Delta)$ (for $C \neq p$); and if $C \in PV^-(I_2)$ then $C \in PV(\Delta)$ by (d). Conclusion: $C \in PV(\Delta)$ and (iv) is proved.

Subcase 1C: $C \neq p$, $p \notin PV(\Delta)$. Put $I := I_1\ [p := I_2]$.

Ad (i): As in Subcase 1B, we have $\Gamma', p \to B \vdash I_1 \wedge (p \to I_2)$; by (d) and $p \notin PV(\Delta)$ we have $p \notin PV^-(I_1)$, so $I_1, (p \to I_2) \vdash I$ by *PS*; now apply *CE* and we get $\Gamma', p \to B \vdash I$, i.e., $\Gamma \vdash I$. Furthermore:

$$\dfrac{\dfrac{I_1, \Delta \vdash p}{I_1[p := I_2], \Delta \vdash I_2 \quad (p \notin PV(\Delta))}\,S \qquad I_2, \Delta \vdash C}{I_1[p := I_2], \Delta \vdash C \text{ i.e., } I, \Delta \vdash C}\,CE$$

Ad (ii):

$$\begin{array}{rcl} PV(I) & = & (PV(I_1) \backslash \{p\}) \cup PV(I_2) \\ & \subseteq & (PV(\Gamma') \cap PV(\Delta, p) \backslash \{p\}) \cup (PV(\Gamma', B) \cap PV(\Delta, C)) \\ & = & PV(\Gamma', B) \cap PV(\Delta, C) \subseteq PV(\Gamma) \cap PV(\Delta, C) \end{array}$$

using (b).

Ad (iii): $I \in NNIL$ follows from (c) and $p \notin PV^-(I_1)$, a consequence of (d) and $p \notin PV(\Delta)$.

Ad (iv): Assume $C \in PV^-(I)$, then $C \in PV^-(I_1) \cup PV^-(I_2)$ (for $p \notin PV^-(I_1)$, see (i)). Now continue as for (iv) under 1B.

Case 2: $A \to B \in \Delta$. Apply $(i \to L1)$ of Schütte's method: it yields an interpolant $I = I_1 \wedge I_2$ and (i)-(iv) follow directly. ❑

As a corollary, we immediately have *NNIL* left-interpolation (using that *NNIL* and $NNIL_0$ coincide modulo *IPC*-provable equivalence). Right- interpolation is somewhat easier. We prove it now.

Lemma 4.3.2 *If* $\Gamma, \Delta \vdash C$*, then there is an* I *with*

i) $\Gamma \vdash I$*;* $I, \Delta \vdash C$

ii) $PV(I) \subseteq PV(\Gamma) \cap PV(\Delta, C)$

iii) if $C \in NNIL_0$ *and* $\Delta \subseteq PV$*, then* $I \in NNIL_0$

iv) if $\Gamma \subseteq PV$*, then* $I = p_1 \wedge \ldots \wedge p_n$ *for some* $p_1, \ldots, p_n \in PV$*.*

Proof. Induction over the length of a derivation of $\Gamma, \Delta \vdash C$.

If $\Gamma, \Delta \vdash C$ is an axiom or the conclusion of a rule different from $\to L$, then apply Schütte's method. If $\Gamma, \Delta \vdash C$ is the conclusion of $\to L$, then we distinguish three cases.

Case 1: $A \to B \in \Gamma$, $C \in NNIL_0$, $\Delta \subseteq PV$. Here $i \to L2$ of Schütte's method prescribes the interpolant $I_1 \to I_2$. This interpolant satisfies (i) and (ii), but in general not (iii) (only if $I_1 \in PV$). However, I_1 is the interpolant of $\Delta, \Gamma' \vdash A$ (with Γ' such that $\Gamma = \Gamma' \cup \{A \to B\}$) and $\Delta \subseteq PV$, so (by (iv) of the induction hypothesis) $I = p_1 \wedge \ldots \wedge p_n$. Now put $I := p_1 \to (\ldots(p_n \to I_2)\ldots)$, then $I \equiv I_1 \to I_2$ so I satisfies (i) and (ii); also $I \in NNIL_0$ for $I_2 \in NNIL_0$ (by induction hypothesis). (iv) is trivially satisfied.

Case 2: $A \to B \in \Gamma$, ($C \notin NNIL_0$ or $\Delta \backslash PV \neq \emptyset$). Now follow $I \to L2$ of Schütte's method, then (i), (ii) are satisfied, (iii) and (iv) are trivially true.

Case 3: $A \to B \notin \Gamma$. Then $A \to B \in \Delta$. Now follow $i \to L1$ of Schütte's method: this yields an interpolant $I = I_1 \wedge I_2$ for which (i), (ii) hold. (iii) is trivially true (for $A \to B \in \Delta$) and (iv) follows by the induction hypothesis. ❑

As a corollary we have *NNIL* right-interpolation.

Remark 4.3.3 (Positive and negative occurrence) Schütte's method yields an interpolant I for $A \vdash B$ with:

$\pm$ $PV^+(I) \subseteq PV^+(A) \cap PV^+(B)$, $PV^-(I) \subseteq PV^-(A) \cap PV^-(B)$

However, our adaptation of Schütte's method used in 4.3.1 does not respect ($\pm$): e.g., in subcase 1B we have $p \in PV^-(I)$, but $p \in PV^+(\Delta) \cup PV^-(C)$ is not excluded. We therefore state the following open problem: does *NNIL* interpolation hold if ($\pm$) is added? ❑

5 Kripke Models

We suppose that the reader is familiar with Kripke models for *IPC* (see for example Troelstra and van Dalen 1988a, or Smoryński 1973). To fix notations: a Kripke model is a structure $\mathbb{K} = \langle K, b, \leq, \mathcal{P}, \Vdash \rangle$, where K is a non-empty set of nodes; $\leq$ is a partial ordering; $b \in K$ is the bottom element w.r.t. $\leq$; $\mathcal{P}$ is a set of propositional variables; $\Vdash$ is the atomic forcing relation on $\mathcal{P}$: it is a relation between nodes and propositional variables in $\mathcal{P}$, satisfying:

$$k{\leq}k' \textit{ and } k \Vdash p \Rightarrow k' \Vdash p \quad \textit{(persistence).}$$

$\Vdash$ is extended to $\mathcal{L}(\mathcal{P})$ in the standard way. The resulting relation is again persistent. We will say that $\mathbb{K}$ is a $\mathcal{P}$*-model* if its set of propositional variables is $\mathcal{P}$. A model is *finite* if all its components are finite.

Our Kripke models are what is usually called *rooted Kripke models.* In many contexts it is more natural to omit the root. However, for the purposes of the present paper it is more convenient to have all our models rooted.

We write $\mathbb{K} \Vdash A$ for: $b \Vdash A$ (or equivalently: $\forall k \in K\ k \Vdash A$). Let $\vec{p}$ be a finite set of propositional variables. We will write $\mathbb{K}(\vec{p})$ for the result of restricting the atomic forcing of $\mathbb{K}$ to $\vec{p}$. For any $k{\in}K$, $\mathbb{K}[k]$ is the model $\langle K', k, \leq', \mathcal{P}, \Vdash' \rangle$, where $K' := \{k' \mid k{\leq}k'\}$ and where $\leq'$ and $\Vdash'$ are the restrictions of $\leq$ respectively $\Vdash$ to K'. (We will often simply write $\leq$ and $\Vdash$ for $\leq'$ and $\Vdash'$.) We write $Th(\mathbb{K}) := \{A{\in}\mathcal{L} \mid \mathbb{K} \Vdash A\}$, $Th_X(\mathbb{K}) := \{A{\in}X \mid \mathbb{K} \Vdash A\}$, where X is a set of formulas. We will often write $Th_X(k)$ for $Th_X(\mathbb{K}[k])$.

The central result connecting structures and language is:

Theorem 5.1 (Kripke Completeness) *We have:*

$$IPC \vdash A \Leftrightarrow \textit{ For all (finite) } PV(A)\textit{-models } \mathbb{K}: \ \mathbb{K} \Vdash A. \qquad \square$$

6 Subsimulations

We start by repeating some well-known notions about relations. Define:

- $x(R \circ S)y :\Leftrightarrow \exists z\ xRzSy$
- $x\widehat{R}y :\Leftrightarrow yRx$
- $ID_X \subseteq X \times X$ and $xID_Xy :\Leftrightarrow x = y$, i.e., $ID_X = \{\langle x, x\rangle \mid x \in X\}$

Let $\mathbb{K}$ an $\mathbb{M}$ be $\mathcal{P}$-models. A relation R on $K \times M$ has *the zig-property (w.r.t* $\mathbb{K}$ *and* $\mathbb{M}$ *)* if:

- $kRm \Rightarrow \forall p{\in}\mathcal{P}(k \Vdash p \Leftrightarrow m \Vdash p)$
- $k' \geq kRm \Rightarrow \exists m'\ k'Rm' \geq m$, i.e., $\geq \circ R \subseteq R \circ \geq$

We will say that R is *zig.* We do *not* require that R preserves roots, i.e., $b_{\mathbb{K}}Rb_{\mathbb{M}}$. We will also call R a *subsimulation of* $\mathbb{K}$ *in* $\mathbb{M}$. The "sub" witnesses that roots are not necessarily preserved. Note that the empty relation is a subsimulation between any two models.

R is *total* if $\forall k{\in}K\ \exists m{\in}M\ kRm$. If R is zig and root-preserving we say that R is *+-zig*. We will say that a +-zig R is a *simulation*. Note that simulations are, in our context, automatically total. R is *zag* if $\widehat{R}$ is zig, etcetera. Define:

- $R : \mathbb{K} \preceq \mathbb{M} :\Leftrightarrow R$ is a total subsimulation of $\mathbb{K}$ in $\mathbb{M}$
- $\mathbb{K} \preceq \mathbb{M} :\Leftrightarrow \exists R\ R : \mathbb{K} \preceq \mathbb{M}$
- $R : \mathbb{K} \preceq^{+} \mathbb{M} :\Leftrightarrow R$ is a simulation of $\mathbb{K}$ in $\mathbb{M}$
- $\mathbb{K} \preceq^{+} \mathbb{M} :\Leftrightarrow \exists R\ R : \mathbb{K} \preceq^{+} \mathbb{M}$

The existence of simulations and the existence of total subsimulations is related in essentially simple ways. It is good to have total subsimulations since they are "definable" in a way in which subsimulations are not (see section 7). It is good to have simulations, since they are better for constructions on models. We give two connections between the two notions. First note that :

$$\mathbb{K} \preceq \mathbb{M} \Rightarrow \textit{ for all } k{\in}K \textit{ there is an } m{\in}M :\ \mathbb{K}[k] \preceq^{+} \mathbb{M}[m].$$

Moreover: $\mathbb{K} \preceq \mathbb{M} \Leftrightarrow$ for some $m \in M$: $\mathbb{K} \preceq^{+} \mathbb{M}[m]$. Secondly take $\mathbb{K}^{+}$ to be the result of adding a new root $\mathfrak{b}$ to $\mathbb{K}$ with:

- $\mathfrak{b} \Vdash p :\Leftrightarrow \mathbb{K} \Vdash p$ *and* $\mathbb{M} \Vdash p$

We have: $\mathbb{K} \preceq \mathbb{M} \Leftrightarrow \mathbb{K}^{+} \preceq^{+} \mathbb{M}$. We leave the simple verifications to the reader.

We list the basic facts about (sub)simulations.

Theorem 6.1 *Let $\mathfrak{P}$ be a set of subsimulations. Then $\bigcup\mathfrak{P}$ is a subsimulation. It follows that the set of subsimulations has a maximum. Moreover if one of the elements of $\mathfrak{P}$ is total (root-preserving), then $\bigcup\mathfrak{P}$ is total (root-preserving).*

Proof. trivial. ❑

Note that R defined by: $kRm :\Leftrightarrow \mathbb{K}[k] \preceq^{+} \mathbb{M}[m]$, is the maximum subsimulation.

Theorem 6.2 *We have:*

- $R : \mathbb{K} \preceq^{(+)} \mathbb{M}$ *and* $S : \mathbb{M} \preceq^{(+)} \mathbb{N} \Rightarrow R \circ S : \mathbb{K} \preceq^{(+)} \mathbb{N}$
- $ID_K : \mathbb{K} \preceq^{+} \mathbb{K}$

Proof. Trivial. ❑

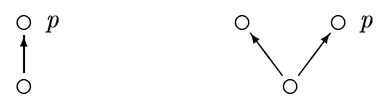
6.2 tells us that $\preceq^{(+)}$ is a preorder. We write $\equiv^{(+)}$ for the induced equivalence relation. Note that if $\mathbb{K} \equiv \mathbb{M}$ and $R : \mathbb{K} \preceq \mathbb{M}$, then $R \cup \{\langle b_{\mathbb{K}}, b_{\mathbb{M}}\rangle\}$ is a simulation. It follows that $\mathbb{K} \equiv^{+} \mathbb{M}$. So $\equiv$ and $\equiv^{+}$ coincide.

A relation R between $\mathbb{K}$ and $\mathbb{M}$ is a *bisimulation* if R and $\widehat{R}$ are both subsimulations. If a bisimulation is total and surjective, we can always extend it to preserve roots. We write:

- $R : \mathbb{K} \simeq \mathbb{M}$ for: R is a total, surjective bisimulation between $\mathbb{K}$ and $\mathbb{M}$
- $\mathbb{K} \simeq \mathbb{M} :\Leftrightarrow \exists R\ R : \mathbb{K} \simeq \mathbb{M}$

It is easy to see that $\simeq$ is an equivalence relation and that: $\mathbb{K} \simeq \mathbb{M} \Rightarrow \mathbb{K} \equiv \mathbb{M}$. Bisimulations are closed under unions, so the set of bisimulations between $\mathbb{K}$ and $\mathbb{M}$ has a maximum. Note that R with $kRm :\Leftrightarrow \mathbb{K}[k] \simeq \mathbb{M}[m]$, is the maximal bisimulation between $\mathbb{K}$ and $\mathbb{M}$.

Example 6.3 The following is an example of two models $\mathbb{U}$ and $\mathbb{V}$ with $\mathbb{U} \equiv \mathbb{V}$ but not $\mathbb{U} \simeq \mathbb{V}$.



$\mathbb{U}$ $\qquad\qquad\qquad\qquad$ $\mathbb{V}$

(If an atom is not displayed at a node, then it is not forced.) ❑

In section 7 we will see that, if we restrict ourselves to $\vec{p}$-models, then the number of $\equiv$-equivalence classes is finite (in contrast to the number of $\simeq$-equivalence classes). In the two subsequent theorems we relate total subsimulations and simulations with the behaviour of models on the formulas of *IPC*. A converse of 6.4 (for $\vec{p}$-models) will be proved in section 7. (6.5 has a number of somewhat weakened converses, but we won't prove them in this paper.)

Theorem 6.4 *Let R be zig. Then: $kRm \Rightarrow Th_{NNIL}(m) \subseteq Th_{NNIL}(k)$. It follows immediately that: $\mathbb{K} \preceq \mathbb{M} \Rightarrow Th_{NNIL}(\mathbb{M}) \subseteq Th_{NNIL}(\mathbb{K})$.*

Proof. By induction on $NNIL_0$. E.g., suppose $kRm \Vdash (p \to A)$ for $A \in NNIL_0$ and $k \leq k' \Vdash p$. Then for some m': $k'Rm' \geq m$ and hence, $m' \Vdash p$. Since $m' \geq m \Vdash (p \to A)$, it follows that $m' \Vdash A$ and hence by the Induction Hypothesis: $k' \Vdash A$. We may conclude that $k \Vdash (p \to A)$. ❑

Theorem 6.5 $\mathbb{K} \simeq \mathbb{M} \Rightarrow Th(\mathbb{K}) = Th(\mathbb{M})$.

Proof. By induction on $\mathcal{L}$. ❑

Define:

- $\mathbb{K} \preceq_1 \mathbb{M} :\Leftrightarrow \exists F(F : \mathbb{K} \preceq \mathbb{M}$ and F is a function)
- $\mathbb{K} \subseteq \mathbb{M} :\Leftrightarrow K \subseteq M$, $\leq_{\mathbb{K}} \subseteq \leq_{\mathbb{M}}$ and $\Vdash_{\mathbb{K}} = \Vdash_{\mathbb{M}} \restriction K$
- $\mathbb{K} \subseteq_{full} \mathbb{M} :\Leftrightarrow \mathbb{K} \subseteq \mathbb{M}$, $\leq_{\mathbb{K}} = \leq_{\mathbb{M}} \cap (K \times K)$
- $\mathbb{K} \subseteq_{ini} \mathbb{M} :\Leftrightarrow \mathbb{K} \subseteq_{full} \mathbb{M}$ and for all $m \leq_{\mathbb{M}} m' \in K$: $m \in K$ or $m = b_{\mathbb{M}}$

For all these notions we have the obvious +-versions, e.g., for $\mathbb{K} \subseteq^{+} \mathbb{M}$ we demand that $b_{\mathbb{K}} = b_{\mathbb{M}}$. Note:

$$\mathbb{K} \subseteq^{+}_{ini} \mathbb{M} \Leftrightarrow \mathbb{K} \subseteq^{+}_{full} \mathbb{M} \textit{ and for all } m \leq_{\mathbb{M}} m' \in K : \ m \in K,$$

and:

$$\mathbb{K} \subseteq_{ini} \mathbb{M} \Rightarrow \mathbb{K} \subseteq_{full} \mathbb{M} \Rightarrow \mathbb{K} \subseteq \mathbb{M} \Rightarrow \mathbb{K} \preceq_1 \mathbb{M} \Rightarrow \mathbb{K} \preceq \mathbb{M}.$$

We will now prove the central result of the present section. It will give us (in section 7) both the desired result on uniform interpolation and the desired analogue of Łoś's Theorem.

Theorem 6.6 (Lifting Theorem) *Let $\vec{q}$, $\vec{p}$ and $\vec{r}$ be disjoint sets of variables. Let $\mathbb{K}$ be a $\vec{q},\vec{p}$-model and let $\mathbb{M}$ be a $\vec{p},\vec{r}$-model. Suppose $\mathbb{K}(\vec{p}) \preceq^{(+)} \mathbb{M}(\vec{p})$. Then there are $\vec{q},\vec{p},\vec{r}$-models $\mathbb{K}'$, $\mathbb{M}'$ such that: $\mathbb{K} \simeq \mathbb{K}'(\vec{q},\vec{p})$, $\mathbb{K}' \subseteq^{(+)}_{full} \mathbb{M}'$ and $\mathbb{M} \simeq \mathbb{M}'(\vec{p},\vec{r})$.*

Proof. We first give the proof for the +-case. Assume $R : \mathbb{K}(\vec{p}) \preceq^{+} \mathbb{M}(\vec{p})$. We construct the promised new models. We first specify $\mathbb{K}'$. (We index the various relations to keep track of where we are. Later we will omit these indices.) Define:

- $K' := \{\langle k,m\rangle \mid kRm\}$ (So K' is just R viewed as a set of pairs.)
- $\langle k,m\rangle \leq'_{\mathbb{K}} \langle k',m'\rangle :\Leftrightarrow k \leq_{\mathbb{K}} k'$ and $m \leq_{\mathbb{M}} m'$
- $b_{\mathbb{K}'} := \langle b_{\mathbb{K}}, b_{\mathbb{M}}\rangle$
- $\mathcal{P}_{\mathbb{K}'} := \vec{q},\vec{p},\vec{r}$
- $\langle k,m\rangle \Vdash_{\mathbb{K}'} s :\Leftrightarrow k \Vdash_{\mathbb{K}} s$ or $m \Vdash_{\mathbb{M}} s$

Note that:

$s \in \vec{q},\vec{p} \Rightarrow (\langle k,m\rangle \Vdash_{\mathbb{K}'\, s} \Leftrightarrow k \Vdash_{\mathbb{K}} s)$
$s \in \vec{p},\vec{r} \Rightarrow (\langle k,m\rangle \Vdash_{\mathbb{K}'} s \Leftrightarrow m \Vdash_{\mathbb{K}} s)$

Define B by: $kB\langle k',m\rangle :\Leftrightarrow k = k'$. It is immediate that $B : \mathbb{K} \simeq \mathbb{K}'(\vec{q},\vec{p})$. We specify $\mathbb{M}'$.

- $M' := \{\langle k,m\rangle \mid \exists m' {\in} M \ kRm' \leq_{\mathbb{M}} m\}$
- $\langle k,m\rangle \leq_{\mathbb{M}'} \langle k',m'\rangle :\Leftrightarrow k \leq_{\mathbb{K}} k'$ *and* $m \leq_{\mathbb{M}} m'$
- $b_{\mathbb{M}'} := \langle b_{\mathbb{K}}, b_{\mathbb{M}}\rangle$
- $\mathcal{P}_{\mathbb{M}'} := \vec{q},\vec{p},\vec{r}$
- $\langle k,m\rangle \Vdash_{\mathbb{M}'} s :\Leftrightarrow k \Vdash_{\mathbb{K}} s$ or $m \Vdash_{\mathbb{M}} s$

Note that:

$q \in \vec{q} \Rightarrow (\langle k,m\rangle \Vdash_{\mathbb{M}'} q \Leftrightarrow k \Vdash_{\mathbb{K}} q)$
$p \in \vec{p} \Rightarrow (k \Vdash_{\mathbb{K}} p \Rightarrow \langle k,m\rangle \Vdash_{\mathbb{M}'} p)$
$s \in \vec{p},\vec{r} \Rightarrow (\langle k,m\rangle \Vdash_{\mathbb{M}'} s \Leftrightarrow m \Vdash_{\mathbb{M}} s)$
$\langle k,m\rangle \in K'$ *and* $m \leq_{\mathbb{M}} m' \Rightarrow \langle k,m'\rangle \in K'$ *and* $\langle k,m\rangle \leq_{\mathbb{M}'} \langle k,m'\rangle$

Define C by: $mC\langle k,m'\rangle :\Leftrightarrow m = m'$. It is easily seen that $C : \mathbb{M} \simeq \mathbb{M}'(\vec{p},\vec{r})$. Finally it is immediate that: $\mathbb{K}' \subseteq^{+}_{full} \mathbb{M}'$.

We turn to the case without $+$. Assume $R : \mathbb{K}(\vec{p}) \preceq \mathbb{M}(\vec{p})$. Without loss of generality we may assume that there is precisely one element m_0 in M with $b_{\mathbb{K}} R m_0$. Extend $\mathbb{K}$ with a new bottom $\mathfrak{b}$ to $\mathbb{K}^+$ so that $R \cup \{\langle \mathfrak{b}, b_{\mathbb{M}} \rangle\} : \mathbb{K}^+ \preceq^+ \mathbb{M}$. We may take the forcing at $\mathfrak{b}$ for the elements of $\vec{q}$ arbitrary within the bounds dictated by persistence. Now apply the construction described above to $\mathbb{K}^+$ and $\mathbb{M}$ to obtain models $\mathbb{K}^{+\prime}$ and $\mathbb{M}'$. Finally we drop the bottom of $\mathbb{K}^{+\prime}$ to obtain a model $\mathbb{K}'$ with bottom $\langle b_{\mathbb{K}}, m_0 \rangle$. It is easy to see that the pair $\mathbb{K}'$, $\mathbb{M}'$ satisfies the conditions of the theorem. ❑

6.6 goes through even in case some of $\vec{q}$, $\vec{p}$ and $\vec{r}$ are infinite. However, such a generalization does not seem very urgent. An immediate consequence of the proof is that for any $\vec{q}$-model $\mathbb{K}$ and for any $\vec{r}$-model $\mathbb{M}$ with $\vec{q}$ and $\vec{r}$ disjoint, there is a $\vec{q}, \vec{r}$-model $\mathbb{N}$ such that $\mathbb{K} \simeq \mathbb{N}(\vec{q})$ and $\mathbb{M} \simeq \mathbb{N}(\vec{r})$. (Take R in the proof the universal relation between K and M.)

Corollary 6.7 *Suppose K and M are $\vec{p}$-models. Then:*

$$\mathbb{K} \preceq^{(+)} \mathbb{M} \Leftrightarrow \exists \mathbb{K}', \mathbb{M}' \ \mathbb{K} \simeq \mathbb{K}' \subseteq^{(+)}_{full} \mathbb{M}' \simeq \mathbb{M}.$$

Proof. "$\Rightarrow$" Is immediate from 6.6, taking $\vec{q} = \vec{r} = \emptyset$. "$\Leftarrow$" Suppose that: $\mathbb{K} \simeq \mathbb{K}' \subseteq^{(+)}_{full} \mathbb{M}' \simeq \mathbb{M}$. It follows that: $\mathbb{K} \preceq^+ \mathbb{K}' \preceq^{(+)} \mathbb{M}' \preceq^+ \mathbb{M}$. Hence: $\mathbb{K} \preceq^{(+)} \mathbb{M}$. ❑

Corollary 6.8 *Suppose $\mathbb{K}$ and $\mathbb{M}$ are $\vec{p}$-models. Then:*

$$\mathbb{K} \preceq^{(+)} \mathbb{M} \Leftrightarrow \exists \mathbb{K}' \ \mathbb{K} \simeq \mathbb{K}' \preceq^{(+)}_1 \mathbb{M}.$$

Proof. Note, by inspecting the proof of 6.6, that in 6.7 the total, surjective bisimulation C between $\mathbb{M}'$ and $\mathbb{M}$ is in fact a function (and thus a p-morphism). ❑

We can improve our result to obtain models embedded via $\subseteq^{(+)}_{ini}$.

Theorem 6.9 (Strengthened Lifting Theorem) *Let $\vec{q}$, $\vec{p}$ and $\vec{r}$ be disjoint sets of variables. Let $\mathbb{K}$ be a $\vec{q}, \vec{p}$-model and let $\mathbb{M}$ be a $\vec{p}, \vec{r}$-model. Suppose $\mathbb{K}(\vec{p}) \preceq^{(+)} \mathbb{M}(\vec{p})$. Then there are $\vec{q}, \vec{p}, \vec{r}$-models $\mathbb{K}'$, $\mathbb{M}'$ such that: $\mathbb{K} \simeq \mathbb{K}'(\vec{q}, \vec{p}), \mathbb{K}' \subseteq^{(+)}_{ini} \mathbb{M}'$ and $\mathbb{M} \simeq \mathbb{M}'(\vec{p}, \vec{r})$.*

Proof. We just specify the new models for the the $+$-case and leave the routine verification and the extension to the case without $+$ to the reader. $\mathbb{K}'$ is given as follows:

- $K' := \{\langle k, m, m \rangle \mid kRm\}$
- $\langle k, m, m \rangle \leq_{\mathbb{K}'} \langle k', m', m' \rangle :\Leftrightarrow k \leq_{\mathbb{K}} k'$ *and* $m \leq_{\mathbb{M}} m'$
- $b_{\mathbb{K}'} := \langle b_{\mathbb{K}}, b_{\mathbb{M}}, b_{\mathbb{M}} \rangle$
- $\mathcal{P}_{\mathbb{K}'} := \vec{q}, \vec{p}, \vec{r}$
- $\langle k, m, m \rangle \Vdash_{\mathbb{K}'} s :\Leftrightarrow k \Vdash_{\mathbb{K}} s$ or $m \Vdash_{\mathbb{M}} s$.

$\mathbb{M}'$ is the following model:

- $M' := \{\langle k,m,n\rangle \mid kRm \leq_{\mathbb{M}} n\}$
- $\langle k,m,n\rangle \leq_{\mathbb{M}'} \langle k',m',n'\rangle :\Leftrightarrow \begin{array}{l} m=n,\ k\leq_{\mathbb{K}}k' \textit{ and } m\leq_{\mathbb{M}}m', \\ or: k=k',\ m=m' \textit{ and } n\leq_{\mathbb{M}}n' \end{array}$
- $b'_{\mathbb{M}} := \langle b_{\mathbb{K}}, b_{\mathbb{M}}, b_{\mathbb{M}}\rangle$
- $\mathcal{P}_{\mathbb{M}'} := \vec{q},\vec{p},\vec{r}$
- $\langle k,m,n\rangle \Vdash \mathbb{M}'s :\Leftrightarrow k \Vdash \mathbb{K}s$ or $n \Vdash \mathbb{M}s$. ❑

Corollary 6.10 *Suppose* $\mathbb{K}$ *and* $\mathbb{M}$ *are* $\vec{p}$*-models. Then:*

$$\mathbb{K} \preceq^{(+)} \mathbb{M} \Leftrightarrow \exists \mathbb{K}', \mathbb{M}'\ \mathbb{K} \simeq \mathbb{K}' \subseteq^{(+)}_{ini} \mathbb{M}' \simeq \mathbb{M}.$$

Proof. Like the proof of 6.7. ❑

Let $\lhd$ be any relation between models. A formula A is $\lhd$-*robust* if

- *for all* $\mathbb{M}$ $(\mathbb{M} \Vdash A \Rightarrow$ *for all* $\mathbb{N} \lhd \mathbb{M}:\ \mathbb{N} \Vdash A)$

Theorem 6.11 *The following are equivalent:*

i) A *is* $\preceq$*-robust*
ii) A *is* $\preceq_1$*-robust*
iii) A *is* $\subseteq$*-robust*
iv) A *is* $\subseteq_{full}$*-robust*
v) A *is* $\subseteq_{ini}$*-robust*

Proof. Trivially $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v)$. We prove $(v) \Rightarrow (i)$. Suppose A is $\subseteq_{ini}$-robust and that $\mathbb{K} \preceq \mathbb{M} \Vdash A$. By 6.9 there are $\mathbb{K}'$, $\mathbb{M}'$ with $\mathbb{K} \simeq \mathbb{K}' \subseteq_{ini} \mathbb{M}' \simeq \mathbb{M}$. By bisimulation $\mathbb{M}' \Vdash A$. By robustness $\mathbb{K}' \Vdash A$. By bisimulation again: $\mathbb{K} \Vdash A$. ❑

7 *NNIL* and Subsimulations

7.1 Normal forms

To each model $\mathbb{K}$ and finite set of atoms $\vec{p}$, we assign $NNIL(\vec{p})$-formulas $\nu_{\mathbb{K}}(\vec{p})$ and $\eta_{\mathbb{K}}(\vec{p})$ as follows:

- $\nu_{\mathbb{K}}(\vec{p}) := \bigvee\{A \in NNIL(\vec{p}) \mid \mathbb{K} \nVdash A\}$
- $\eta_{\mathbb{K}}(\vec{p}) := \bigwedge\{A \in NNIL(\vec{p}) \mid \mathbb{K} \Vdash A\}$

Define also:

- $\rho_{\mathbb{K}}(\vec{p}) := \bigvee\{p \in \vec{p} \mid \mathbb{K} \nVdash p\}$
- $\pi_{\mathbb{K}}(\vec{p}) := \bigwedge\{p \in \vec{p} \mid \mathbb{K} \Vdash p\}$

If from the context it is clear that we are considering a model $\mathbb{K}$ we write $\nu_k(\vec{p})$ for $\nu_{\mathbb{K}[k]}(\vec{p})$ and similarly for η, ρ and π.

Theorem 7.1.1 $\mathbb{K} \nVdash \nu_{\mathbb{K}}(\vec{p})$ *and* $\mathbb{K} \Vdash \eta_{\mathbb{K}}(\vec{p})$

Proof. Obvious. ❑

Theorem 7.1.2 *Consider two $\vec{p}$-models $\mathbb{K}$ and $\mathbb{M}$. Then:*

$$\mathbb{K} \preceq \mathbb{M} \Leftrightarrow Th_{NNIL(\vec{p})}(\mathbb{M}) \subseteq Th_{NNIL(\vec{p})}(\mathbb{K})$$

Proof. "$\Rightarrow$" is immediate from 6.4. "$\Leftarrow$" Suppose $Th_{NNIL(\vec{p})}(\mathbb{M}) \subseteq Th_{NNIL(\vec{p})}(\mathbb{K})$. Define R by:

- $kRm :\Leftrightarrow Th_{\vec{p}}(k) \subseteq Th_{\vec{p}}(m)$ *and* $Th_{NNIL(\vec{p})}(m) \subseteq Th_{NNIL(\vec{p})}(k)$.

Then R is total, zig. To show that R is total consider any k in K. Clearly, $\mathbb{K} \nVdash \pi_k(\vec{p}) \to \nu_k(\vec{p})$ and hence by assumption (since $(\pi_k(\vec{p}) \to \nu_k(\vec{p})) \in NNIL(\vec{p})$): $\mathbb{M} \nVdash \pi_k(\vec{p}) \to \nu_k(\vec{p})$. It follows that for some m: $m \Vdash \pi_k(\vec{p})$ and $m \nVdash \nu_k(\vec{p})$. Ergo: kRm.

To prove that R is zig suppose kRm. It follows that:

$$Th_{NNIL(\vec{p})}(\mathbb{M}[m]) \subseteq Th_{NNIL(\vec{p})}(\mathbb{K}[k]).$$

Hence by the previous argument R restricted to the domains of $\mathbb{K}[k]$ and $\mathbb{M}[m]$ is total. But this gives us precisely the zig-property. ❑

Theorem 7.1.3 *Let $\mathbb{K}$ and $\mathbb{M}$ be $\vec{p}$-models. Then:*

$$\mathbb{K} \preceq \mathbb{M} \Leftrightarrow \mathbb{M} \nVdash \nu_{\mathbb{K}}(\vec{p}) \Leftrightarrow \mathbb{K} \Vdash \eta_{\mathbb{M}}(\vec{p}).$$

Proof. Immediate from 7.1.2. ❑

Theorem 7.1.4 *Let $\mathbb{K}$ and $\mathbb{M}$ be $\vec{p}$-models. Then:*

$$\mathbb{K} \preceq \mathbb{M} \Leftrightarrow \vdash \nu_{\mathbb{K}}(\vec{p}) \to \nu_{\mathbb{M}}(\vec{p}) \Leftrightarrow \vdash \eta_{\mathbb{K}}(\vec{p}) \to \eta_{\mathbb{M}}(\vec{p}).$$

Proof. Suppose $\mathbb{K} \preceq \mathbb{M}$. If $\mathbb{N} \nVdash \nu_{\mathbb{M}}(\vec{p})$, then $\mathbb{M} \preceq \mathbb{N}$ and hence $\mathbb{K} \preceq \mathbb{N}$, so $\mathbb{N} \nVdash \nu_{\mathbb{K}}(\vec{p})$. By the Completeness Theorem, we may conclude: $\vdash \nu_{\mathbb{K}}(\vec{p}) \to \nu_{\mathbb{M}}(\vec{p})$. For the converse assume $\vdash \nu_{\mathbb{K}}(\vec{p}) \to \nu_{\mathbb{M}}(\vec{p})$. Since $\mathbb{M} \nVdash \nu_{\mathbb{M}}(\vec{p})$, it follows that $\mathbb{M} \nVdash \nu_{\mathbb{K}}(\vec{p})$. Hence by 7.1.3: $\mathbb{K} \preceq \mathbb{M}$.

Suppose $\mathbb{K} \preceq \mathbb{M}$. If $\mathbb{N} \Vdash \eta_{\mathbb{K}}(\vec{p})$, then $\mathbb{N} \preceq \mathbb{K}$ and hence $\mathbb{N} \preceq \mathbb{M}$. It follows that $\mathbb{N} \Vdash \eta_{\mathbb{M}}(\vec{p})$. By the Completeness Theorem: $\vdash \eta_{\mathbb{K}}(\vec{p}) \to \eta_{\mathbb{M}}(\vec{p})$. For the converse suppose $\vdash \eta_{\mathbb{K}}(\vec{p}) \to \eta_{\mathbb{M}}(\vec{p})$. Since $\mathbb{K} \Vdash \eta_{\mathbb{K}}(\vec{p})$, it follows that $\mathbb{K} \Vdash \eta_{\mathbb{M}}(\vec{p})$, and hence that $\mathbb{K} \preceq \mathbb{M}$. ❑

Theorem 7.1.5 *The number of $\equiv$-equivalence classes of $\vec{p}$-models is finite.*

Proof. By 7.1.4 for $\vec{p}$-models $\mathbb{K}$ and $\mathbb{M}$: $\mathbb{K} \equiv \mathbb{M} \Leftrightarrow \vdash \nu_{\mathbb{K}}(\vec{p}) \leftrightarrow \nu_{\mathbb{M}}(\vec{p})$. But there are only finitely many $NNIL(\vec{p})$-formulas modulo provable equivalence. ❑

7.2 Interpolation

We want to prove the uniform interpolation theorem. Clearly, if the uniform left-interpolant $A^*(\vec{p})$ exists, then it is equivalent to $\bigvee\{B \in NNIL(\vec{p}) \mid \ \vdash B \to A\}$. So we define:

- $A^*(\vec{p}) := \bigvee\{B{\in}NNIL(\vec{p}) \mid\ \vdash B \to A\}$

and prove that this formula is the uniform left-interpolant. Similarly, we define:

- $A^\circ(\vec{p}) := \bigwedge\{B{\in}NNIL(\vec{p}) \mid\ \vdash A \to B\}$

and prove that it is the uniform right-interpolant.

Theorem 7.2.1 (Uniform Interpolation Theorem) *NNIL satisfies uniform interpolation.*

Proof. Consider formulas A and B and a finite set of propositional variables $\vec{p}$. Let $PV(A) \subseteq \vec{p},\vec{r}$ and $PV(B) \subseteq \vec{q},\vec{p}$ for $\vec{q},\vec{p},\vec{r}$ disjoint. Suppose $\vdash A \to B$.

We show that $B^*(\vec{p})$ is the uniform *NNIL* left-interpolant of B. Clearly, $\vdash B^*(\vec{p}) \to B$, so it is sufficient to show:

$$\textit{if } A \in NNIL, \textit{ then } \vdash A \to B^*(\vec{p}).$$

Suppose $A \in NNIL$ and $\nvdash A \to B^*(\vec{p})$. Let $\mathbb{M}$ be a $\vec{p},\vec{r}$-model such that $\mathbb{M} \Vdash A$ and $\mathbb{M} \nVdash B^*(\vec{p})$. Suppose $\eta_{\mathbb{M}}(\vec{p}) \vdash B$, then $\eta_{\mathbb{M}}(\vec{p}) \vdash B^*(\vec{p})$ by the definition of $B^*(\vec{p})$. But then $\mathbb{M} \Vdash B^*(\vec{p})$. Quod non. Ergo $\eta_{\mathbb{M}}(\vec{p}) \nvdash B$. By the Completeness Theorem there is a $\vec{q},\vec{p}$-model $\mathbb{K}$ such that $\mathbb{K} \Vdash \eta_{\mathbb{M}}(\vec{p})$ and $\mathbb{K} \nVdash B$. By 7.1.3: $\mathbb{K}(\vec{p}) \preceq \mathbb{M}(\vec{p})$. We apply the Lifting Theorem 6.6 to obtain $\vec{q},\vec{p},\vec{r}$-models $\mathbb{K}'$, $\mathbb{M}'$ such that: $\mathbb{K} \simeq \mathbb{K}'(\vec{q},\vec{p})$, $\mathbb{K}' \preceq \mathbb{M}'$ and $\mathbb{M} \simeq \mathbb{M}'(\vec{p},\vec{r})$. We find $\mathbb{K}' \nVdash B$ and $\mathbb{M}' \Vdash A$. Since $\mathbb{K}' \preceq \mathbb{M}'$ and $A \in NNIL$ we get: $\mathbb{K}' \Vdash A$. Ergo $\nvdash A \to B$. A contradiction. So $\vdash A \to B^*(\vec{p})$.

Let A be —again— an arbitrary formula. We show that $A^\circ(\vec{p})$ is the uniform *NNIL* right-interpolant of A. Clearly, $\vdash A \to A^\circ(\vec{p})$, so it is sufficient to show:

$$\textit{if } B \in NNIL, \textit{ then } \vdash A^\circ(\vec{p}) \to B.$$

Suppose $B \in NNIL$ and $\nvdash A^\circ(\vec{p}) \to B$. Let $\mathbb{K}$ be a $\vec{q},\vec{p}$-model such that $\mathbb{K} \Vdash A^\circ(\vec{p})$ and $\mathbb{K} \nVdash B$. Suppose $A \vdash \nu_{\mathbb{K}}(\vec{p})$, then $A^\circ(\vec{p}) \vdash \nu_{\mathbb{K}}(\vec{p})$ by the definition of $A^\circ(\vec{p})$. But then $\mathbb{K} \Vdash \nu_{\mathbb{K}}(\vec{p})$. Quod non. Ergo $A \nvdash \nu_{\mathbb{K}}(\vec{p})$. By the Completeness Theorem there is a $\vec{p},\vec{r}$-model $\mathbb{M}$ such that $\mathbb{M} \Vdash A$ and $\mathbb{M} \nVdash \nu_{\mathbb{K}}(\vec{p})$. By 7.1.3: $\mathbb{K}(\vec{p}) \preceq \mathbb{M}(\vec{p})$. We apply the Lifting Theorem 6.6 to obtain $\vec{q},\vec{p},\vec{r}$-models $\mathbb{K}'$, $\mathbb{M}'$ such that: $\mathbb{K} \simeq \mathbb{K}'(\vec{q},\vec{p}), \mathbb{K}' \preceq \mathbb{M}'$ and $\mathbb{M} \simeq \mathbb{M}'(\vec{p},\vec{r})$. We find $\mathbb{K}' \nVdash B$ and $\mathbb{M}' \Vdash A$. Since $\mathbb{K}' \preceq \mathbb{M}'$ we get: $\mathbb{M}' \nVdash B$. Ergo $\nvdash A \to B$. A contradiction. So $\vdash B \to A^*(\vec{p})$. ❑

Note that by the results of section 3 it also follows that the *NNIL* approximants A^* and A° exist. Moreover if $PV(A) \subseteq \vec{p}$, then:

$$\vdash A^* \leftrightarrow A^*(\vec{p}) \textit{ and } \vdash A^\circ \leftrightarrow A^\circ(\vec{p}).$$

We give an alternative characterization of $A^*(\vec{p})$ and $A^\circ(\vec{p})$.

Theorem 7.2.2 *Consider $A \in \mathcal{L}(\vec{p},\vec{q})$. We have:*

i) $\vdash A^*(\vec{p}) \leftrightarrow \bigwedge\{\nu_{\mathbb{K}}(\vec{p}) \mid \mathbb{K} \nVdash A\}$

ii) $\vdash A^\circ(\vec{p}) \leftrightarrow \bigvee\{\eta_{\mathbb{K}}(\vec{p}) \mid \mathbb{K} \Vdash A\}$

Proof. (i) "$\rightarrow$" Suppose $\mathbb{K} \nVdash A$, then $\mathbb{K} \nVdash A^*(\vec{p})$ and hence by the definition of $\nu_{\mathbb{K}}(\vec{p})$: $\vdash A^*(\vec{p}) \rightarrow \nu_{\mathbb{K}}(\vec{p})$. Ergo: $\vdash A^*(\vec{p}) \rightarrow \bigwedge\{\nu_{\mathbb{K}}(\vec{p}) \mid \mathbb{K} \nVdash A\}$.

"$\leftarrow$" Suppose that $\nvdash \bigwedge\{\nu_{\mathbb{K}}(\vec{p}) \mid \mathbb{K} \nVdash A\} \rightarrow A$. Then for some $\mathbb{M}$:

$$\mathbb{M} \Vdash \bigwedge\{\nu_{\mathbb{K}}(\vec{p}) \mid \mathbb{K} \nVdash A\} \text{ and } \mathbb{M} \nVdash A.$$

But then $\mathbb{M} \Vdash \nu_{\mathbb{M}}(\vec{p})$, quod non. Hence $\vdash \bigwedge\{\nu_{\mathbb{K}}(\vec{p}) \mid \mathbb{K} \nVdash A\} \rightarrow A$ and so by the definition of $A^*(\vec{p})$: $\vdash \bigwedge\{\nu_{\mathbb{K}}(\vec{p}) \mid \mathbb{K} \nVdash A\} \rightarrow A^*(\vec{p})$.

(ii) "$\leftarrow$" Suppose $\mathbb{K} \Vdash A$, then $\mathbb{K} \Vdash A^\circ(\vec{p})$ and hence by the definition of $\eta_{\mathbb{K}}(\vec{p}) :\vdash \eta_{\mathbb{K}}(\vec{p}) \rightarrow A^\circ(\vec{p})$. Ergo: $\vdash \bigvee\{\eta_{\mathbb{K}}(\vec{p}) \mid \mathbb{K} \Vdash A\} \rightarrow A^\circ(\vec{p})$.

"$\rightarrow$" Suppose that $\nvdash A \rightarrow \bigvee\{\eta_{\mathbb{K}}(\vec{p}) \mid \mathbb{K} \Vdash A\}$. Then for some $\mathbb{M}$:

$$\mathbb{M} \Vdash A \text{ and } \mathbb{M} \nVdash \bigvee\{\eta_{\mathbb{K}}(\vec{p}) \mid \mathbb{K} \Vdash A\}.$$

But then $\mathbb{M} \nVdash \eta_{\mathbb{M}}(\vec{p})$, quod non. Hence $\vdash A \rightarrow \bigvee\{\eta_{\mathbb{K}}(\vec{p}) \mid \mathbb{K} \Vdash A\}$ and so by the definition of $A^\circ(\vec{p})$: $\vdash A^\circ(\vec{p}) \rightarrow \bigvee\{\eta_{\mathbb{K}}(\vec{p}) \mid \mathbb{K} \Vdash A\}$. ❑

7.3 Modal Operators

Define:

- $\mathbb{M} \Vdash \bigcirc A :\Leftrightarrow \forall \mathbb{K} \preceq \mathbb{M}\ \mathbb{K} \Vdash A$
- $\mathbb{M} \Vdash \bigcirc_1 A :\Leftrightarrow \forall \mathbb{K} \preceq_1 \mathbb{M}\ \mathbb{K} \Vdash A$
- $\mathbb{M} \Vdash \Diamond A :\Leftrightarrow \exists \mathbb{K} \succeq \mathbb{M}\ \mathbb{K} \Vdash A$

Theorem 7.3.1 *Consider* $\mathbb{M}$ *and* $A \in \mathcal{L}(\vec{p})$. *Then:*

i) $\mathbb{M} \Vdash \bigcirc_1 A \Leftrightarrow \mathbb{M} \Vdash \bigcirc A \Leftrightarrow \mathbb{M} \Vdash A^*$

ii) $\mathbb{M} \Vdash \Diamond A \Leftrightarrow \mathbb{M} \Vdash A^\circ$

Proof. (i) The first equivalence is immediate by 6.8. We prove the second equivalence. "$\Leftarrow$" By 6.4. "$\Rightarrow$" Suppose $\mathbb{M} \nVdash A^*$. We have $\mathbb{M} \nVdash A^*(\vec{p})$. Then by 7.2.2 for some $\mathbb{K}$ with $\mathbb{K} \nVdash A$: $\mathbb{M} \nVdash \nu_{\mathbb{K}}(\vec{p})$. Clearly, we may assume that $\mathbb{K}$ is a $\vec{p}$-model. It follows that $\mathbb{K} \preceq \mathbb{M}(\vec{p})$. For some (possibly infinite) $\vec{r}$, disjoint from $\vec{p}$, $\mathbb{M}$ is a $\vec{p},\vec{r}$-model. By the Lifting Theorem 6.6 we can find a $\vec{p},\vec{r}$-model $\mathbb{K}'$ such that: $\mathbb{K} \simeq \mathbb{K}'(\vec{p})$ and $\mathbb{K}' \preceq \mathbb{M}$. Hence there is a $\mathbb{K}'$ with $\mathbb{K}' \nVdash A$ and $\mathbb{K}' \preceq \mathbb{M}$. Ergo $\mathbb{M} \nVdash \bigcirc A$.

Ad (ii): "$\Rightarrow$" Suppose $\exists \mathbb{K} \succeq \mathbb{M}\ \mathbb{K} \Vdash A$. Then $\mathbb{K} \Vdash A^\circ(\vec{p})$ and hence $\mathbb{M} \Vdash A^\circ(\vec{p})$, i.e., $\mathbb{M} \Vdash A^\circ$. "$\Leftarrow$" Suppose $\mathbb{M} \Vdash A^\circ$. By 7.2.2 for some $\mathbb{K}$ with $\mathbb{K} \Vdash A$: $\mathbb{M} \Vdash \eta_{\mathbb{K}}(\vec{p})$. We may assume that $\mathbb{K}$ is a $\vec{p}$-model. We find $\mathbb{M}(\vec{p}) \preceq \mathbb{K}$. For some (possibly infinite) $\vec{q}$, disjoint from $\vec{p}$, $\mathbb{M}$ is a $\vec{q},\vec{p}$-model. By the Lifting Theorem 6.6 we can find a $\vec{q},\vec{p}$-model $\mathbb{K}'$ such that: $\mathbb{K} \simeq \mathbb{K}'(\vec{p})$ and $\mathbb{M} \preceq \mathbb{K}'$. Hence there is a $\mathbb{K}'$ with $\mathbb{K}' \Vdash A$ and $\mathbb{M} \preceq \mathbb{K}'$. Ergo $\mathbb{M} \Vdash \Diamond A$. ❑

Question 7.3.2 Is there a reasonable, complete set of inference rules for $\bigcirc$? The same question for $\Diamond$ and for $\bigcirc$ and $\Diamond$ together. ❑

Example 7.3.3 Define $\mathbb{M} \Vdash \triangle A :\Leftrightarrow \forall \mathbb{K} \subseteq \mathbb{M}\ \mathbb{K} \Vdash A$. Consider the model $\mathbb{U}$ of 6.3. Clearly, $\mathbb{U} \Vdash \triangle((\neg\neg p \to p) \to (p \vee \neg p))$, but $\mathbb{U} \nVdash (p \vee \neg p)$. Moreover:

$$((\neg\neg p \to p) \to (p \vee \neg p))^* = (p \vee \neg p).$$

(One way of quickly seeing this, is by inspecting the Rieger-Nishimura Lattice, see Troelstra and van Dalen 1988a, pp 49-50. The only *NNIL* (p)-formulas are (modulo provable equivalence): $\bot$, p, $\neg p$, $p \vee \neg p$) and $\top$. Alternatively one may apply the algorithm specified in Visser 1985 or Visser 1994. A third way is to use the argument presented in 7.5.3.) So 7.3.1(i) does not generalize to $\triangle$. ❑

7.4 An Analogue of Łoś's Theorem

Theorem 7.4.1 *A is $\preceq$-robust $\Leftrightarrow$ A is in NNIL.*

Proof. "$\Leftarrow$" is immediate by 6.4. "$\Rightarrow$" Let A be a $\preceq$-robust formula in $\mathcal{L}(\vec{p})$. Suppose $\mathbb{K} \Vdash A$, then $\mathbb{K} \Vdash \bigcirc A$ and hence by 7.3.1: $\mathbb{K} \Vdash A^*$. Ergo $\vdash A \leftrightarrow A^*$. So A is in *NNIL*. ❑

Note that by 6.11 we can replace $\preceq$-robustness in the conclusion of 7.4.1 by each of: $\preceq_1$-robustness, $\subseteq$-robustness, $\subseteq_{full}$-robustness, $\subseteq_{ini}$-robustness.

Corollary 7.4.2 *We have:*

$$A \text{ is a NNIL-formula} \quad \Leftrightarrow \quad \begin{array}{l} I(A,x) \text{ is } T\text{-provably equivalent to a} \\ \Pi_1\text{-formula of predicate logic.} \end{array}$$

Proof. "$\Rightarrow$" By 2.1. "$\Leftarrow$" Suppose $I(A,x)$ is T-provably equivalent to a Π_1-formula of predicate logic. Then $I(A,x)$ is preserved under taking submodels of models of T. But this means that A is $\subseteq$-robust. By the above consequence of 7.4.1 it follows that A is in *NNIL*. ❑

In appendix A we show how 7.4.2 can be improved.

7.5 Anti-model-descriptions

Let $\mathbb{K}$ be a finite model. There is a pleasant way of characterizing $\nu_{\mathbb{K}}(\vec{p})$ as a formula giving an "anti-description". Remember:

- $\rho_{\mathbb{K}}(\vec{p}) := \bigvee\{p \in \vec{p} \mid \mathbb{K} \nVdash p\}$
- $\pi_{\mathbb{K}}(\vec{p}) := \bigwedge\{p \in \vec{p} \mid \mathbb{K} \Vdash p\}$

Define:

- $\alpha_{\mathbb{K}}(\vec{p}) := \pi_{\mathbb{K}}(\vec{p}) \to (\rho_{\mathbb{K}}(\vec{p}) \vee \bigvee\{\alpha_{\mathbb{K}[k]}(\vec{p}) \mid b_{\mathbb{K}} < k\})$

We again employ the convention of the empty conjunction being $\top$ and the empty disjunction being $\bot$. Modulo provable equivalence $\alpha_{\mathbb{K}}(\vec{p})$ can be written more efficiently, by restricting ourselves to the immediate strict $<$-successors of $b_{\mathbb{K}}$ in the last disjunction of the definition. Clearly, $\alpha_{\mathbb{K}}(\vec{p}) \in$ *NNIL*$(\vec{p})$.

Theorem 7.5.1 $\mathbb{K} \nVdash \alpha_{\mathbb{K}}(\vec{p})$.

Proof. We prove by induction on the converse of $<$, that $k \nVdash \alpha_k(\vec{p})$. Suppose $k \Vdash \alpha_k(\vec{p})$. Since, clearly, $k \Vdash \pi_k(\vec{p})$ and $k \nVdash \nu_k(\vec{p})$, it follows that $k \Vdash \bigvee\{\alpha_{k'}(\vec{p}) \mid k < k'\}$ and hence, for some $k' > k$, $k \Vdash \alpha_{k'}(\vec{p})$. This contradicts the induction hypothesis. ❑

Theorem 7.5.2 $\vdash \alpha_{\mathbb{K}}(\vec{p}) \leftrightarrow \nu_{\mathbb{K}}(\vec{p})$.

Proof. "$\rightarrow$" Is immediate by 7.5.1 and the definition of $\nu_{\mathbb{K}}(\vec{p})$.

"$\leftarrow$" We prove by induction on de depth of k in $\mathbb{K}$ that for all $A \in$ *NNIL*$(\vec{p})$: $k \nVdash A \Rightarrow \vdash A \rightarrow \alpha_k(\vec{p})$. Consider k. The proof proceeds by a subinduction on A in $NNIL_0(\vec{p})$. The cases of atoms, conjunction and disjunction are straightforward. Suppose A is of the form $p \rightarrow B$. In case $k \Vdash p$ we find: $k \nVdash B$. Hence by our subinduction hypothesis: $\vdash B \rightarrow \alpha_k(\vec{p})$. Since p is a conjunct of $\pi_k(\vec{p})$, we find: $\vdash (p \rightarrow B) \rightarrow \alpha_k(\vec{p})$. In case $k \nVdash p$ there is a $k' > k$ such that $k' \Vdash p$ and $k' \nVdash B$. It follows by our main induction hypothesis that $\vdash (p \rightarrow B) \rightarrow \alpha_{k'}(\vec{p})$. Hence by the definition of $\alpha_k(\vec{p})$: $\vdash (p \rightarrow B) \rightarrow \alpha_k(\vec{p})$. ❑

Example 7.5.3 Consider the characterization of A^* of 7.2.2. Note that by 7.1.4 we can restrict the conjunction to $\nu_{\mathbb{K}}(\vec{p})$ for $\mathbb{K}$ a representative of a $\preceq$-minimal $\equiv$-equivalence class X such that $\mathbb{M} \nVdash A$ for some $\mathbb{M} \in X$. This insight allows us to use 7.5.2 for actual computation of $A^*(\vec{p})$. We compute e.g.,

$$A := ((\neg\neg p \rightarrow p) \rightarrow (p \vee \neg p))^*.$$

Consider the models $\mathbb{U}$ and $\mathbb{V}$ of example 6.3. It is easy to see that any model $\mathbb{K}$ such that $\mathbb{K} \nVdash A$ has $\mathbb{V}$ as a submodel. Moreover $\mathbb{V} \nVdash A$ and $\mathbb{U} \equiv \mathbb{V}$. So:

$$A^* = \alpha_{\mathbb{U}} = \bigwedge \emptyset \rightarrow (\bigvee\{p\} \vee \bigvee\{(\bigwedge\{p\} \rightarrow (\bigvee \emptyset \vee \bigvee \emptyset))\}) = (p \vee \neg p).$$

❑

8 Beyond *NNIL*

Can we extend our results to the higher complexity classes? It turns out that the characterization of the complexity classes in terms of an appropriate notion of simulation extends in an immediate way. How to do this is sketched in appendix B. In this section we show that *yes* we can get uniform right interpolation for Π_2, but *no* we cannot get uniform left interpolation for Π_2, and *no* we cannot get uniform right interpolation for Π_3. These classes are PV-finite by 2.3 and closed under disjunction and conjunction. So by the results of section 3, it follows that Π_2 does not have *IPL* and that Π_3 does not have *IPR*.

Let $\mathbb{K}$ and $\mathbb{M}$ be $\vec{p}$-models. A pair of relations R, S is a *2-subsimulation* between $\mathbb{K}$ and $\mathbb{M}$, if R is a subsimulation between $\mathbb{K}$ and $\mathbb{M}$, and S is a subsimulation between $\mathbb{M}$ and $\mathbb{K}$ and $R \subseteq \widehat{S}$. We write:

- $R, S : \mathbb{K} \preceq_2 \mathbb{M} :\Leftrightarrow R, S$ is a 2-subsimulation between $\mathbb{K}$ and $\mathbb{M}$ and R is total
- $R, S : \mathbb{K} \preceq_2^+ \mathbb{M} :\Leftrightarrow R, S : \mathbb{K} \preceq_2 \mathbb{M}$, and R is root preserving
- $\mathbb{K} \preceq_2 \mathbb{M} :\Leftrightarrow \exists R, S\ \mathbb{K} \preceq_2 \mathbb{M}$
- $\mathbb{K} \preceq_2^+ \mathbb{M} :\Leftrightarrow \exists R, S\ \mathbb{K} \preceq_2^+ \mathbb{M}$

Note that we have: $\mathbb{K} \preceq_2 \mathbb{M} \Leftrightarrow \exists m{\in}M\ \mathbb{K} \preceq_2^+ \mathbb{M}[m]$. Note that if R is root preserving, then S is also root preserving.

Theorem 8.1 *Suppose $\mathbb{K}$ and $\mathbb{M}$ are $\vec{p}$-models. Then:*

$$\mathbb{K} \preceq_2 \mathbb{M} \Leftrightarrow Th_{\Pi_2(\vec{p})}(\mathbb{M}) \subseteq Th_{\Pi_2(\vec{p})}(\mathbb{K}).$$

Proof. This theorem is a special case of B.4.3. ❑

Define:

- $A^\bullet(\vec{p}) := \bigwedge\{B{\in}\Pi_2(\vec{p}) \mid A \vdash B\}$
- $A^\bullet := A^\bullet(PV(A))$

Theorem 8.2 *Let $A \in \mathcal{L}$ and $B \in \Pi_2$. Suppose $A \vdash B$, then $A^\bullet \vdash B$.*

Proof. Let $\vec{q} \supseteq PV(A) \cup PV(B)$, $\vec{p} := PV(A)$. Suppose $A^\bullet \nvdash B$. Let $\mathbb{K}$ be a $\vec{q}$-model such that $\mathbb{K} \Vdash A^\bullet$ and $\mathbb{K} \nVdash B$. Suppose

$$A, \bigwedge\{C{\in}\Pi_1(\vec{p}) \mid b_\mathbb{K} \Vdash C\} \vdash \bigvee\{D{\in}\Pi_2(\vec{p}) \mid b_\mathbb{K} \nVdash D\}.$$

It follows that:

$$A^\bullet \vdash \bigwedge\{C{\in}\Pi_1(\vec{p}) \mid b_\mathbb{K} \Vdash C\} \to \bigvee\{D{\in}\Pi_2(\vec{p}) \mid b_\mathbb{K} \nVdash D\}.$$

But this contradicts the fact that $b_\mathbb{K} \Vdash A^\bullet(\vec{p})$. Hence:

$$A, \bigwedge\{C{\in}\Pi_1(\vec{p}) \mid b_\mathbb{K} \Vdash C\} \nvdash \bigvee\{D{\in}\Pi_2(\vec{p}) \mid b_\mathbb{K} \nVdash D\}.$$

Let $\mathbb{M}$ be a $\vec{p}$-model such that:

$$\mathbb{M} \Vdash A, \mathbb{M} \Vdash \bigwedge\{C{\in}\Pi_1(\vec{p}) \mid b_\mathbb{K} \Vdash C\} \text{ and } \mathbb{M} \nVdash \bigvee\{D{\in}\Pi_2(\vec{p}) \mid b_\mathbb{K} \nVdash D\}.$$

It follows that for some R, S: $R, S : \mathbb{K}(\vec{p}) \preceq_2^+ \mathbb{M} \Vdash A$. We will construct a $\vec{q}$-model $\mathbb{N}$ such that: (a) $\mathbb{M} \simeq \mathbb{N}(\vec{p})$, (b) $\mathbb{K} \preceq_2 \mathbb{N}$. It immediately follows that $\mathbb{N} \Vdash A$ and hence $\mathbb{N} \Vdash B$. But then $\mathbb{K} \Vdash B$. A contradiction. Ergo $A^\bullet \vdash B$.

We take:

- $N := \{\langle m, k\rangle \mid mSk\}$
- $\langle m, k\rangle \leq_\mathbb{N} \langle m', k'\rangle :\Leftrightarrow m \leq_\mathbb{M} m'$ *and* $k \leq_\mathbb{K} k'$
- $b_\mathbb{N} := \langle b_\mathbb{M}, b_\mathbb{K}\rangle$
- $\mathcal{P}_\mathbb{N} := \vec{q}$

- $\langle m, k\rangle \Vdash_{\mathbb{N}} q :\Leftrightarrow k \Vdash_{\mathbb{K}} q$

Since S is total, zig from $\mathbb{M}$ to $\mathbb{K}(\vec{p})$, it follows that $\mathbb{M} \simeq \mathbb{N}(\vec{p})$. Define S' from $\mathbb{N}$ to $\mathbb{K}$ by:

- $\langle m, k\rangle S'k' :\Leftrightarrow k = k'$

By definition S' is total, zig. Finally R' from $\mathbb{K}$ to $\mathbb{N}$ by:

- $kR'\langle m, k'\rangle :\Leftrightarrow k = k'Rm$

It is immediate that $R' \subseteq \widehat{S'}$ (if $\langle m, k\rangle \in N$, then mSk). Also it is easy to see that R' is total, zig. ❑

It follows that $A^{\bullet}$ is the Π_2 right-approximant of A. Moreover $(.)^{\bullet}$ is preserving. Hence, by 3.9 and 3.10, we have uniform Π_2 right-interpolation. By 3.5, it follows that Π_2 is closed under Pitts universal quantification.

Let X be any formula class. We write: $\exists X := \{\exists pA \mid A \in X,\ p \in PV\}$ and $\forall X := \{\forall pA \mid A \in X,\ p \in PV\}$.

Theorem 8.3 $\exists\Pi_2 = \forall\Pi_3 = \mathcal{L}$.

Proof. Suppose $A \in \mathcal{L}(\vec{p})$. Let $\vec{q}$ be a set of variables disjoint from $\vec{p}$ that is in 1-1 correspondence with the subformulas of the form $(B \to C)$ of A. Let the correspondence be $\mathfrak{q}$. We define $\mathcal{T} : Sub(A) \to Sub(A)$ as follows:

- $\mathcal{T}$ commutes with atoms, conjunction and disjunction
- $\mathcal{T}(B \to C) := \mathfrak{q}(B \to C)$

Define:

- $EQ := \bigwedge\{\mathfrak{q}(B \to C) \leftrightarrow (\mathcal{T}(B) \to \mathcal{T}(C)) \mid (B \to C) \in Sub(A)\}$

Note that EQ is Π_2. Finally we put:

- $A^{\#} := \exists\vec{q}(EQ \wedge \mathcal{T}(A))$
- $A^{\$} := \forall\vec{q}(EQ \to \mathcal{T}(A))$

Note that $A^{\#} \in \exists\Pi_2$ and $A^{\$} \in \forall\Pi_3$. By elementary reasoning in second order propositional logic we find: $\vdash A \leftrightarrow A^{\#}$ and $\vdash A \leftrightarrow A^{\$}$. ❑

Theorem 8.3 illustrates that in general the growth of implicational complexity in constructing the Pitts interpolants is necessary.

Theorem 8.4 Π_2 *does not satisfy (uniform) left interpolation and that* Π_3 *does not satisfy (uniform) right interpolation.*

Proof. We treat the case of Π_2. The other case is similar. Suppose Π_2 satisfies left interpolation. Since Π_2 is PV-finite and closed under disjunction, it is, by 3.10, the image of a PVP-interior operation. By 3.5, it follows that Π_2 is closed under Pitts existential quantification. Quod non, by 8.3. ❑

A *IPC* as a Fragment of Predicate Logic

In this appendix we characterize the formulas of *IPL* as first order formulas which are persistent and preserved under bisimulation. The result is a variation on Johan van Benthem's characterization of the formulas of modal propositional logic as the first order formulas that are preserved under bisimulation. See van Benthem 1976.

In 2.1 we introduced the theory *Krip* in predicate logic of Kripke models. Let $A(x)$ be any $\mathcal{L}_{Krip}$-formula in one variable.

We say that $A(x)$ is *persistent* if for any *Krip*-model (or equivalently: *IPC*-model) $\mathbb{K}$ and for any k, k' in $\mathbb{K}$:

$$(k \leq k' \textit{ and } \mathbb{K} \models A(k)) \Rightarrow \mathbb{K} \models A(k').$$

We say that $A(x)$ is *preserved under bisimulations* if for all *Krip*-models $\mathbb{K}$ and $\mathbb{M}$ and for all bisimulations R between $\mathbb{K}$ and $\mathbb{M}$, we have:

$$\textit{if } kRm \textit{ and } \mathbb{K} \models A(k), \textit{ then } \mathbb{M} \models A(m).$$

We say that $A(x)$ is *upwards preserved under (.)[.]* if for all *Krip*-models $\mathbb{K}$ and all k in $\mathbb{K}$:

$$\textit{if } \mathbb{K} \models A(k), \textit{ then } \mathbb{K}[k] \models A(k).$$

We say that $A(x)$ is *downwards preserved under (.)[.]* if for all *Krip*-models $\mathbb{K}$ and all k in $\mathbb{K}$:

$$\textit{if } \mathbb{K}[k] \models A(k), \textit{ then } \mathbb{K} \models A(k).$$

Theorem A.1 *Suppose $A(x)$ is (1) persistent and (2) preserved under bisimulations, then there is a $B \in \mathcal{L}$, such that $Krip \vdash A(x) \leftrightarrow I(B, x)$.*

We postpone the proof a bit to take a closer look at (1) and (2) first. Note that (2) is equivalent to the conjunction of:

2a) $A(x)$ is preserved under total, surjective bisimulations
2b) $A(x)$ is downwards preserved under [.](.)
2c) $A(x)$ is upwards preserved under [.](.)

We give separating examples for the four conditions (1), ..., (2c), Each example is designated by the one condition it doesn't satisfy.

$\neg(1)$ $\neg P(x)$
$\neg(2a)$ $\forall y(x \leq y \to y \leq x)$
$\neg(2b)$ $\forall y P(y)$
$\neg(2c)$ $\exists y \neg P(y)$

Proof of A.1. Suppose $A(x)$ satisfies (1) and (2). Let:

- $\Delta(x) := \{I(B, x) \mid Krip \vdash A(x) \to I(B, x)\}$

If $Krip, \Delta(x) \vdash A(x)$, we are easily done by compactness. If $Krip, \Delta(x) \nvdash A(x)$, then, by results in Chang and Keisler 1977, chapters 4 and 5 (alternatively: see van Benthem 1991a, chapter 15 or de Rijke 1993, chapter 6), there is an ω-saturated *Krip*-model $\mathbb{K}$ and an element k of $\mathbb{K}$, such that $\mathbb{K} \models \Delta(k)$ and $\mathbb{K} \models \neg A(k)$. Let:

- $\Theta(x) := \{A(x)\} \cup \{I(B,x) \mid k \Vdash_{\mathbb{K}} B\} \cup \{\neg I(C,x) \mid k \nVdash_{\mathbb{K}} C\}$

We claim that $\Theta(x)$ is consistent. If not, there are finite sets of *IPC*-formulas X and Y such that for every B in X: $k \Vdash_{\mathbb{K}} B$ and for every C in Y: $k \nVdash_{\mathbb{K}} C$ and such that:

$$\mathit{Krip} \vdash \neg(A(x) \wedge \bigwedge\{I(B,x) \mid B \in X\} \wedge \bigwedge\{\neg I(C,x) \mid C \in Y\}).$$

Hence:

$$\mathit{Krip} \vdash A(x) \to (\bigwedge\{I(B,x) \mid B \in X\} \to \bigvee\{I(C,x) \mid C \in Y\}).$$

By predicate logic, we find that:

$$\mathit{Krip} \vdash \forall y{\geq}x A(y) \to \forall y{\geq}x(\bigwedge\{I(B,y) \mid B \in X\} \to \bigvee\{I(C,y) \mid C \in Y\}).$$

By the persistence of $A(x)$, it follows that:

$$\mathit{Krip} \vdash A(x) \to \forall y{\geq}x(\bigwedge\{I(B,y) \mid B \in X\} \to \bigvee\{I(C,y) \mid C \in Y\}).$$

Finally, by the definition if I, we get:

$$\mathit{Krip} \vdash (A(x) \to I((\bigwedge X \to \bigvee Y), x).$$

But, then, on the one hand, $(\bigwedge X \to \bigvee Y) \in \Delta(x)$, and, on the other hand, $\mathbb{K} \not\models I((\bigwedge X \to \bigvee Y), k)$. A contradiction.

Let $\mathbb{M}$ be an ω-saturated *Krip*-model of $\Theta(x)$, say $\mathbb{M} \models \Theta(m)$. We define R between the nodes of $\mathbb{K}$ and $\mathbb{M}$ as follows:

- $k'Rm' :\Leftrightarrow \forall B{\in}\mathcal{L}(k' \Vdash_{\mathbb{K}} B \Leftrightarrow m' \Vdash_{\mathbb{M}} B)$

We claim that R is a bisimulation. Suppose e.g., $k'' \geq k'Rm'$. Take:

- $\Gamma(x) := \{x \geq m'\} \cup \{I(B,x) \mid k'' \Vdash_{\mathbb{K}} B\} \cup \{\neg I(C,x) \mid k'' \nVdash_{\mathbb{K}} C\}$

We claim that $\Gamma(x)$ is finitely satisfiable in $\mathbb{M}$. By ω-saturatedness it will follow that there is an $m'' \geq m$ satisfying $\Gamma(x)$. Hence $k''Rm''$.

Consider any finite X and Y such that for all B in X: $k'' \Vdash_{\mathbb{K}} B$ and for all C in Y: $k'' \nVdash_{\mathbb{K}} C$. Then evidently $k' \nVdash_{\mathbb{K}} (\bigwedge X \to \bigvee Y)$. Since $k'Rm'$, it follows that: $m' \nVdash_{\mathbb{M}} (\bigwedge X \to \bigvee Y)$. We may conclude that for some $m'' \geq m'$: $m'' \Vdash_{\mathbb{M}} \bigwedge X$ and $m'' \nVdash_{\mathbb{M}} \bigvee Y$. Evidently m'' satisfies: $\{x \geq m\} \cup \{I(B,x) \mid B \in X\} \cup \{\neg I(C,x) \mid C \in Y\}$.

Since kRm and $\mathbb{M} \models A(m)$, we have $\mathbb{K} \models A(k)$. A contradiction. So $\mathit{Krip}, \Delta(x) \vdash A(x)$. ❑

Suppose $A(x)$ satisfies (1) and (2) and is Π_1 in *Krip*. By A.1, $A(x)$ is *Krip*-provably equivalent to $I(B,x)$ for some $B \in \mathcal{L}$. $A(x)$ is preserved under taking submodels (for predicate logic). Hence B will be preserved under taking submodels (for *IPC*). Hence, by the consequence of 7.4.1,

B is *IPC*-provably equivalent to a *NNIL*-formula C. Ergo $A(x)$ is *Krip*-provably equivalent to $I(C, x)$ with C in *NNIL*. Thus the Π_1-formulas of *Krip*, satisfying (1) and (2), correspond precisely to the *NNIL*-formulas.

B Appendix: *Zigzag*-simulations

In this appendix we describe the notion of simulation appropriate for the Π- hierarchy of *IPC*.

B.1 Extended Numbers

It is pleasant, but not strictly necessary, to have some rules for calculating with infinity, at hand. Let $\omega^+ := \omega \cup \{\infty\}$. We give ω^+ the obvious ordering. We let α, β, ... range over ω^+ and we let m, n, ... range over ω. Define:

- $+$ has its usual meaning on ω
- $\infty + \alpha := \alpha + \infty := \infty$
- $\alpha \dot{-} \beta := 0$ if $\alpha \leq \beta$, $m \dot{-} n := m - n$ if $n < m$, $\infty \dot{-} n := \infty$

Note that:

$\flat$ $\alpha \leq \beta + \gamma \Leftrightarrow \alpha \dot{-} \beta \leq \gamma$

It follows e.g., that for $X \subseteq \omega^+$:

$$sup(X) \dot{-} \alpha = sup(\{\beta \dot{-} \alpha \mid \beta \in X\}).$$

Another important principle —easily verified— is:

$\natural$ If $min(\beta, \gamma) < \infty$, then $(\alpha + \beta) \dot{-} \gamma = (\alpha \dot{-} (\gamma \dot{-} \beta)) + (\beta \dot{-} \gamma)$

Immediate consequences are:

$\natural$1 if $min(\beta, \gamma) < \infty$ and $\gamma \leq \beta$, then $(\alpha + \beta) \dot{-} \gamma = \alpha + (\beta \dot{-} \gamma)$

$\natural$2 $(\alpha + n) \dot{-} n = \alpha$

Finally we have:

$\sharp$ if $\alpha \leq \beta$, then $\alpha + (\beta \dot{-} \alpha) = \beta$

B.2 Basics of *Zigzag*-simulations

A *zigzag-simulation*[5] R between $\mathcal{P}$-models $\mathbb{K}$ and $\mathbb{M}$ is a quaternary relation between K, { *zig,zag* }, ω^+ and M, satisfying the conditions below. We will consider R also as a $\{zig, zag\} \times \omega^+$-indexed set of binary relations between K and M, writing $kR_{\mathfrak{z},\alpha}m$ for $\langle k, \mathfrak{z}, \alpha, m \rangle \in R$. We put: $\widehat{zig} := zag$ and $\widehat{zag} := zig$. We give the conditions:

i) $kR_{zig,\alpha}m \Rightarrow Th_{\mathcal{P}}(k) \supseteq Th_{\mathcal{P}}(m)$, $kR_{zag,\alpha}m \Rightarrow Th_{\mathcal{P}}(k) \subseteq Th_{\mathcal{P}}(m)$

ii) $\alpha > 0$ and $k' \geq kR_{zig,\alpha}m \Rightarrow$ there is an m' with $k'R_{zig,\alpha}m' \geq m$

[5] The designation *zigzag-connection* has been used to refer to total bisimulations, in van Benthem 1984. Since, however, this name for bisimulations didn't catch, we feel justified to employ a variant of it for a different notion. *zigzag*-simulations are closely related to, but different from bounded bisimulations.

iii) $\alpha > 0$ and $kR_{zag,\alpha}m \leq m' \Rightarrow$ there is a k' with $k \leq k'R_{zag,\alpha}m'$
iv) $\alpha > 0$ and $kR_{\mathfrak{z},\alpha}m \Rightarrow kR_{\widehat{\mathfrak{z}},\alpha \dot{-} 1}m$

We call (ii) the *zig-property* and (iii) the *zag-property*.

If we set $k \succeq_{\mathcal{P}} m :\Leftrightarrow Th_{\mathcal{P}}(k) \supseteq Th_{\mathcal{P}}(m)$, we can also formulate our conditions as:

i) $R_{zig,\alpha} \subseteq \succeq_{\mathcal{P}}$, $R_{zag,\alpha} \subseteq \widehat{\succeq_{\mathcal{P}}}$
ii) $\alpha > 0 \Rightarrow \geq \circ R_{zig,\alpha} \subseteq R_{zig,\alpha} \circ \geq$
iii) $\alpha > 0 \Rightarrow R_{zag,\alpha} \circ \leq \subseteq \leq \circ R_{zag,\alpha}$
iv) $\alpha > 0 \Rightarrow R_{\mathfrak{z},\alpha} \subseteq R_{\widehat{\mathfrak{z}},\alpha \dot{-} 1}$

We write:

- $kR_{\alpha}m :\Leftrightarrow kR_{zig,\alpha}m$ *and* $kR_{zag,\alpha}m$
- $kRm :\Leftrightarrow kR_{\infty}m$

Note that by (iv) it follows that: $kR_{\mathfrak{z},\infty}m \Leftrightarrow kR_{\infty}m$.

A binary relation R between $\mathbb{K}$ and $\mathbb{M}$ is a *bisimulation* between $\mathbb{K}$ and $\mathbb{M}$ iff $R^{+} := \{\langle k, \mathfrak{z}, \infty, m\rangle \mid kRm\}$ is a *zigzag*-simulation. We will simply confuse bisimulations R with the corresponding *zigzag*-simulations R^{+}.

Suppose R is an *zigzag*-simulation between $\mathbb{K}$ and $\mathbb{M}$ and that S is an *zigzag*-simulation between $\mathbb{M}$ and $\mathbb{N}$. The composition $R \circ S$ is given by:

- $(R \circ S)_{\mathfrak{z},\alpha} := R_{\mathfrak{z},\alpha} \circ S_{\mathfrak{z},\alpha}$

It is easily seen that *zigzag*-simulations are closed under composition.

Suppose $\mathcal{R}$ is a set of *zigzag*-simulations between $\mathbb{K}$ and $\mathbb{M}$. It is easy to verify that $\bigcup \mathcal{R}$ is again a *zigzag*-simulation between $\mathbb{K}$ and $\mathbb{M}$. It follows that there is always a maximal *zigzag*-simulation between two models.

Suppose R is a *zigzag*-simulation between $\mathbb{K}$ and $\mathbb{M}$. The inverse $\widehat{R}$ is given by:

- $(\widehat{R})_{\mathfrak{z},\alpha} := \widehat{(R_{\widehat{\mathfrak{z}},\alpha})}$

where $\widehat{(.)}$ is the usual inverse of binary relations. Clearly, *zigzag*-simulations are closed under $\widehat{(.)}$.

Consider a *zigzag*-simulation R between $\mathbb{K}$ and $\mathbb{M}$. Define $R[\alpha]$ by:

- $kR[\alpha]_{\mathfrak{z},\beta}m :\Leftrightarrow kR_{\mathfrak{z},\alpha+\beta}m$

We say that R is *downwards closed* if for all $\alpha \leq \beta$: $R_{\mathfrak{z},\beta} \subseteq R_{\mathfrak{z},\alpha}$. The *downwards closure* $R{\downarrow}$ of a *zigzag*-simulation R is the smallest downwards closed relation extending it.

Note that if R is downwards closed we automatically have for $\beta > 0$:

$$\textit{for all } \alpha \leq \beta \dot{-} 1 : R_{\mathfrak{z},\beta} \subseteq R_{\widehat{\mathfrak{z}},\alpha}.$$

Theorem B.2.1 *let R be a zigzag-simulation. We have:*

i) $R[\alpha]$ *is an zigzag-simulation* *ii) The downwards closure of R is a zigzag-simulation*

Proof. (i) We verify the zig-property of $R[\alpha]$. Suppose $\beta > 0$ and $k' \geq kR[\alpha]_{zig,\beta}m$. It follows that $k' \geq kR_{zig,\alpha+\beta}m$. Hence there is an $m' \geq m$ with $k'R_{zig,\alpha+\beta}m'$. Hence $k'R[\alpha]_{zig,\beta}m'$. The zag-property is analogous. Finally we have for $\beta > 0$:

$$\begin{array}{rcl} kR[\alpha]_{\mathfrak{z},\beta}m & \Rightarrow & kR_{\mathfrak{z},\alpha+\beta}m \\ & \Rightarrow & kR_{\widehat{\mathfrak{z}},(\alpha+\beta)\dot{-}1}m \\ & \Rightarrow & kR_{\widehat{\mathfrak{z}},\alpha+(\beta\dot{-}1)}m \\ & \Rightarrow & kR[\alpha]_{\widehat{\mathfrak{z}},\beta\dot{-}1}m \end{array}$$

ii) It is sufficient to show that: $R{\downarrow} = \bigcup\{R[\alpha] \mid \alpha \in \omega^+\}$. This is immediate from:

$$\begin{array}{rcl} \exists\gamma\ \beta{\leq}\gamma \textit{ and } kR_{\mathfrak{z},\gamma}m & \Leftrightarrow & \exists\gamma\ \beta{\leq}\gamma \textit{ and } kR_{\mathfrak{z},\beta+(\gamma\dot{-}\beta)}m \\ & \Leftrightarrow & \exists\gamma\ \beta{\leq}\gamma \textit{ and } kR[\gamma \dot{-} \beta]_{\mathfrak{z},\beta}m \\ & \Leftrightarrow & \exists\alpha\ kR[\alpha]_{\mathfrak{z},\beta}m \end{array}$$

The first equivalence is by $\sharp$. To prove the "$\Leftarrow$"-direction of the third equivalence, we need to show that for all α there is a $\gamma \geq \beta$, such that $\beta + \alpha = \beta + (\gamma \dot{-} \beta)$. In case $\beta < \infty$, we can take $\gamma := \alpha + \beta$ (by $\natural 2$). If $\beta = \infty$, take, $\gamma := \beta$ ❑

For $k{\in}\mathbb{K}$, we define $\uparrow k := \{k'{\in}K \mid k \leq k'\}$. Suppose R is a *zigzag*-simulation between $\mathbb{K}$ and $\mathbb{M}$. Let $k \in K$ and $m \in M$. Then $R{\restriction}(\uparrow k \times \uparrow m)$, the restriction of R to $\uparrow k \times \uparrow m$, is a *zigzag*-simulation between $\mathbb{K}[k]$ and $\mathbb{M}[m]$.

B.3 Preorders based on *Zigzag*-simulations

Let $\mathbb{K}$ and $\mathbb{M}$ be *rooted* $\mathcal{P}$-models. Define:

- $R : \mathbb{K} \preceq_\alpha \mathbb{M} \Leftrightarrow R$ is a *zigzag*-simulation between $\mathbb{K}$ and $\mathbb{M}$ and $\exists m{\in}M\ b_{\mathbb{K}}R_{zig,\alpha}m$
- $\mathbb{K} \preceq_\alpha \mathbb{M} \Leftrightarrow \exists R\ R : \mathbb{K} \preceq_\alpha \mathbb{M}$
- $\mathbb{K} \simeq_\alpha \mathbb{M} :\Leftrightarrow \mathbb{K} \preceq_\alpha \mathbb{M}$ and $\mathbb{M} \preceq_\alpha \mathbb{K}$

Theorem B.3.1 *$\preceq_\alpha$ is a partial preordering.*

Proof. Clearly, $ID : \mathbb{K} \preceq_\alpha \mathbb{K}$, where:

- $ID := \{\langle k, \mathfrak{z}, \alpha, k\rangle \mid \alpha \in \omega^+, \mathfrak{z} \in \{zig, zag\}, k \in K\}$

Moreover if $R : \mathbb{K} \preceq_\alpha \mathbb{M}$ and $S : \mathbb{M} \preceq_\alpha \mathbb{N}$, then $R \circ S : \mathbb{K} \preceq_\alpha \mathbb{N}$. ❑

Theorem B.3.2

$$\mathbb{K} \simeq_\alpha \mathbb{M} \Leftrightarrow \exists R \begin{array}{l} R \textit{ is a zigzag-simulation between } \mathbb{K} \textit{ and } \mathbb{M} \textit{ and} \\ b_{\mathbb{K}}R_{zig,\alpha}b_{\mathbb{M}} \textit{ and } b_{\mathbb{K}}R_{zag,\alpha}b_{\mathbb{M}}. \end{array}$$

Proof. "$\Leftarrow$" Trivial. "$\Rightarrow$" Suppose $S : \mathbb{K} \preceq_\alpha \mathbb{M}$ and $T : \mathbb{M} \preceq_\alpha \mathbb{K}$. Take:

- $R := \{\langle b_\mathbb{K}, zig, \alpha, b_\mathbb{M}\rangle, \langle b_\mathbb{K}, zag, \alpha, b_\mathbb{M}\rangle\} \cup S \cup \widehat{T}$

We have to show that R is a *zigzag*-simulation. For some m, $b_\mathbb{K} S_{zig,\alpha} m$. Hence $Th_\mathcal{P}(b_\mathbb{K}) \supseteq Th_\mathcal{P}(m) \supseteq Th_\mathcal{P}(b_\mathbb{M})$. Similarly in the other direction. It follows that $Th_\mathcal{P}(b_\mathbb{K}) = Th_\mathcal{P}(b_\mathbb{M})$. We leave the rest of the verification to the reader. ❑

Theorem B.3.3

$$\mathbb{K} \preceq_\infty \mathbb{M} \Leftrightarrow \exists m{\in}M\ \mathbb{K} \simeq_\infty \mathbb{M}[m]$$

Proof. left to the reader. ❑

B.4 Main Results

Theorem B.4.1 *Suppose that R is a zigzag-simulation between the $\mathcal{P}$-models $\mathbb{K}$ and $\mathbb{M}$. Then:*

$$\begin{array}{rcl} kR_{zig,\alpha}m & \Rightarrow & Th_{\Pi_\alpha(\mathcal{P})}(k) \supseteq Th_{\Pi_\alpha(\mathcal{P})}(m) \\ kR_{zag,\alpha}m & \Rightarrow & Th_{\Pi_\alpha(\mathcal{P})}(k) \subseteq Th_{\Pi_\alpha(\mathcal{P})}(m) \end{array}$$

Proof. By induction on A in $\mathcal{L}(\mathcal{P})$, simultaneous for all k, m, α, *zig* and *zag*. The cases of atoms, conjunction and disjunction are trivial. Suppose e.g., $kR_{zig,\alpha}m, (B \to C) \in \Pi_\alpha$ and $k \nVdash (B \to C)$. Then for some $k' \geq kk' \Vdash B$ and $k' \nVdash C$. By the zig-property there is an $m' \geq m$ such that $k'R_{zig,\alpha}m'$ and hence by the induction hypothesis $m' \nVdash C$. Moreover B will be in $\Pi_{\alpha \dot{-} 1}(\mathcal{P})$, so by the Induction Hypothesis applied for $\alpha \dot{-} 1$, noting that $k'R_{zag,\alpha \dot{-} 1}m'$, we find: $m' \Vdash B$. Ergo $m \nVdash (B \to C)$. ❑

Theorem B.4.2 *Suppose $\mathbb{K}$ and $\mathbb{M}$ are $\mathcal{P}$-models. Then:*

$$\mathbb{K} \preceq_\alpha \mathbb{M} \Rightarrow Th_{\Pi_\alpha(\mathcal{P})}(\mathbb{M}) \subseteq Th_{\Pi_\alpha(\mathcal{P})}(\mathbb{K}).$$

Proof. Left to the reader. ❑

Theorem B.4.3 *Suppose $\mathbb{K}$ and $\mathbb{M}$ are $\vec{p}$-models. Then:*

$$Th_{\Pi_n(\vec{p})}(\mathbb{M}) \subseteq Th_{\Pi_n(\vec{p})}(\mathbb{K}) \Rightarrow \mathbb{K} \preceq_n \mathbb{M}.$$

Proof. Suppose $\mathbb{K}$ and $\mathbb{M}$ are $\vec{p}$-models and $Th_{\Pi_n(\vec{p})}(\mathbb{M}) \subseteq Th_{\Pi_n(\vec{p})}(\mathbb{K})$. We want to prove: $\mathbb{K} \preceq_n \mathbb{M}$. Define:

- $kR_{zig,i}m :\Leftrightarrow Th_{\Pi_i(\vec{p})}(k) \supseteq Th_{\Pi_i(\vec{p})}(m)$, and $i > 0 \Rightarrow Th_{\Pi_{i-1}(\vec{p})}(k) \subseteq Th_{\Pi_{i-1}(\vec{p})}(m)$
- $kR_{zag,i}m :\Leftrightarrow Th_{\Pi_i(\vec{p})}(k) \subseteq Th_{\Pi_i(\vec{p})}(m)$ and $i > 0 \Rightarrow Th_{\Pi_{i-1}(\vec{p})}(k) \supseteq Th_{\Pi_{i-1}(\vec{p})}(m)$

We check that $R$is an *zigzag*-simulation.

Suppose e.g., $i > 0$ and $kR_{zig,i}m$. We verify the zig-property. Suppose $k \leq k'$. Let:

- $\eta_i(k') := (\bigwedge\{B{\in}\Pi_{i-1}(\vec{p}) \mid k' \Vdash B\} \to \bigvee\{C{\in}\Pi_i(\vec{p}) \mid k' \nVdash C\})$

Clearly, $k \nVdash \eta_i(k')$ and $\eta_i(k') \in \Pi_i(\vec{p})$. Ergo $m \nVdash \eta_i(k')$. But, then, for some $m' \geq m$:

$$m' \Vdash \bigwedge\{B{\in}\Pi_{i-1}(\vec{p}) \mid k' \Vdash B\} \textit{ and } m' \nVdash \bigvee\{C{\in}\Pi_i(\vec{p}) \mid k' \nVdash C\}.$$

It follows that $k'R_{zig,i}m'$.

We leave the rest of the verification to the reader. Since $b_{\mathbb{M}} \nVdash \eta_n(b_{\mathbb{K}})$, we can find an m such that: $b_{\mathbb{K}}R_{zig,i}m$. ❑

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