Meeting Strength
in substructural logics

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Abstract.

This paper contributes to the theory of hybrid substructural logics, i.e. weak logics given by a Gentzen-style proof theory in which there is only a *limited* possibility to use structural rules. Following the literature, we use an operator to mark formulas to which the extra structural rules may be applied. New in our approach is that we do not see this ∇ as a modality, but rather as the *meet* of the marked formula with a special type Q. In this way we can make the specific structural behaviour of marked formulas more explicit.

The main motivation for our approach is that we can provide a nice, intuitive semantics for hybrid substructural logics. Soundness and completeness for this semantics are proved; besides this we consider some proof-theoretical aspects like cut-elimination and embeddings of the 'strong' system in the hybrid one.

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1 Introduction

If we drop some or all of the structural rules from a Gentzen-type sequent calculus for let's say intuitionistic logic, the arising logic will be resource-conscious: for instance, in the absence of Contraction or Weakening, the number of times that a premisse is used, becomes relevant; if the rule of Permutation is absent, the ordering of the premisses. In recent years, such so-called *substructural* logics have received a lot of attention, partly for their theoretical interest, but also because of applications in e.g.

- computer science: linear logic, cf. GIRARD [9], TROELSTRA [21],
- linguistics: Lambek Calculus, cf. LAMBEK [14] for the original article, or MOORTGAT [15], VAN BENTHEM [2] or MORRILL [16] for recent developments,
- philosophy: relevance logic, cf. DUNN [7].

There is a bewildering variety of substructural logics, as we may drop any subset of structural rules from a standard derivation system. Of this landscape, WANSING [24] draws a partial map in the form of a lattice, set-inclusion of the derivable sequents being the ordering.

In general however one is not after systems where structural rules are categorically absent or present; certainly for applications such a rigidity would be unsatisfactory. For instance, the ordering of premisses in Lambek's Calculus reflects the fact that in natural languages the meaning of a sentence generally depends on the word order. But this dependency is not ubiquitous, and varies from one language to another; hence the adequacy becomes questionable of a formalism like the Lambek Calculus, which has no access to the rule of permutation at all. Besides that, a *restricted* version of Permutation (and Contraction) provides an elegant tool to describe some linguistic phenomena like (parasitic) gaps in relative clauses, cf. MORRILL [16] for details. Apart from such considerations from the field of applications, there are mathematical motivations as well to study *hybrid* systems. As an example, if one wants to *embed* a strong logic into a weak one, the latter needs to have have at least a restricted access to the structural rules that the strong one has.

Linear logic had this hybridity built in from the beginning, so let us now have a look at this site of the substructural landscape in some more detail. Girard used *operators*, the so-called exponentials (! and ?) as devices to encapsulate stronger logics. Operators need logical rules in the Gentzen paradigm: Girard gave the following left and right rule for ! (we consider intuitionistic linear logic):

$$\frac{X_1, A, X_2 \longrightarrow B}{X_1, !A, X_2 \longrightarrow B}\ [!L] \qquad \text{and} \qquad \frac{!X \longrightarrow A}{!X \longrightarrow !A}\ [!R]$$

where $!X$ denotes $!A_1, \ldots, !A_n$ if $X = A_1, \ldots, A_n$. Because of the $S4$-like character of these rules, the shriek ! is often referred to as a *modality*.

Weakening and Contraction now are only allowed on formulas marked with a !:

$$\frac{X \longrightarrow B}{X, !A \longrightarrow B}\ [W!] \qquad \text{and} \qquad \frac{X, !A, !A \longrightarrow B}{X, !A \longrightarrow B}\ [C!]$$

It was a very natural move for researchers interested in developing other hybrid substructural logics to look at linear logic for inspiration. The issue was addressed in wide scope in DOŠEN (cf. [5, 6]) who discusses the general picture, concentrating on proof-theoretical properties like embeddability. Independently, the idea was taken up by Morrill et alii in [17], who were interested in an extension of the Lambek Calculus with restricted Permutation, and in YETTER [25] where an extension of cyclic linear logic is treated.

The point that we want to make here is that there are some problems involved with a straightforward adaptation of the proof calculus for ! from linear logic to other substructural logics. To discuss these problems, let us assume that we add an operator $\Box$ to a substructural logic (for a precise definition of a substructural logic we refer to the next section), giving it the operational rules of !, i.e.

$$\frac{X_1, A, X_2 \longrightarrow B}{X_1, \Box A, X_2 \longrightarrow B}\ [\Box L] \qquad \text{and} \qquad \frac{\Box X \longrightarrow A}{\Box X \longrightarrow \Box A}\ [\Box R]$$

where $\Box X$ denotes $\Box A_1, \ldots, \Box A_n$ if $X = A_1, \ldots, A_n$. We will argue that the right rule of the 'modality' is not as natural as it seems to be. Wording it somewhat boldly:

- Having side effects on the meaning of other operators, $[\Box R]$ is not intuitive for all substructural logics, in particular not for those lacking the rule of Permutation.
- Some natural and appealing semantics for substructural logics do not allow for an equally natural and appealing interpretation of $\Box$, if $[\Box L]$ and $[\Box R]$ are its operational rules.

These topics are best illustrated via the Lambek Calculus (again, cf. the next section for definitions), a system *lacking* the rule of Permutation. Suppose that we want to add Contraction and Weakening to the Lambek Calculus, and that we introduce a !-like operator $\Box$, for which we have given rules $[\Box L]$, $[\Box R]$, $[W\Box]$ and $[C\Box]$:

$$\frac{X \longrightarrow B}{X, \Box A \longrightarrow B}\ [W\Box] \qquad\qquad \frac{X, \Box A, \Box A \longrightarrow B}{X, \Box A \longrightarrow B}\ [C\Box]$$

Note that according to the resource-conscious character of Lambek's Calculus, the *naive meaning* of a formula $\Box A$ will be "any list of information packages with A", here represented as: $A \ldots A$. Now consider the following derivation

$$\frac{\dfrac{\dfrac{\dfrac{A \longrightarrow A}{\Box A \longrightarrow A}\ [\Box L] \quad \dfrac{B \longrightarrow B}{\Box B \longrightarrow B}\ [\Box L]}{\Box A, \Box B \longrightarrow A \circ B}\ [\circ R]}{\Box A, \Box B \longrightarrow \Box(A \circ B)}\ [\Box R]}{\Box A \circ \Box B \longrightarrow \Box(A \circ B)}\ [\circ L]$$

which is a proof of the sequent $\Box A \circ \Box B \longrightarrow \Box(A \circ B)$. Our naive understanding of this sequent is: "$A \ldots AB \ldots B$ gives $AB \ldots AB$". Now in our opinion, it is counterintuitive to have this sequent as a theorem, *unless we add some kind of permutation rule for* $\Box$*-ed formulas.* The crucial step in the derivation is the application of the rule $[\Box R]$: it is here where the A's and B's are shuffled.

Note that we do not argue against having $[\Box R]$ itself as *part* of a hybrid system; we only feel that it should not be part of the *basic* hybrid system, and certainly not (a side effect of) the rule of proof for $\Box$.

Related to this problem, and in some sense formalizing it, is the second issue that we want to raise, namely that of the *semantics* of hybrid substructural logics. As we already mentioned, there are strong linguistic reasons for adding restricted permutation to the Lambek Calculus. Let us suppose, that we add a $\Box$ to the set of operators, with the logical rules given above. The rules allowing permutation of boxed formulas then could be

$$\frac{X_1, B, \Box A, X_2 \longrightarrow C}{X_1, \Box A, B, X_2 \longrightarrow C}\ [P'\Box]$$

(The double bar indicates that we have both the downward and the upward rule.)

The problem however is to give an intuitive *semantics* for the arising system $L\Box$. The Lambek calculus L itself is known to have nice interpretations: for instance[1], it is sound and complete with respect to semigroup semantics, cf. BUSZKOWSKI [3]. If we view $\Box$ as a *modality*, the obvious way to interpret $\Box$ would be via some accessibility relation. Some results are known in this direction, cf. KURTONINA [13] for a completeness result of $L\Box$ with respect to semigroup-like

[1] L has other interesting interpretations, which we will not discuss here.

relational structures expanded with an accessibility relation. In DE PAIVA [18], a category-theoretic interpretation is given which was inspired, again, by linear logic. However, it is not immediately clear what *intuitive* meaning one can assign to these proposed interpretations for the $\Box$-operator.

Indeed, more natural from the *applicational* point of view seems to be the *subalgebra interpretation* of HEPPLE [10] and MORRILL [17]. The boxed Lambek calculus $L\Box$ is interpreted in semigroups $\mathfrak{G}$ having a designated 'commuting subalgebra' $\mathfrak{G}'$ (i.e. consisting of elements g' satisfying $(\forall x \in G) \quad g' \cdot x = x \cdot g'$). Here the meaning function assigns to a boxed formula $\Box A$ the *intersection* of the meaning of A with the universe of the subalgebra. In other words, boxed formulas are special pieces of information, with a special (commutative) semantic behaviour.

Unfortunately, the rules given above, although sound, are not sufficient to prove completeness with respect to this subalgebra semantics. This was shown in VERSMISSEN [23]; replacing $[\Box R]$ by

$$\frac{X_1 \longrightarrow \Box B_1 \quad \ldots \quad X_n \longrightarrow \Box B_n \quad X_1, \ldots, X_n \longrightarrow A}{X_1, \ldots, X_n \longrightarrow \Box A} \; [\Box R']$$

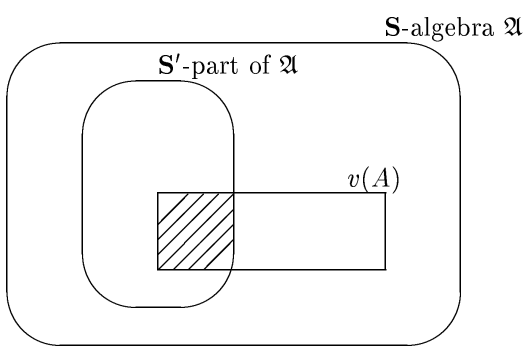
he can prove completeness for the subalgebra interpretation.

To analyze the rule $[\Box R']$, let us drop for a moment the association of $\Box$ with modal logic, and read $\Box A$ as 'a special A'. Now $[\Box R']$ says the following: if X proves an A, it proves that A is special if it can be decomposed into sequences proving (some other[2]) formulas to be special. Our idea is now to make this 'specialness' *explicit* by adding a special type Q to the language, and reading $\Box A$ as some sort of *meet* of Q and A. In the semigroup semantics, Q is then assigned a special subset of the semigroup, and $[\Box R']$ can be decomposed into

$$\frac{X \longrightarrow Q \quad X \longrightarrow A}{X \longrightarrow \Box A} \qquad \text{and} \qquad \frac{X_1 \longrightarrow Q \quad X_2 \longrightarrow Q}{X_1, X_2 \longrightarrow Q}$$

where intuitively, the latter rule states that the Q-elements of the semigroup indeed form a subalgebra. Adding the $\Box$-permutation rule $[P'\Box]$ ensures that this subalgebra consists of commuting elements.

Let us finish this introduction with putting these semantic considerations on the Lambek Calculus in a more general perspective. DOŠEN [4] describes an algebraic semantics for substructural logics, in which the resource-conscious intuitions concerning various substructural logics are somehow made explicit. Different kinds of algebras correspond to different substructural logics — we will go into details in section 4, the point that we want to make here is that these correspondences are such that when we consider two substructural logics, $\mathbf{S}$ and $\mathbf{S}'$, of which $\mathbf{S}'$ is stronger than $\mathbf{S}$, then the algebras for $\mathbf{S}'$ form a subclass of those for $\mathbf{S}$. Now if we want to have 'parts' of $\mathbf{S}$ that do allow all structural rules of $\mathbf{S}'$, what could be more natural than look at *subalgebras* of $\mathbf{S}$-algebras that are themselves algebras for $\mathbf{S}'$? The interpretation of a 'special' formula ∇A will then consist of that part of the interpretation of A which belongs to the $\mathbf{S}'$-part of the algebra, viz.



[2]One may restrict the rule by demanding that the B_i's be subformulas of formulas in the X_i's and A.

In this sense, the linguistic motivation for the subalgebra interpretation of the strengthening operator, has a nice mathematical counterpart.

OVERVIEW In the next section we give some preliminary definitions on substructural logics and their semantics. In section three we give our basic proof-theoretical definitions, and in section four we discuss the semantics for our approach. The following two sections are devoted to proof-theoretical properties of our systems: cut-elimination and embeddings. In the last section we draw some conclusions and we raise some questions.

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2 Preliminaries

In this section we give some basic definitions and results concerning substructural logics. For more information, the reader is referred to DOŠEN [4].

The idea of using a special meet operator to strengthen logics lacking some structural rules is not confined to one particular site in the substructural landscape. In this paper, we want to be as general as possible, for instance abstracting away from the particular connectives[3] of the system under consideration. However, we will confine ourselves to logics meeting the following constraints:

Definition 2.1 *The* language *of the systems considered will consist of* basic types*,* brackets *and* connectives *from the following set:* $\backslash$ *(left slash,* $/$ *(right slash),* $\circ$ *(concatenation, or multiplicative product),* $\wedge$ *(meet),* $\vee$ *(join),* $\top$ *(top) and* $\mathbf{1}$ *(one). From this alphabet* formulas *are built up, in the usual way.*

The set of terms *is defined by induction*[4]*: every formula of the language is a term, and so is the empty term* Λ*. If* X *and* Y *are terms, then so is* (X, Y)*. We assume familiarity with notions like* subterms*, and* substitutions*.* $X[Y]$ *denotes a term* X *in which* Y *occurs as a subterm. Finally, a* sequent *is a pair* $X \longrightarrow A$ *consisting of a term* X *and a formula* A*.*

As variables we take capitals $A, B, C, A', \ldots$ *ranging over formulas, capitals* $X, Y, Z, X', \ldots$ *ranging over terms.*

Definition 2.2 *We associate with every connective of the list given above a standard pair of derivation rules, a left rule (or rule of use), and a right rule (or rule of proof).*

[3]Note that the behaviour of the logical operators is partly determined by the structural rules; for instance, in the presence of Permutation, the right and the left slash of the Lambek Calculus collapse into the linear implication.

[4]Characteristic of substructural logics is that terms are *structured*; in the most general case, i.e. if we even allow for the absence of the rule of Associativity, we need the given definition of terms and sequents.

$$\frac{X[B] \longrightarrow C \quad Y \longrightarrow A}{X[Y, A\backslash B] \longrightarrow C}\ [\backslash L] \qquad \frac{(A, X) \longrightarrow B}{X \longrightarrow A\backslash B}\ [\backslash R]$$

$$\frac{X[B] \longrightarrow C \quad Y \longrightarrow A}{X[B/A, Y] \longrightarrow C}\ [/L] \qquad \frac{(X, A) \longrightarrow B}{X \longrightarrow B/A}\ [/R]$$

$$\frac{X[A, B] \longrightarrow C}{X[A \circ B] \longrightarrow C}\ [\circ L] \qquad \frac{X \longrightarrow A \quad Y \longrightarrow B}{(X, Y) \longrightarrow A \circ B}\ [\circ R]$$

$$\frac{X[A_i] \longrightarrow C}{X[A_0 \wedge A_1] \longrightarrow C}\ [\wedge L] \qquad \frac{X \longrightarrow A \quad X \longrightarrow B}{X \longrightarrow A \wedge B}\ [\wedge R]$$

$$\frac{X[A] \longrightarrow C \quad X[B] \longrightarrow C}{X[A \vee B] \longrightarrow C}\ [\vee L] \qquad \frac{X \longrightarrow A_i}{X \longrightarrow A_0 \vee A_1}\ [\vee R]$$

$$\textit{no left rule for } \top \qquad \frac{}{X \longrightarrow \top}\ [\top R]$$

$$\frac{X[\Lambda] \longrightarrow A}{X[\mathbf{1}] \longrightarrow A}\ [\mathbf{1}L] \qquad \frac{}{\Lambda \longrightarrow \mathbf{1}}\ [\mathbf{1}R]$$

Definition 2.3 *Without defining what a structural rule* is, *we just mention the ones that we will focus on in this paper. These are Associativity* $[A]$, *Permutation* $[P]$, *Contraction* $[C]$, *Expansion* $[E]$ *and Weakening* $[W]$, *given by*[5]

$$\frac{X[(Y_1, Y_2), Y_3] \longrightarrow B}{X[Y_1, (Y_2, Y_3)] \longrightarrow B}\ [A^l] \qquad \frac{X[Y_1, (Y_2, Y_3)] \longrightarrow B}{X[(Y_1, Y_2), Y_3] \longrightarrow B}\ [A^r]$$

$$\frac{X[Y_2, Y_1] \longrightarrow B}{X[Y_1, Y_2] \longrightarrow B}\ [P] \qquad \frac{X[Y, Y] \longrightarrow B}{X[Y] \longrightarrow B}\ [C]$$

$$\frac{X[Y] \longrightarrow B}{X[Y, Y] \longrightarrow B}\ [E] \qquad \frac{X[\Lambda] \longrightarrow B}{X[Y] \longrightarrow B}\ [W]$$

Besides these, we need structural rules to ensure that Λ *indeed functions as the empty sequence:*

$$\frac{X[Y] \longrightarrow B}{X[\Lambda, Y] \longrightarrow B}\ [I\Lambda^l] \quad \frac{X[\Lambda, Y] \longrightarrow B}{X[Y] \longrightarrow B}\ [E\Lambda^l] \quad \frac{X[Y] \longrightarrow B}{X[Y, \Lambda] \longrightarrow B}\ [I\Lambda^r] \quad \frac{X[Y, \Lambda] \longrightarrow B}{X[Y] \longrightarrow B}\ [E\Lambda^r]$$

Now we are in the position to give a definition of the logical systems we will be treating in this paper.

Definition 2.4 *In this paper we will understand with a* substructural logic, *a Gentzen-type derivation system of the following type.*

Let C *be a set of connectives,* Ξ *a set of structural rules containing the* Λ*-rules*[6]. *Then* $\mathbf{S}_{\Xi}(C)$ *is the logic consisting of the following groups of (axioms and) rules:*

1. *the basics: the axiom of Identity and the Cut-rule:*

$$\frac{}{A \longrightarrow A}\ [Id] \qquad \frac{Y \longrightarrow A \quad X[A] \longrightarrow B}{X[Y] \longrightarrow B}\ [Cut].$$

[5] Note that these rules act on *terms*, not on *formulas*; in the presence of the product $\circ$ and the top $\top$ one can give an equivalent system where the structural rules operate on formulas, cf. DOŠEN [4].

[6] At the moment, we are not interested in calculi where the 'empty' sequent has non-standard behaviour.

2. operational *rules for the connectives of C, as given in definition 2.2.*

3. *of course, the structural rules of the system are those in Ξ.*

Notions like derivability and theoremhood are defined as usual.

EXAMPLES: In this terminology, the Lambek Calculus finds it place as $\mathbf{S}_{\{A\}}(\{\circ, \backslash, /\})$ (at least if we relax the condition that the empty term is not allowed as the antecedent of a sequent). $\mathbf{S}_{\{A,P\}}(C)$ will be the C-fragment of intuitionistic linear logic, etc...

In the sequel we will consider extensions of such systems, and investigate some mathematical properties like cut-elimination and semantics. With respect to the first, it is of interest whether the Cut-rule can be eliminated from the system. We state the following fact (cf. DOŠEN [4] for a proof).

Theorem 2.5 *Let X be a substructural logic such that the structural rules of X are among $[A]$, $[P]$, $[C]$, $[M]$ and the Λ-rules. Then applications of $[Cut]$ can be removed from proofs in X.*

Now we turn to providing some basic information on the semantics of substructural logics. We follow DOŠEN [4] (cf. section 4 for motivations):

Definition 2.6 *A* resource algebra[7] *is a structure $(W, \cdot, \cap, 1)$ where $1 \in W$, the set W is closed under the binary operations $\cdot$ and $\cap$, the structure $(W, \cap)$ is a semilattice, and $\cdot$ is distributive over $\cap$. Define the partial ordering $\leq$ by '$x \leq y$ iff $x \cap y = x$'.*

A resource model *is a pair $(\mathfrak{F}, v_0)$ with $\mathfrak{F}$ a resource algebra and v_0 a valuation, i.e. function mapping propositional variables into subsets of the universe of $\mathfrak{F}$. Such a v_0 is understood to satisfy*

$$(Heredity\ \ v_0) \qquad\qquad x_1 \cap x_2 \in v_0(q) \iff \big(x_1 \in v_0(q)\ \&\ x_2 \in v_0(q)\big).$$

The function v_0 can be extended to a map v for all *formulas and terms. We only give the following clauses:*

$$\begin{array}{lcl}
v(A_1/A_2) & = & \{w \in W | (\forall y \in v(A_2))\ w \cdot y \in v(A_2)\} \\
v(A_1 \backslash A_2) & = & \{w \in W | (\forall y \in v(A_1))\ y \cdot w \in v(A_2)\} \\
v(A_1 \circ A_2) & = & \{w \in W | \exists y_1 \in v(A_1)\ \exists y_2 \in v(A_2)\ y_1 \cdot y_2 \leq w\} \\
v(\mathbf{1}) & = & \{w \in W | 1 \leq w\} \\
v(A_1 \wedge A_2) & = & \{w \in W | w \in v(A_1)\ \&\ w \in v(A_2)\} \\
v(A_1 \vee A_2) & = & \{w \in W | w \in v(A_1)\ or\ w \in v(A_2)\ or\ \exists y_i \in v(A_i) y_1 \cap y_2 \leq w\} \\
v(\top) & = & W \\
 & & \\
v(\Lambda) & = & \{w \in W | 1 \leq w\} \\
v(X_1, X_2) & = & \{w \in W | \exists y_1 \in v(X_1) \exists y_2 \in v(X_2)\ y_1 \cdot y_2 \leq w\}.
\end{array}$$

Because of the modal flavour of the definition, we will also use terminology and notation from modal logic, like possible world *for elements of the universe of the resource algebra,* truth *of a formula A at a world x of a model $\mathfrak{M}$, notation: $\mathfrak{M}, x \models A$, for membership of x in $v(A)$. A sequent $X \longrightarrow A$ is valid in a model $\mathfrak{M}$ if $v(X) \subseteq v(A)$,* valid in a class K *of frames if it holds in every model on a frame of* K, valid *or* holds in a frame $\mathfrak{F}$ *if it is valid in the class $\{\mathfrak{F}\}$.*

We need the following technical lemma later on:

Proposition 2.7 *Let $\mathfrak{M} = (\mathfrak{F}, v_o)$ be a resource model, and let Y be a term; then for all x_1, x_2 in $\mathfrak{F}$:*

$$(Heredity\ \ v) \qquad\qquad x_1 \cap x_2 \in v(Y) \iff \big(x_1 \in v(Y)\ \&\ x_2 \in v(Y)\big).$$

[7] These algebras are called *semi-lattice-ordered groupoids* or *slogs* in DOŠEN [4].

3 The basic idea

The basic idea of our approach is very simple: we introduce two new symbols to the language: a constant type Q and a unary operator ∇. Intuitively, a formula ∇A should be read as: an A having special structural behaviour. The proof rules for ∇ will be very simple (cf. the definition below): ∇A will be indistinguishable from the *meet* of Q and A[8]. The 'special structural behaviour' is coded in the formula Q: Q is intended to be a type containing meta-information rather than information proper.

We already mentioned in the introduction that our strategy to make a substructural logic *hybrid* is to select a new set of structural rules and only permit the application of these rules on formulas/terms that are special. For instance, the naive idea to allow Permutation only on marked items would be to introduce

$$\frac{X[\nabla A_2, \nabla A_1] \longrightarrow B}{X[\nabla A_1, \nabla A_2] \longrightarrow B}\ [P\nabla],$$

and likewise for the other structural rules. The disadvantage of this approach is that we face problems with cut-elimination. For instance, the following derivation would involve a necessary application of $[Cut]$:

$$\dfrac{\dfrac{Q \longrightarrow Q \quad Q \longrightarrow Q}{Q \longrightarrow \nabla Q}\ [\nabla R] \qquad \dfrac{\dfrac{\dfrac{\dfrac{Q \longrightarrow Q \quad \nabla A \longrightarrow \nabla A}{Q, \nabla A \longrightarrow Q \circ \nabla A}\ [\circ R]}{\nabla Q, \nabla A \longrightarrow Q \circ \nabla A}\ [\nabla L]}{\nabla A, \nabla Q \longrightarrow Q \circ \nabla A}\ [P\nabla]}{}}{\nabla A, Q \longrightarrow Q \circ \nabla A}\ [Cut]$$

For, a cut-free proof of $\nabla A, Q \longrightarrow Q \circ \nabla A$ should end in an application of $[\circ R]$, which is clearly impossible. We will follow the usual way out here, by compiling the $[Cut]$-rule into the hybridization rules.

Recall from the introduction that one of our objections against $S4$-like *modalities* as structural operators, was the right rule of proof. We were looking for a formalization in which we can choose *explicitly* whether the multiplicative addition of special formulas will be special, or semantically, whether the special formulas are interpreted in a subalgebra or merely in a subset of the resource algebra. Again, the naive solution

$$\frac{X_1 \longrightarrow Q \quad X_2 \longrightarrow Q}{(X_1, X_2) \longrightarrow Q} \qquad\qquad \frac{}{\Lambda \longrightarrow Q}$$

will cause problems with cut-elimination, so the Q-rules[9] of our system will be slightly more complex as well.

Definition 3.1 *Let* **S** *be a substructural logic as defined in 2.4. We assume that we add the following* new *connectives to the language: a constant* Q *and a unary* ∇*, for which we define some operational rules.*

The operator ∇ *has two left rules:*

$$\frac{X[A] \longrightarrow B}{X[\nabla A] \longrightarrow B}\ [\nabla L, 1] \qquad\qquad \frac{X[Q] \longrightarrow B}{X[\nabla A] \longrightarrow B}\ [\nabla L, 2]$$

[8] If we have a meet-operator in the language (with standard logical rules), then the equivalence of ∇A and $Q \wedge A$ will be easily provable, and ∇A may be read as an abbreviation. However, we feel it philosophically more sound to have the structural-behaviour-operator ∇ as an *primitive* connective of the language. Besides that, there are situations where having an unrestricted meet-operator in the systems is less attractive: for instance, adding a meet-operator to the Lambek calculus, will pump up the recognizing power, cf. some results in KANAZAWA [12], and the recent proof of PENTUS [19] that all languages recognized by a Lambek Grammar are context free.

[9] The second rule is needed to ensure that the Q-part of the algebra is not only closed under $\cdot$, but also contains the basic piece of information 1.

and the following right rule:

$$\frac{X \longrightarrow A \quad X \longrightarrow Q}{X \longrightarrow \nabla A}\ [\nabla R]$$

For Q, we have two rules:

$$\frac{X_1 \longrightarrow Q \quad X_2 \longrightarrow Q \quad X[Q] \longrightarrow A}{X[(X_1, X_2)] \longrightarrow A}\ [Q\circ] \qquad\qquad \frac{X[Q] \longrightarrow A}{X[\Lambda] \longrightarrow A}\ [Q1]$$

The (Q-)hybridization rules $[A^lQ], \ldots, [WQ]$ *are defined as the ordinary structural rules, with the proviso that they have as an extra premiss that the terms involved derive Q, for instance Permutation:*

$$\frac{X[Y_2, Y_1] \longrightarrow B \quad Y_1 \longrightarrow Q \quad Y_2 \longrightarrow Q}{X[Y_1, Y_2] \longrightarrow B}\ [PQ]$$

The extension of **S** *with the left and right rule for ∇ will be called* $\mathbf{S}\nabla$*, and the extension of* $\mathbf{S}\nabla$ *with the Q-rules:* $\mathbf{S}Q$*. If we add, furthermore, a set Ξ of hybridization rules to either* $\mathbf{S}\nabla$ *or* $\mathbf{S}Q$*, we will denote the resulting* hybrid substructural logic *by* $\mathbf{S}\nabla_\Xi$*, resp,* $\mathbf{S}Q_\Xi$*.*

Example Let **ILL0** be intuitionistic linear logic without exponentials or quantifiers (cf. Troelstra [21], then $\mathbf{ILL0}Q_{CW}$ is our version of propositional intuitionistic linear logic. To give the reader some feeling for our approach, we show how to derive the ∇-version of $[R!]$ in our system:

$$\frac{\nabla X \longrightarrow A}{\nabla X \longrightarrow \nabla A}\ (*)$$

where $\nabla X = \nabla A_1, \ldots, \nabla A_n$ if $X = A_1, \ldots, A_n$ (note that we use multiset notation for terms). First we show by induction on n, that $\nabla A_1, \ldots, \nabla A_n \longrightarrow Q$. For the induction step, we prove

$$\frac{\nabla A_1, \ldots, \nabla A_{n-1} \longrightarrow Q \quad \dfrac{Q \longrightarrow Q}{\nabla A_n \longrightarrow Q}\ [\nabla L]}{\nabla A_1, \ldots, \nabla A_n \longrightarrow Q}\ [Q]$$

so now we can prove $(*)$ by one application of $[\nabla R]$. In fact, it is straightforward to show that propositional intuitionistic linear logic can be embedded in $\mathbf{ILL0}Q_{CW}$. Note however, that our approach yields somewhat *more* than propositional intuitionistic linear logic, as $\mathbf{ILL0}Q_{CW} \vdash A \wedge \nabla B \longrightarrow \nabla(A \wedge B)$, while $A \wedge !B \longrightarrow !(A \wedge B)$ is not provable in the latter system.

Remark By no means does definition 3.1 exhaust the possibilities to hybridize a substructural logic. For instance, the attentive reader will have noticed the subtle difference between the rules $[P'\Box]$ on page 2, and $[P\nabla]$ on page 7. In the approach of definition 3.1, a proper reformulation of $[P'\Box]$ would have been

$$\frac{X[Y_2, Y_1] \longrightarrow B \quad Y_2 \longrightarrow Q}{X[Y_1, Y_2] \longrightarrow B}\ [P^lQ] \qquad\qquad \frac{X[Y_2, Y_1] \longrightarrow B \quad Y_1 \longrightarrow Q}{X[Y_1, Y_2] \longrightarrow B}\ [P^rQ]$$

In other words, the question is whether we allow special terms to jump over arbitrary terms, or only over terms that are special themselves. Both approaches seem to deserve investigation — in Venema [22] we investigated in detail a hybrid version of the Lambek Calculus with $[P^lQ]$ and $[P^rQ]$ as the hybridization rules, cf. also the remark at the end of section 4.

4 Semantics

Of course, a judgement of the intuitive appeal of an interpretation is not a mathematical ordeal; besides, one will not have a rigid opinion of the ideal semantics for a certain system. Nevertheless, some semantics are more equal than others, and we want to start this section with a defense of the Došen groupoid semantics, of which we already gave a formal definition in the preliminaries. Here we mention a reading of this semantics which renders its interpretation as much more than the technical, algebraic tool that it may seem to be at first sight. Wansing [24] develops a so-called *informational interpretation*, in which the universe W of a resource algebra $(W, \cdot, \cap, 1)$ is a set of information pieces, 1 is the initial (empty) set of information, $\cap$ is the intersection of information and $\cdot$ is the addition operation. The valuation v is to be read as a support function: $x \in v(A)$ iff x supports the information A.

The naturalness of the various clauses in the truth definition 2.6 are argued for by both Došen and Wansing. For instance, the odd-looking clause for the disjunction ($\vee$) is defended by Došen by pointing out an analogy in the Birkhoff representation of lattices by sets, while Wansing reasons that "In the case of $\vee$ it makes perfectly good sense to require that $(A \vee B)$ is not only true at pieces of information a at which A is true or at which B is true but also at pieces of information which prolong the intersection of pieces of information b_1 and b_2 such that A is true at b_1 and B is true at b_2. Thus, $(A \vee B)$ should also be true at information pieces which prolong so to speak the common content of information pieces b_1, b_2 with A true at b_1 and B true at b_2."

We agree with Wansing that this informational interpretation is quite intuitive; in fact, it is maybe even the most obvious one when one has the *resource-conscious* character of substructural logics in mind, where the ethereal notion of information is caught by the resource-bounded carriers. The very nice thing about Došen's semantics is that it can make our intuitions about substructural logics mathematically explicit. In particular, much in the style of modal logic, there is a correspondence theory for substructural logics, where every structural rule of inference finds a counterpart in a *condition* on resource algebras. For instance, the rule of Contraction is sound precisely in those algebras where one given information package x gives at least as much information as two of it added together $(x \cdot x)$: x is re-usable. To be more precise:

Definition 4.1 *Below we give a table in which we list the* corresponding resource equations *of some structural rules:*

$[A^l]$	$x \cdot (y \cdot z)$	$\leq$	$(x \cdot y) \cdot z$
$[A^r]$	$(x \cdot y) \cdot z$	$\leq$	$x \cdot (y \cdot z)$
$[P]$	$x \cdot y$	$\leq$	$y \cdot x$
$[E]$	x	$\leq$	$x \cdot x$
$[C]$	$x \cdot x$	$\leq$	x
$[W]$	1	$\leq$	x
$[I\Lambda^l]$	x	$\leq$	$1 \cdot x$
$[E\Lambda^l]$	$1 \cdot x$	$\leq$	x
$[I\Lambda^r]$	x	$\leq$	$x \cdot 1$
$[E\Lambda^r]$	$x \cdot 1$	$\leq$	x

Let Ξ be a set of structural rules, then $\phi(\Xi)$ is the set of corresponding formulas. For a class K of resource algebras and a set of formulas Φ, K_Φ is the class of resource algebras in K validating all the formulas in Φ, and that G is the class of all resource algebras.

The following theorem can be seen as a general soundness and completeness theorem for substructural logics:

Theorem 4.2 (Došen) *Let Ξ be a set of structural rules. Then (for any set of connectives)* $\mathbf{S}_\Xi$ *is sound and complete with respect to $\mathsf{G}_{\phi(\Xi)}$, i.e. for every sequent $X \longrightarrow A$ we have*

$$\mathbf{S}_\Xi \vdash X \longrightarrow A \iff \mathsf{G}_{\phi(\Xi)} \models X \longrightarrow A.$$

Proof
For the soundness part of the proof, we refer to Došen [4]; for the completeness part, we just mention how in that same article *the canonical model* for $\mathbf{S}_\Xi$ is defined.

First of all, a possible world (i.e. an element of the universe of the resource algebra) is any set x for which there is a term X such that $x = \{A \mid \mathbf{S}_\Xi \vdash X \longrightarrow A\}$. The initial information set 1 consists of all those formulas that can be derived form the empty term Λ, the intersection operation $\cap$ is just set intersection, and if x_1, x_2 consist of those formulas that can be derived from resp. X_1 and X_2, then $x_1 \cdot x_2$ is defined as the set $\{A \mid \mathbf{S}_\Xi \vdash (X_1, X_2) \longrightarrow A\}$. Finally, the canonical valuation is given by $V_0(p) = \{x \mid p \in x\}$.

The crucial lemma in the proof is then the *canonical lemma* stating that for any formula A and possible world X of the canonical frame, we have

$$x \models A \iff A \in x.$$

Finally, one has to prove that for the respective substructural logics, the canonical frame is indeed in the corresponding class of resource algebras. We will give one example, Contraction: suppose that Ξ contains $[C]$, then we have to show that in the canonical resource algebra, $\exists x(x \cdot x \leq x)$. Thereto, let x be an element of the model, and let X be the corresponding term such that $x = \{A \mid \mathbf{S}_\Xi \vdash X \longrightarrow A\}$. By definition, $x \cdot x = \{A \mid \mathbf{S}_\Xi \vdash (X, X) \longrightarrow A\}$. So, it is immediate that $x \cdot x$ is a subset of x, and thus by definition of $\leq$, we find $x \cdot x \leq x$. □

So now we come to the main and motivating part of this paper: defining the semantics for our hybrid systems, and proving the proof theory sound and complete with respect to it. Let us first repeat the motivation already given in the introduction, but now in a more explicit terminology. If $\Xi \subset \Xi'$ are two sets of structural rules, and C a set of connectives, then the resource algebras corresponding to $\mathbf{S}_{\Xi'}(C)$ form a subclass of those for $\mathbf{S}_\Xi(C)$. In other words, extra structural rules impose extra conditions on resource algebras. So what could be a more natural semantics for the hybrid system $\mathbf{S}_\Xi(C)\nabla_{\Xi'}$, than resource algebras in which *all* information carriers are governed by the $\phi(\Xi)$-laws, and *some special ones* (to be 'precise', the carriers of Q) by the $\phi(\Xi')$-laws.

This inspires the following definition:

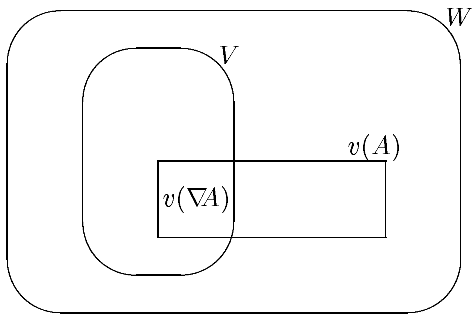
Definition 4.3 *Let Φ, Ψ be formulas in the (algebraic) language of resource algebras. A (Φ, Ψ)-hybrid resource algebra is a quintuple $(W, V, \cdot, \cap, 1)$ satisfying the conditions $A1 \ldots A4$ below:*
$(A1)$ $(W, \cdot, \cap, 1)$ *is a resource algebra,*
$(A2)$ $(W, \cdot, \cap, 1) \models \Phi$,
$(A3)$ $V \subset W$ *is closed under* $\cap$,
$(A4)$ $(V, \cdot, \cap, 1) \models \Psi$,
$(A5)$ $(V, \cdot, \cap, 1)$ *is a resource algebra,*
If moreover, $(W, V, \cdot, \cap, 1)$ satisfies condition $A5$, too, we call $(W, V, \cdot, \cap, 1)$ a (Φ, Ψ)-oval resource algebra.

Models are defined in the obvious way, and the definition of the interpretation function is extended with the following clauses:

$$\begin{array}{rcl} v(Q) & = & V \\ v(\nabla A) & = & \{y \mid y \in V \;\&\; y \in v(A)\}, \end{array}$$

if $((W, V, \cdot, \cap, 1), v)$ is the model at hand.

The idea of the subset/subalgebra interpretation is best illustrated via a picture:



Note that $(W, V, \cdot, \cap, 1)$ is a (Φ, Ψ)-oval resource algebra iff $(W, \cdot, \cap, 1)$ is a Φ-resource algebra and $(V, \cdot, \cap, 1)$ is a Ψ-resource algebra.

Theorem 4.4 *Let $\Xi \subset \Xi'$ be sets of structural rules. Then*
(i) the hybrid system $\mathbf{S}_\Xi(C)\nabla_{\Xi'}$ is sound and complete with respect to $(\phi(\Xi), \phi(\Xi'))$-hybrid resource algebras.
(ii) the hybrid system $\mathbf{S}_\Xi(C)Q_{\Xi'}$ is sound and complete with respect to $(\phi(\Xi), \phi(\Xi'))$-oval resource algebra.

Proof
(i) To start with **soundness**, we show that all the axioms and rules of $\mathbf{S}_{\Xi'}(C)\nabla_{\Xi'}$ (which we abbreviate by $\mathbf{S}$) are valid resp. sound in $(\phi(\Xi), \phi(\Xi')$-hybrid resource algebras. This is straightforward to check for the logical rules and axioms, and for the operational rules for the old connectives.

Of the rules for ∇, we only treat $[\nabla R]$. Let $\mathfrak{F} = (W, V, \cdot, \cap, 1)$ be an arbitrary algebra in the class. Suppose that v_0 is a valuation on $\mathfrak{F}$ such that $v(X) \subseteq v(Q)$ and $v(X) \subseteq v(A)$. Then obviously $v(X) \subseteq v(Q) \cap v(A) = V \cap v(A) = v(\nabla A)$.
The soundness of the structural rules in Ξ follows from 4.2.

Finally we consider the hybridization rules, by example of Expansion:

$$\frac{Y \longrightarrow Q \quad X[Y] \longrightarrow A}{X[Y, Y] \longrightarrow A} \; [EQ]$$

Suppose that Ξ' contains the rule of Expansion, and Ξ does not, and that we have a valuation v_0 for which $v(X[Y]) \subseteq v(A)$. We have to show that $v(X[Y, Y]) \subseteq v(A)$. To do so, it suffices to establish that

$$(*) \qquad v(X[(Y, Y)]) \subseteq v(X[Y])$$

which we will do by induction to the complexity of X. We only treat the basic case, where $X = (Y, Y)$. Let y be in $v(Y, Y)$. By unraveling the definition of $v(Y)$, we find that $y \geq s \cdot t$ for some s, t with (i) $s, t \in v(Y)$. By $\vdash Y \longrightarrow Q$ we have $v(Y) \subset V$, so (ii) $s, t \in V$. By (ii) and condition $(A3)$, we have that (iii) $s \cap t \in V$. This implies by $(A4)$ that $(s \cap t) \cdot (s \cap t) \geq s \cap t$. Therefore, we find by (A1) that (iv) $s \cap t \leq s \cdot t \leq y$. By (Heredity v) and (i) we obtain (v) $y \in v(Y)$.

For **completeness**, we give a canonical $(\phi(\Xi), \phi(\Xi'))$-hybrid resource algebra $\mathfrak{G}$ and a canonical valuation v_0 on $\mathfrak{G}$ such that for any sequent $X \longrightarrow A$ which is not provable in $\mathbf{S}$, we have

$v(X) \not\subseteq v(A)$. $\mathfrak{G} = (W, V, \cdot, \cap, 1)$ is defined as follows:

$$\begin{array}{rcl} W & = & \{x \mid \text{for some term } X,\ x = \{A \mid \mathbf{S} \vdash X \longrightarrow A\}\} \\ V & = & \{x \mid Q \in x\} \\ x \cap y & = & \{A \mid A \in x \ \&\ A \in y\} \\ x \cdot y & = & \{x \mid \mathbf{S} \vdash X, Y \longrightarrow A\} \\ 1 & = & \{A \mid \mathbf{S} \vdash \Lambda \longrightarrow A\} \end{array}$$

where in the clause for $\cdot$, we adopted the convention that x and X are related by $x = \{A \mid \mathbf{S} \vdash X \longrightarrow A\}$. (In the sequel we will do the same, without warning.) The canonical valuation v_0 is given by

$$v_0(p) = \{x \mid p \in x\}.$$

Let $\mathfrak{M}$ be the canonical model $(\mathfrak{G}, v_0)$.

First we prove that $\mathfrak{G}$ is a $(\phi(\Xi), \phi(\Xi'))$-hybrid resource algebra. The conditions $(A1)$ and $(A2)$ follow from Došen's completeness proof, cf. 4.2. For $(A3)$, suppose that $x_1, x_2 \in V$; then by definition of V, $Q \in x_1, x_2$, so by definition of $\cap$, $Q \in x_1 \cap x_2$.
Condition $(A4)$ again is proved by example: Permutation. Suppose that $[P] \in \Xi' - \Xi$, then we have to show that

$$(\dagger) \qquad \mathfrak{G} \models (\forall x_1 x_2 \in V)\ x_1 \cdot x_2 \leq x_2 \cdot x_1.$$

As $\leq$ is set-inclusion, it suffices to prove that every formula A provable from (X_1, X_2) is also provable from (X_2, X_1), provided that from both X_i's, Q is provable. But this is precisely what the rule $[PQ]$ says.

Second, we need the Truth Lemma

$$(\ddagger) \qquad \mathfrak{M}, x \models A \iff A \in x$$

which is proved by induction on the complexity of A.

We only consider the cases where $A = Q$ or $A = \nabla B$, referring the reader to section 3.4 of Došen [4] for the other cases. Now, with respect to the first case: $\mathfrak{M}, x \models Q$ iff $x \in V$ (by definition of $\models$) iff $Q \in x$ (by definition of V). And for the second case we have:

	$\mathfrak{M}, x \models \nabla B$	
iff	$x \in V$ & $\mathfrak{M}, x \models B$	(by definition of $\models$)
iff	$Q \in x$ & $B \in x$	(by the induction hypothesis)
iff	$\mathbf{S} \vdash X \longrightarrow Q$ and $\mathbf{S} \vdash X \longrightarrow B$	(by the x, X-convention)
iff	$\mathbf{S} \vdash X \longrightarrow \nabla B$	(proof theory)
iff	$\nabla B \in x$	(by the x, X-convention)

Now we are finished, for assume that $\mathbf{S} \not\vdash X \longrightarrow A$, and let x be the set $\{A \mid \mathbf{S} \vdash X \longrightarrow A\}$. It is easy to show that $x \in v(X)$, while $x \notin v(A)$. So we find that $v(X)$ is not a subset of $v(A)$.

(ii) This proof builds on the proof in **(i)**.

For **soundness**, we have to prove that the Q-rules are sound in $(\phi(\Xi), \phi(\Xi'))$-resource resource algebra.

For $[Q\circ]$, suppose that v_0 is a valuation on a $(\phi(\Xi), \phi(\Xi'))$-oval resource algebra $\mathfrak{F} = (W, V, \cdot, \cap, 1)$ such that $v(X_i) \subset V$ and $v(Y[Q]) \subset v(A)$. We have to show $(*)$ $v(Y[X_1, X_2]) \subset v(A)$.

It follows easily by (A5) (closure of V under $\cdot$) that $v(X_1, X_2) \subset V$. By induction on Y one can easily infer that this implies $(*)$.

To show $[Q1]$ is likewise simple, now using the fact that $1 \in V$.

For **completeness**, we assume that we have defined a canonical algebra $\mathfrak{S}$ like in the proof of **(i)**; it will be clear that it suffices to show that this $\mathfrak{S}$ is a $(\phi(\Xi), \phi(\Xi'))$-oval resource algebra. In other words, we have to establish (i) V is closed under $\cdot$, (ii) $1 \in V$, (iii) $(V, \cap)$ is a semilattice, and (iv) $\cdot$ is distributive over $\cap$.

For (ii), note that the logic contains the theorem $\Lambda \longrightarrow Q$. By definition of 1 then $Q \in 1$, so by definition of v_0 we find $1 \in V$. For (i), let x_1, x_2 be in V. By definition we have proofs for $X_1 \longrightarrow Q$ and $X_2 \longrightarrow Q$, so with the axiom $Q \longrightarrow Q$, one application of $[Q\circ]$ gives a proof of $X_1, X_2 \longrightarrow Q$. By definition of $\cdot$ then, $Q \in x_1 \cdot x_2$, so $x_1 \cdot x_2$ is in V by definition of V. The conditions (iii) and (iv) follow immediately from the fact that $(V, \cdot, \cap, 1)$ is a subalgebra of $(W, \cdot, \cap, 1)$ $\Box$

REMARK Similar results can be proved for hybridization rules like $[P^l Q]$ and $[P^r Q]$ discussed at the end of section 3. For instance, adding precisely these two rules to a logic $\mathbf{S}_\Xi$ will yield a logic which is sound and complete with respect to oval resource algebras satisfying

$$(\phi(P')) \qquad \forall x \in V \forall y \in W \ \ x \cdot y = y \cdot x,$$

i.e. the subalgebra consists of elements that commute with arbitrary elements of the bigger algebra.

5 Cut-elimination

For several reasons, among which are resource-consciousness and proof-theoretical elegance, the logical rule of $[Cut]$ is less attractive. Thus the question becomes relevant whether it can be eliminated from the logic, whether every provable sequent has a $[Cut]$-free derivation. For some families of systems we will answer this question in the affirmative. In order to give a more elegant proof of cut-elimination, we consider a seemingly stronger version of the $[Cut]$-rule:

Definition 5.1 *Let $Y(Z)$ denote a term Y with a positive number of occurrences of Z. In the following we asume that our $[Cut]$-rule has the following form:*

$$\frac{X \longrightarrow A \quad Y(A) \longrightarrow B}{Y(X) \longrightarrow B}\ [Cut]$$

Put into words, the version of the $[Cut]$-rule given above says that any number of occurrences of A in Y may be cut at once. Note that this rule is not really stronger than the usual $[Cut]$-rule — we even gave it the same name.

In the sequel we will show that for suitable combinations of a substructural logic $\mathbf{S}$ and a set Ξ of Q-hybridization rules, the resulting system $\mathbf{S}Q_\Xi$ (and thus $\mathbf{S}\nabla_\Xi$) can do without $[Cut]$. First we need some definitions:

Definition 5.2 *Proof-trees are defined as usual, we denote "Π is a proof for $X \longrightarrow A$" by: $\frac{\Pi}{X \longrightarrow A}$. The* tree-depth *of a proof (tree) is defined as follows: the tree-depth of an axiom is zero, and if Π is of the form $\dfrac{\frac{\Pi_1}{P_1} \ \cdots \ \frac{\Pi_n}{P_n}}{C_n}$, then $t(\Pi) = 1 + \max\{t(\Pi_i) \mid 1 \leq i \leq n\}$.*

The complexity $c(A)$ *of a formula A is the total number of occurrences of connectives in A.*

Definition 5.3 *All formulas in the application of a rule $[R]$ are called side formulas, except in the following cases:*
(1) $[R]$ is an operational rule, and A is the formula introduced; then A is called the main *formula.*
(2) $[R]$ is $[Cut]$, and A is the formula not appearing in the conclusion; then A is called the cut formula.

We leave it to the reader to give a formal definition of a multiple *resp.* single *cut formula.*

The main lemma needed to eliminate $[Cut]$ from the system is the following:

Lemma 5.4 *Let $\Xi \subset \Xi'$ be sets of structural rules such that $\Xi, \Xi' \subseteq \{[A], [P], [C], [W]\}$. If a theorem has an $\mathbf{S}_\Xi Q_{\Xi'}$-proof with a single application of $[Cut]$, then the sequent is also cut-free provable.*

Proof.
Abbreviate $\mathbf{S}_\Xi Q_{\Xi'}$ by T. We define a cut-degree $d(\Pi)$ of proofs Π with one application of $[Cut]$: let Π_0 be the subproof of Π ending in the application of $[Cut]$. Assume that the daughters of Π_0 are Π_1 and Π_2, and that A is the cut formula. Then $(c(A), t(\Pi_1) + t(\Pi_2))$ is the cut-degree of Π. Assume that cut-degrees are lexicographically ordered.

We will now prove the lemma, by induction on the degree of the proof given in the assumption of the lemma. So, assume that the proof has a subproof Π of the form

$$\frac{\dfrac{\Pi_1}{X \longrightarrow A}\,[LR] \qquad \dfrac{\Pi_2}{Y(A) \longrightarrow B}\,[RR]}{Y(X) \longrightarrow B}\,[Cut]$$

(Note our convention concerning the bracketing $(\cdot)$!)

The idea of the proof is of course to move $[Cut]$ upwards into Π_1 or Π_2 (whence it will eventually disappear) in such a way that the cut-degree of the transformed proof has decreased. This calculation will not always be made explicitly, nor will we always explicitly apply the induction hypothesis to the transformed proof.

To decide which action to take, we make a case distinction. First, divide the axioms/rules of T into the following groups:

I the identity axiom, and the operational rules for the 'old' connectives and ∇,

II the structural rules in Ξ,

III the Q-rules,

IV the hybridization rules (i.e. from Ξ').

Our main distinction however, is whether the cut formula A is main or side formula in the rules $[LR]$ and $[RR]$, and single or multiple:

A The cut formula is single, and main formula in both $[LR]$ and $[RR]$. In this case we have to do with rules from I in both the left and right proof. As an example we treat the case where A is of the form ∇C (the other cases are standard); transform the proof

$$\frac{\dfrac{\dfrac{\Pi_{11}}{X \longrightarrow C} \quad \dfrac{\Pi_{12}}{X \longrightarrow Q}}{X \longrightarrow \nabla C}\,[\nabla R] \qquad \dfrac{\dfrac{\Pi_{20}}{Y[P] \longrightarrow B}}{Y[\nabla C] \longrightarrow B}\,[\nabla L]}{Y[X] \longrightarrow B}\,[Cut] \quad \rightsquigarrow \quad \frac{\dfrac{\Pi_{1i}}{X \longrightarrow P} \quad \dfrac{\Pi_{20}}{Y[P] \longrightarrow B}}{Y[X] \longrightarrow B}\,[Cut]$$

where either $P = C$ and $i = 2$, or $P = Q$ and $i = 1$. Note that the cut-degree of this proof is indeed less than that of the original one, as the complexity of the cut formula has decreased.

A' The cut formula is multiple, and main formula in both $[LR]$ and $[RR]$. Now the idea is first to cut the left premisse of $[Cut]$ with the premisse(s) of $[RR]$, thus leaving a $[Cut]$ of a single main case- kind. After removing this $[Cut]$ in the way described in **A**, we have a proof with two applications of $[Cut]$, like in the following example:

$$\frac{\dfrac{\dfrac{\Pi_{11}}{X \longrightarrow C} \quad \dfrac{\Pi_{12}}{X \longrightarrow Q}}{X \longrightarrow \nabla C}\,[\nabla R] \qquad \dfrac{\dfrac{\Pi_{20}}{Y(\nabla C)[P] \longrightarrow B}}{Y(\nabla C)[\nabla C] \longrightarrow B}\,[\nabla L]}{Y(X)[X] \longrightarrow B}\,[Cut]$$

$\rightsquigarrow$

$$\frac{\dfrac{\Pi_{1i}}{X \longrightarrow P} \qquad \dfrac{\dfrac{\dfrac{\Pi_{11}}{X \longrightarrow C} \quad \dfrac{\Pi_{12}}{X \longrightarrow Q}}{X \longrightarrow \nabla C}\,[\nabla L] \qquad \dfrac{\Pi_{20}}{Y(\nabla C)[P] \longrightarrow B}}{Y(X)[P] \longrightarrow B}\,[Cut]}{Y(X)[X] \longrightarrow B}\,[Cut]$$

Now we first remove the upper $[Cut]$, which is possible by Induction Hypothesis, as its depth is less than that of the original $[Cut]$. Then we remove the second $[Cut]$, which the Induction Hypothesis allows us to do because of the decreased complexity of the cut formula.

B The cut formula is side formula of $[LR]$. Look at which group $[LR]$ is from: if $[LR]$ is from I or II, the procedure is standard.

In case we are dealing with $[Q\circ]$, transform

$$\dfrac{\dfrac{\dfrac{\Pi_{11}}{X_1 \longrightarrow Q} \quad \dfrac{\Pi_{12}}{X_2 \longrightarrow Q} \quad \dfrac{\Pi_{13}}{X[Q] \longrightarrow A}}{X[(X_1, X_2)] \longrightarrow A}\,[Q\circ] \quad \dfrac{\Pi_2}{Y(A) \longrightarrow B}}{Y(X[(X_1, X_2)]) \longrightarrow B}\,[Cut]$$

into

$$\dfrac{\dfrac{\Pi_{11}}{X_1 \longrightarrow Q} \quad \dfrac{\Pi_{12}}{X_2 \longrightarrow Q} \quad \dfrac{\dfrac{\Pi_{13}}{X[Q] \longrightarrow A} \quad \dfrac{\Pi_2}{Y(A) \longrightarrow B}}{Y(X[Q]) \longrightarrow B}\,[Cut]}{Y(X[(X_1, X_2)]) \longrightarrow B}\,[Q\circ]$$

In the other case of III, $[LR] = [Q1]$, and our move is:

$$\dfrac{\dfrac{\dfrac{\Pi_{10}}{X[Q] \longrightarrow A}}{X[\Lambda] \longrightarrow A}\,[Q1] \quad \dfrac{\Pi_2}{Y(A) \longrightarrow B}}{Y(X[\Lambda]) \longrightarrow B}\,[Cut] \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Pi_{10}}{X[Q] \longrightarrow A} \quad \dfrac{\Pi_2}{Y(A) \longrightarrow B}}{Y(X[Q]) \longrightarrow B}\,[Cut]}{Y(X[\Lambda]) \longrightarrow B}\,[Q1]$$

For the case where the left rule applied was from group IV, we consider an example: suppose that $[LR] = [CQ]$:

$$\dfrac{\dfrac{\dfrac{\Pi_{11}}{Z \longrightarrow Q} \quad \dfrac{\Pi_{12}}{X[Z, Z] \longrightarrow A}}{X[Z] \longrightarrow B}\,[CQ] \quad \dfrac{\Pi_2}{Y(A) \longrightarrow B}}{Y(X[Z]) \longrightarrow B}\,[Cut]$$

is transformed into

$$\dfrac{\dfrac{\Pi_{11}}{Z \longrightarrow Q} \quad \dfrac{\dfrac{\Pi_{12}}{X[Z, Z] \longrightarrow A} \quad \dfrac{\Pi_2}{Y(A) \longrightarrow B}}{Y(X[Z, Z]) \longrightarrow B}\,[Cut]}{Y(X[Z]) \longrightarrow B}\,[CQ]$$

Note that indeed the cut degree of the proof has decreased, (although the complexity of the cut formula is still the same).

C The cut formula is a side formula in $[RR]$. Now distinguish cases, according to which group $[RR]$ is from:

The cases of I and II are well-known again (for A not identical to ∇B) and straightforward in the case of $A = \nabla B$.

If $[RR] = [Q\circ]$, make a further case distinction as to where in $Y[(Y_1, Y_2)]$ the formula A occurs. We only treat the most complex case, where the cut formula occurs both inside Y_1 and inside Y_2, and outside (Y_1, Y_2).

$$\dfrac{\dfrac{\Pi_1}{X \longrightarrow A} \quad \dfrac{\dfrac{\Pi_{21}}{Y_1(A) \longrightarrow Q} \quad \dfrac{\Pi_{22}}{Y_2(A) \longrightarrow Q} \quad \dfrac{\Pi_{23}}{Y(A)[Q] \longrightarrow B}}{Y(A)[(Y_1(A), Y_2(A))] \longrightarrow B}\,[Q\circ]}{Y(X)[(Y_1(X), Y_2(X))] \longrightarrow B}\,[Cut]$$

which we transform into

$$\dfrac{\dfrac{\dfrac{\Pi_1}{X \longrightarrow A} \quad \dfrac{\Pi_{2i}}{Y_i(A) \longrightarrow Q}}{Y_i(X) \longrightarrow Q}\,[Cut] \quad \dfrac{\dfrac{\Pi_1}{X \longrightarrow A} \quad \dfrac{\Pi_{23}}{Y(A)[Q] \longrightarrow B}}{Y(X)[Q] \longrightarrow B}\,[Cut]}{Y(X)[(Y_1(X), Y_2(X))] \longrightarrow B}\,[Q\circ]$$

The case where $[RR] = [Q\wedge]$ is rather straightforward, and is left to the reader.

So we are left with the situation where the right rule was one of the new hybridization rules. Again, we confine ourselves to the example of Contraction. Note that the cut formula A may occur inside the contracted term, outside of it, or both. It is this third (and most complex) case that we treat here:

$$\dfrac{\dfrac{\Pi_1}{X \longrightarrow A} \quad \dfrac{\dfrac{\Pi_{21}}{Z(A) \longrightarrow Q} \quad \dfrac{\Pi_{22}}{Y(A)[Z(A), Z(A)] \longrightarrow B}}{Y(A)[Z(A)] \longrightarrow B}\,[CQ]}{Y(X)[Z(X)] \longrightarrow B}\,[Cut]$$

is replaced by

$$\dfrac{\dfrac{\dfrac{\Pi_1}{X \longrightarrow A} \quad \dfrac{\Pi_{21}}{Z(A) \longrightarrow Q}}{Z(X) \longrightarrow Q}\,[Cut] \quad \dfrac{\dfrac{\Pi_1}{X \longrightarrow A} \quad \dfrac{\Pi_{22}}{Y(A)[Z(A), Z(A)] \longrightarrow B}}{Y(X)[Z(X), Z(X)] \longrightarrow B}\,[Cut]}{Y(X)[Z(X)] \longrightarrow B}\,[CQ]$$

Note that this is the only case where the multiplicity of the cut formula increases. As this does not affect the cut degree, we may apply the Induction Hypothesis to both applications of $[Cut]$ in the transformed proof, as their depth is less than that of the original $[Cut]$.

□

Theorem 5.5 *Let $\Xi \subset \Xi'$ be sets of structural rules such that $\Xi, \Xi' \subseteq \{[A], [P], [C], [W]\}$. Any theorem of $\mathbf{S}_\Xi Q_{\Xi'}$-proof has a cut-free proof.*

Proof Using lemma 5.4, we can successively remove all applications of $[Cut]$ from a given $\mathbf{S}_\Xi Q_{\Xi'}$-proof. □

6 Embeddings

In the same way that intuitionistic and classical logic can be faithfully embedded in linear logic, we can show that our hybrid logic $\mathbf{S}_\Xi(Q)Q_{\Xi'}$ is not weaker in expressive power than $\mathbf{S}_\Xi$. The basic idea behind our embeddings is very simple to illustrate via the semantics of section 4: let $\mathbf{S}$, $\mathbf{SQ}'$ and $\mathbf{S}'$ be abbreviations for $\mathbf{S}_\Xi(C)$, $\mathbf{S}_\Xi(C)Q_{\Xi'}$ and $\mathbf{S}_{\Xi'}(C)$ respectively. By our completeness theorem 4.4, every $\mathbf{SQ}'$-structure has a subalgebra (consisting of Q-elements) which is in itself an appropriate structure for $\mathbf{S}'$. In other words, in a certain semantic sense, the 'Q-part of the hybrid logic $\mathbf{SQ}'$ *is* $\mathbf{S}'$. Our embeddings can be seen a proof-theoretical counterpart and implementation of this idea to identify $\mathbf{S}'$ with the Q-part of $\mathbf{SQ}'$.

First we define translations from the formulas of $\mathbf{S}_\Xi$ to the formulas of $\mathbf{S}_\Xi(C)Q_{\Xi'}$, then we prove the embedding theorem.

Definition 6.1 *Let C be a set of connectives; define the following translations $(\cdot)^\nabla$ and $(\cdot)'$ from C-formulas to $L \cup \{\nabla, Q\}$-formulas:*

$$\begin{array}{rcl} p^\nabla & = & \nabla p \\ (\top)^\nabla & = & \nabla \top \\ (\mathbf{1})^\nabla & = & \nabla \mathbf{1} \\ (A \wedge B)^\nabla & = & \nabla(A^\nabla \wedge B^\nabla) \\ (A \vee B)^\nabla & = & \nabla(A^\nabla \vee B^\nabla) \\ (A/B)^\nabla & = & \nabla(A^\nabla / B^\nabla) \\ (A \backslash B)^\nabla & = & \nabla(A^\nabla \backslash B^\nabla) \\ (A \circ B)^\nabla & = & \nabla(A^\nabla \circ B^\nabla) \end{array} \qquad \begin{array}{rcl} p' & = & p \\ (\top)' & = & Q \\ (\mathbf{1})' & = & \nabla \mathbf{1} \\ (A \wedge B)' & = & (A' \vee B') \\ (A \vee B)' & = & (\nabla A' \vee \nabla B') \\ (A/B)' & = & (A'/\nabla B') \\ (A \backslash B)' & = & (\nabla A' \backslash B') \\ (A \circ B)' & = & (\nabla A' \circ \nabla B') \end{array}$$

Both are extended to terms by putting

$$\begin{array}{rcl} \Lambda^\nabla & = & \Lambda \\ (X,Y)^\nabla & = & (X^\nabla, Y^\nabla) \end{array} \qquad \begin{array}{rcl} \Lambda^+ & = & \Lambda \\ A^+ & = & \nabla A' \\ (X,Y)^+ & = & (X^+, Y^+) \end{array}$$

Theorem 6.2 *Let C be a set of connectives containing $\top$, and let $\Xi \subset \Xi'$ be sets of structural rules. Then for any sequent $X \longrightarrow A$ we have that*

$$\begin{array}{rcl} \mathbf{S}_{\Xi'}(C) \vdash X \longrightarrow A & \iff & \mathbf{S}_{\Xi}(C)Q_{\Xi'} \vdash X^\nabla \longrightarrow A^\nabla \\ \mathbf{S}_{\Xi'}(C) \vdash X \longrightarrow A & \iff & \mathbf{S}_{\Xi}(C)Q_{\Xi'} \vdash X^+ \longrightarrow A' \end{array}$$

Proof.
Let us fix Ξ and Ξ', and abbreviate $\mathbf{S}_{\Xi'}(C)$ by $\mathbf{S}'$, $\mathbf{S}_{\Xi}(C)Q_{\Xi'}$ by $\mathbf{SQ}'$. We will only prove the theorem for the more parsimonious translation $(\cdot)'$.

$\boxed{\Rightarrow}$ First, the following claim will be needed later on:

$(*)$ For all terms X: $\mathbf{SQ}' \vdash X^+ \longrightarrow Q$.

One can easily prove $(*)$ by term-induction (note that the Q-rules are essential here).

The correctness of the embedding will be proved by induction on the derivation of $X \longrightarrow A$ in $\mathbf{S}'$.

For the atomic step, assume that $X \longrightarrow A$ is an axiom of $\mathbf{S}'$. We are dealing with one of the following three cases:

(i) X is of the form A. Then $\mathbf{SQ}' \vdash \nabla A \longrightarrow A$ is easily proved.
(ii) A is of the form T. Then $\mathbf{SQ}' \vdash X^+ \longrightarrow A$ is the claim $(*)$.
(iii) $X = \Lambda$ and $A = 1$. Then $\mathbf{SQ}' \vdash \Lambda^+ \longrightarrow \mathbf{1}'$ follows from $\vdash \Lambda \longrightarrow Q$ and $\vdash \Lambda \longrightarrow 1$.

If $\mathbf{S}'$ derives $X \longrightarrow A$ by application of a structural rule in Ξ, $\mathbf{SQ}'$ has this rule too and the proof is straightforward.

For the case where the last rule applied was in $\Xi' - \Xi$, we give an example:

$$\frac{Y[(Z,Z)] \longrightarrow A}{Y[Z] \longrightarrow A}\ [C]$$

By the inductive hypothesis, $\mathbf{SQ}' \vdash Y^+[(Z^+, Z^+)] \longrightarrow A'$. Now consider the following derivation:

$$\frac{Y^+[Z^+, Z^+] \longrightarrow A' \quad Z^+ \longrightarrow Q}{Y^+[Z^+] \longrightarrow A'}\ [CQ]$$

The only case left is where the last step in the derivation is of an application of an operational rule. Below we will give a few examples (IH stands for: induction hypothesis).

$[/R]$ Assume that $X \longrightarrow B/A$ was derived from $(X, A) \longrightarrow B$. Using the (IH), we find

$$\frac{(X^+, \nabla A') \longrightarrow B'}{X^+ \longrightarrow (B/A)'}\ [/R]$$

[$/L$] If $Y[C/B, X] \longrightarrow A$ is derived from $X \longrightarrow B$ and $Y[C] \longrightarrow A$, then the following is an $\mathbf{SQ}'$-derivation of $Y^+[\nabla(C/B)', X^+] \longrightarrow A'$:

$$\dfrac{\nabla(C'/\nabla B') \longrightarrow \nabla C'/\nabla B' \qquad \dfrac{\dfrac{\dfrac{\text{IH}}{X^+ \longrightarrow B'} \quad \dfrac{(*)}{X^+ \longrightarrow Q}}{X^+ \longrightarrow \nabla B'}\,[\nabla R] \qquad \dfrac{\text{IH}}{Y^+[\nabla C'] \longrightarrow A'}}{Y^+[\nabla C'/\nabla B', X^+] \longrightarrow A'}\,[/L]}{Y^+[\nabla(C'/\nabla B'), X^+] \longrightarrow A'}\,[Cut]$$

where the reader is invited to show that $\nabla(C'/\nabla B') \longrightarrow \nabla C'/\nabla B'$ is a $\mathbf{SQ}'$-theorem.

[$\circ R$] Assume that $X, Y \longrightarrow A \circ B$ is derived from $X \longrightarrow A$ and $Y \longrightarrow B$. Then we obtain:

$$\dfrac{\dfrac{\dfrac{(*)}{X^+ \longrightarrow Q} \quad \dfrac{(IH)}{X^+ \longrightarrow A'}}{X^+ \longrightarrow \nabla A'}\,[\nabla R] \quad \dfrac{\dfrac{(*)}{Y^+ \longrightarrow Q} \quad \dfrac{(IH)}{Y^+ \longrightarrow B'}}{Y^+ \longrightarrow \nabla B'}\,[\nabla R]}{(X, Y)^+ \longrightarrow \nabla A' \circ \nabla B'}\,[\circ R]$$

[$\mathbf{1}L$] Here we use the following transformation:

$$\dfrac{X[\Lambda] \longrightarrow A}{X[\mathbf{1}] \longrightarrow A}\,[\mathbf{1}L] \qquad\qquad \dfrac{\dfrac{\dfrac{\text{IH}}{X^+[\Lambda] \longrightarrow A'}}{X^+[\mathbf{1}] \longrightarrow A'}\,[\mathbf{1}L]}{X^+[\nabla \mathbf{1}] \longrightarrow A'}\,[\nabla L]$$

$\boxed{\Leftarrow}$ The basic idea for the other direction is that in a certain sense, every $\mathbf{SQ}'$-derivation 'is' an $\mathbf{S}'$-derivation. The only problem is that $\mathbf{SQ}'$-formulas are not necessarily $\mathbf{S}'$-formulas, because we added new connectives. So let us start with defining a forgetful translation $(\cdot)^-$ from $\mathbf{SQ}'$-formulas to $\mathbf{S}'$-formulas: for atoms we set $P^- = P$, and we continue with $Q^- = \top$, $(\nabla A)^- = A^-$, and $(\cdot)^-$ is a homomorphism with respect to the other connectives. For terms, we set $\Lambda^- = \Lambda$ and $(X, Y)^- = (X^-, Y^-)$. Our aim now is to establish the following claim:

$$(*) \qquad \mathbf{SQ}' \vdash Y \longrightarrow B \ \Rightarrow\ \mathbf{S}' \vdash Y^- \longrightarrow B^-.$$

The *proof* of this claim is by induction to $\mathbf{SQ}'$-derivations. The basic step, and most of the inductive steps are trivial. We only consider the cases where the last step of the derivation of $Y \longrightarrow B$ was by one of the hybridization rules, and this case is treated by an example (Contraction): suppose that the conclusion is of the form $Y[Z] \longrightarrow B$, and it was derived from

$$\dfrac{Z \longrightarrow Q \quad Y[Z, Z] \longrightarrow B}{Y[Z] \longrightarrow B}\,[CQ]$$

By induction hypothesis, $\mathbf{S}' \vdash Y^-[Z^-, Z^-] \longrightarrow B^-$ (and $\mathbf{S}' \vdash Z^- \longrightarrow \top$, but this we do not need), so by one application of $[C]$ we find that $\mathbf{S}' \vdash Y^-[Z^-] \longrightarrow B^-$.

We are virtually finished now: let $X \longrightarrow A$ be an $\mathbf{S}'$-sequent for which $\mathbf{SQ}' \vdash X^+ \longrightarrow A'$. It is straightforward to show that $(X^+)^- = X$ and $(A')^- = A$, so by $(*)$ we find that $\mathbf{S}' \vdash X \longrightarrow A$, which is what we wanted to prove indeed. $\Box$

REMARK Note that the non-modal kind of embedding defined here, deviates from the tradition in the literature. The connections between our theorem and the results in e.g. DOŠEN [5, 6] remains to be investigated.

7 Conclusions

Accepting the idea to use operators for the task of strengthening a substructural logic, we have asked ourselves the question what the *meaning* of a formula ∇A (∇ the operator) in a resource-bounded derivation system might be. Our answer was, that a formula ∇A is like a labelled formula: the label (∇, but in fact a special type Q) tells us that the information proper, A, may be *used*, qua structural rules, in a way extending the default character of the logic. The novelty of this paper (as far as we know) lies in the fact that we have *implemented* this idea in a fashion inspired by the wish to give a natural *semantics* for the arising hybrid logic. We have separated the information of a formula from its structural behaviour, thus being able to make the structural properties of marked formulas *explicit* by manipulating the proof- and structural rules involving the special type Q.

It seems that this idea can easily be extended to logics having more than two kinds of structural behaviour. In fact, one could introduce a type Q_Ξ for *every* set of structural rules, and allow precisely these rules on sequences that derive Q_Ξ.

We believe our approach to be intuitive and compatible with the paradigm of resource-consciousness in substructural logics. Besides, it enjoys the nice mathematical properties one would want for hybrid substructural logics, like cut-elimination for the basic systems, and embeddability of the 'strong' logic in the hybrid system.

A lot of research remains to be done — we mention a few questions:

1. A huge part of the research into linear logic is of a category-theoretic nature. Recently, the use of modalities in weaker logics has been studied from such a perspective as well, cf. DE PAIVA [18]. What is the category-theoretic side of our approaches?

2. Substructural logics have a type-theoretical side, via (adaptations of) the Curry-Howard interpretation, cf. WANSING [24], VAN BENTHEM [2]. (How) can we assign terms to proofs in our calculi?

3. Besides linear logic itself, Girard also invented a new proof method for it, viz. via *proofnets.* In his dissertation [20], Roorda extended this method to the Lambek calculus. Can we also find proof nets for the extended logic discussed here?

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