APPLICATIONS OF CONSTRUCTIVE LOGIC TO SHEAF CONSTRUCTIONS IN TOPOSES

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1 Introduction

1.1 Abstract

The internal logic of a topos is exploited to give an easy proof of the fact that topologies in a topos form a locale and to give simple internal reformulations of two well-known variants of associated sheaf constructions. We have two applications: (i) we extend Gödel's negative translation to a large class of higher order logics including **HHA** and **IZF** with help of the associated sheaf construction for the double negation topology, (ii) generalizations of our internal notions and constructions are useful to investigate the relations between locales of Lawvere-Tierney topologies, Gabriel-Grothendieck topologies, universal closure operations and locales of localisations of the subcategory of $\mathbb{T}$-algebras in an elementary topos for an arbitrary theory $\mathbb{T}$.

1.2 The internal logic of toposes

Recall that toposes correspond in a natural way with higher order logics ranging from intuitionistic to classic. References to this phenomenon are by now abundant: cf. [Boileau &Joyal], [Johnstone (1)], [Lambek & Scott], [Rosolini] and recently the detailed [Lavendhomme & Lucas]. We prefer the style of [Rosolini]. The ingredients that generate a higher order logic are a set of Sorts, from which the Types can be constructed, a set of Functional Symbols (including possibly constants) together with a set of Axioms. For example, think of *higher order Heyting arithmetic*, **HHA**: sort $\mathbb{N}$, function symbols $0 : \mathbf{1} \to \mathbb{N}$ and $s : \mathbb{N} \to \mathbb{N}$ and the usual Peano axioms. If $L_{\mathbf{E}}$ is the language corresponding with the topos $\mathbf{E}$ there we are interested in these categorical properties Φ of E for which we can find an expression ϕ in $L_{\mathbf{E}}$ such that $\mathbf{E} \models \phi$ if and only if Φ is true for E.
Formulas in these type theories are just terms of type Ω, where Ω is the object of truth values. Recall, for example, that a morphism $j : \Omega \to \Omega$ is a (Lawvere-Tierney) topology in $\mathbf{E}$ if and only if the following three diagrams commute:

$$\begin{tikzcd} 1 \arrow[r, "\top"] \arrow[dr, "\top"'] & \Omega \arrow[d, "j"] \\ & \Omega \end{tikzcd} \qquad \begin{tikzcd} \Omega \arrow[r, "j"] \arrow[dr, "j"'] & \Omega \arrow[d, "j"] \\ & \Omega \end{tikzcd} \qquad \begin{tikzcd} \Omega \times \Omega \arrow[r, "\wedge"] \arrow[d, "j \times j"] & \Omega \arrow[d, "j"] \\ \Omega \times \Omega \arrow[r, "\wedge"] & \Omega \end{tikzcd}$$

(i) (ii) (iii)

We will treat such a morphism $j:\Omega\to\Omega$ as a unary logical operator. We will apply it to terms of type Ω, i.e. formulas.
A morphism $j:\Omega\to\Omega$ satisfies the externally given definition of topology if and only if internally holds, i.e., in **E** is true:

(i) $j\top=\top$

(ii) $\forall\omega\in\Omega\ jj\omega=j\omega$

(iii) $\forall\omega_1,\omega_2\in\Omega\ j(\omega_1\wedge\omega_2)=j\omega_1\wedge j\omega_2$

This internal language should be considered as a convenient shorthand to prove facts on toposes. A drawback is that one hides the categorical properties of the topos necessary for the proofs. On the other hand internal proofs are often more conspicuous and shorter:

1.3 First example

The Lawvere-Tierney topologies of a topos form a locale. This is a well-known fact, but can be proven easily via a constructive proof in the internal language of the topos. E.g. one can check that the proof in [Johnstone (2)] is constructive. Note that such a proof shows that the object $\mathsf{LT\text{-}TOP} := \{j:\Omega^\Omega \mid j \text{ is a LT-topology}\}$ is a locale in **E**. Which is a stonger fact than its corrolary that the global elements of $\mathsf{LT\text{-}TOP}$, that is the external family of morphisms $j:\Omega_{\mathbf{T}}\to\Omega_{\mathbf{T}}$ that are a topology in **E**, form a locale.
This illustrates that one can internalize a notion, like in this case LT-topology, in various degrees!

It is also instructive to consider a proof using the related Grothendieck-topologies:

1.3.1 Definition. A(n internal) Grothendieck topology on Ω is a subobject $J\subseteq\Omega$, such that

(i) $\top\in J$

(ii) $\forall\omega_1,\omega_2\in J\ \omega_1\wedge\omega_2\in J$

(iii) $\forall\omega_1\in J\ \forall\omega_2\in\Omega\ \omega_1\wedge\omega_2\in J\to\omega_2\in J$.

Now consider the following object $\mathsf{G\text{-}TOP} = \{J:[\Omega] \mid J \text{ is a G-topology}\}$ together with the morphisms

$$(-)_j = \mathsf{G\text{-}TOP}\to\mathsf{LT\text{-}TOP}: J\mapsto\{\omega\in\Omega\mid\top=j\omega\}$$
$$(-)_J = \mathsf{LT\text{-}TOP}\to\mathsf{G\text{-}TOP}: j\mapsto\omega\in J$$

(Recall that the formula $\omega\in J$ itself is an element of Ω.)

It is easy to see that these morphisms are each others inverses. One can prove directly that $\mathsf{G\text{-}TOP}$ is a locale:

$$J_1\leq J_2 \;=\; J_1\subseteq J_2$$
$$J_1\wedge J_2 \;=\; J_1\cap J_2$$
$$J_1\Rightarrow J_2 \;=\; \{\omega\in\Omega\mid\forall J\ J\wedge J_1\leq J_2\to\omega\in J\}$$

$$\bigvee_I J_i = \{\omega\in\Omega \mid \forall J\,[(\forall i\in I\; J_i\subseteq J)\to\omega\in J]\}.$$

It is straightforward to check that G-TOP becomes a locale in this way, and one can show that G-TOP and LT-TOP are isomorphic locales.

1.4 Second example

As an exercise the reader can try to give an direct, constructive proof for the following theorem of Mikkelsen: an topos has a natural number object if and only if it allows for constructions of free monoids (cf. [Johnstone (1)]). In this case there seems not to be a direct categorical proof.

As another example of the use of internal logic we will present in the next section internal versions of various constructions of associated sheaf functors.

We have two uses for such an internal version of the associated sheaf functor. First we apply it to derive an extension of the Gödel negative translation and secondly we generalize it to the context of algebraic theories $\mathbb{T}$, and study a notion of topology on the object $\Omega_{\mathbb{T}}$ of subalgebras of $F\mathbb{1}$, the free $\mathbb{T}$-algebra generated by a singleton set.

2 An internal associated sheaf construction

In this section and the following we will perform the associated sheaf construction of both Lawvere and Johnstone inside a fixed topos **E** equipped with a Lawvere-Tierney topology $j:\Omega \rightarrow \Omega$.

2.1 Fact. Let $j:\Omega\rightarrow\Omega$ be a topology in a topos **E** and A an object in **E**. Then

(i) $A \subseteq B$ is j-dense iff $\forall a \in B\ ja \in A \rightarrow a \in A$

(ii) A is j-separated iff $\forall a,b \in A\ ja=b \rightarrow a=b$

(iii) A is j-sheaf iff $\forall S \subseteq A\ [(j\exists! a \in A\ a \in S) \rightarrow \exists! a \in A\ ja \in S]$

(iv) the j-closure of $B \subseteq A$ is $\{a \in A \mid ja \in B\}$

(v) Ω_j, the equalizer of j and the identity 1_Ω, is $\{\omega \in \Omega \mid \omega = j\omega\}$.

Let $sh_j E$ be the topos of j-sheaves for a given topos **E** and topology $j:\Omega\rightarrow\Omega$ in **E**.
The internal notion of j-sheaf makes use of subsets of objects that are almost a singleton set. Let us give them a name.

2.2 Definition. Let A be an object of **E**.

(i) A subobject $S \subseteq A$ is called a j-singleton of A if $j\exists! a \in A\ a \in S$.

(ii) Two j-singletons S and T are defined to be equal (notation $S \approx T$) if $jS=T$.

In the case of the double negation topology $\neg\neg:\Omega\rightarrow\Omega$ this definition has partial predecessors: independently of each other van Dalen and Lifschitz have used a notion of $\neg\neg$-singleton for respectively Dedekind reals and natural numbers. Note that these two objects are $\neg\neg$-separated!
Now given some object we take the quotient of the set of its j-singletons by $\approx$:

2.3 Construction à la Johnstone. Let A be an object of **E** and $f:A\rightarrow B$ a morphism.

(i) $A^+ = \{S \subseteq A \mid S \text{ is a j-singleton of } A\}/\approx$

(ii) $f^+ = A^+ \rightarrow B^+ : [S] \mapsto [\{fa \mid a \in S\}]$

(iii) $L_j A = A^{++}$

(iv) $\gamma = A \rightarrow A^+ : a \mapsto [\{a\}]$.

2.4 Lemma. Let A be an object of **E**.

(i) γ is mono.

(ii) A^+ is j-separated.

(iii) If A is j-separated, then A^+ is a j-sheaf.

(iv) If A is a j-sheaf, then A^+ is isomorphic to A.

(v) $(\ )^+ : \mathbf{E} \rightarrow \mathbf{E}$ is a left exact functor.

(vi) $L_j : \mathbf{E} \rightarrow sh_j\mathbf{E}$ is an associated sheaf functor corresponding to the topology $j:\Omega\rightarrow\Omega$.

The reader acquainted with [Johnstone (1)] should recognize the above construction as the internal version of the double colimit construction presented there (at p.85). Lawvere has a different method of constructing the associated sheaf (cf. exc. 3.4 in [Johnstone (1)] or [Barr & Wells] for categorical versions of Lawvere's construction).

2.5 Construction à la Lawvere. Let A be an object of $\mathbf{E}$.

(i) $M_A=\{S\subseteq A \mid \exists a{:}A\ S=\{c\in A \mid ja=c\}\}$ is the image of $j^A\circ\{\,\}$ in $(\Omega_j)^A$

(ii) $\mathrm{Law}_jA=\{S\subseteq A \mid j\exists a\in A\ S=\{c\in A \mid ja=c\}\}$ is the j-closure of M_A in $(\Omega_j)^A$.

Note that in this internal definition no references are made to $(\Omega_j)^A$. One verifies directly the following lemma:

2.6 Lemma. Let A be an object of $\mathbf{E}$.

(i) $A \twoheadrightarrow M_A$ is universal with respect to morphisms from A into j-separated objects.

(ii) $\mathrm{Law}_j{:}\,\mathbf{E}\rightarrow \mathrm{sh}_j\mathbf{E}$ is an associated sheaf functor corresponding to the topology $j{:}\,\Omega\rightarrow\Omega$.

A sort of mixed categorical/logical account of this construction occurs in [Veit].

3 Gödel's Negative Translation extended to higher order logics

It is a old result of Gödel that classical arithmetic can be embedded in intuitionistic arithmetic (cf. [Gödel] and [Leivant]) via a so-called negative translation. Two well-known facts from topos theory hint at a generalization of Gödel's negative translation for higher order logics:

(i) toposes are in a natural sense nothing but higher order logics,

(ii) a boolean topos $sh_{\neg\neg}\mathbf{E}$ is included in each topos $\mathbf{E}$ is included via the geometric inclusion $L_{\neg\neg}:sh_{\neg\neg}\mathbf{E}\rightarrow\mathbf{E}$ determined by the double negation topology.

So we would like to define for a intuitionistic higher order logic H and language L_H a translation $(-)^G:L_H\rightarrow L_H$ such that:

$$H+\text{Excluded Middle} \vdash \phi \Leftrightarrow H \vdash \phi^G.$$

And we would like to give a semantical proof as follows.

"$\Rightarrow$" Assume $H \nvdash \phi^G$. Using completeness, there is a topos $\mathbf{E}$ modelling H and invalidating ϕ^G. Construct $sh_{\neg\neg}\mathbf{E}$. If $sh_{\neg\neg}\mathbf{E}$ models H and invalidates ϕ we conclude $H+EM \nvdash \phi$.

"$\Leftarrow$" The proof in the other direction should show that in the presence of EM the G-translation trivializes to the identity on L_H.

There are two gaps in this proof:

$$sh_{\neg\neg}\mathbf{E} \vDash \phi \Rightarrow \mathbf{E} \vDash \phi^G \quad\text{and}\quad \mathbf{E} \vDash H \Rightarrow sh_{\neg\neg}\mathbf{E} \vDash H.$$

The second gap will be resolved by putting a condition on the higher order logic H, namely $H \vdash H^G$, that is, H proves its own Gödel translation H^G. For example, in the case of higher order Heyting arithmetic $\mathbf{HHA} \vdash \mathbf{HHA}^G$ is a logical reformulation of the categorical preservation of a natural number object by a left exact functor like the associated sheaf functor.

The solution of the first gap will be built in the Gödel translation. Because $sh_{\neg\neg}\mathbf{E}$ is a subcategory of $\mathbf{E}$, we can reformulate all constructions of $sh_{\neg\neg}\mathbf{E}$ in terms of $\mathbf{E}$. Objects in $sh_{\neg\neg}\mathbf{E}$ are also objects in $\mathbf{E}$, but power objects differ. Let A be in $sh_{\neg\neg}\mathbf{E}$. For instance, the power $[A]$ of A is $\{B\subseteq A \mid B \text{ is j-sheaf}\}$ in $sh_{\neg\neg}\mathbf{E}$ in contrast to $\{B\subseteq A\mid \top\}$ in $\mathbf{E}$ and the description in E of the object $[\,]$ of truth values in $sh_{\neg\neg}\mathbf{E}$ is $\{\omega\in\Omega \mid j\omega=\omega\}$.

An interpretation of a language L_H of type theory H is specified by its interpretation of sorts and functions between types in L_H. So in order to transform an interpretation $[\![\,]\!]$ of L_H in a topos E into an interpretation $[\![\,]\!]^G$ of L_H in the sheaf topos $sh_{\neg\neg}(E)$, it suffices to define the behavior of the new interpretation on the sorts and functions in terms of the old. Clearly for sorts A it is forced that $[\![A]\!]^G=[\![A]\!]^{++}=[\![A^{++}]\!]$. The definition of types from sorts imposes on us the following construction:

3.1 Definition. By induction to the structure of types we define syntactically a subtype A^G for each type A of H.

(i) $B^G = B^{++}$, when B is a sort of H.

(ii) $(B_1 x B_2)^G = (B_1)^G x (B_2)^G$, where B_1 and B_2 are types of H.

(iii) $[\,]^G = \{\omega \in \Omega \mid \neg\neg\omega = \omega\}$. Recall that $[\,]$ also denotes Ω in H.

(iv) $[B]^G = \{C \subseteq B^G \mid C \text{is} \neg\neg\text{sheaf}\}$, where B is type of H.

One verifies by induction that A^G is a j-sheaf for each type A.
To define in a similar way for each function $f{:}A \to B$ a function $f^G{:}A^G \to B^G$ we need a help-function $\eta{:}B \to B^G$.

3.2 Definition. For each type A of H we define a morphism $\eta{:}A \to A^G$ by induction to the structure of A:

(i) $\eta{:}B \to B^G : b \mapsto [\{[\{b\}]\}]$, when B is a sort of H. (compare $y{:}B \to B^+$)

(ii) $\eta{:}(B_1 x B_2) \to (B_1 x B_2)^G : (b_1, b_2) \mapsto (\eta(b_1), \eta(b_2))$, where B_1 and B_2 are types of H.

(iii) $\eta{:}[\,] \to [\,]^G : \omega \mapsto \neg\neg\omega$, when B is $[\,]$.

(iv) $\eta{:}[B] \to [B]^G : C \mapsto \{\eta(c) \mid c \in C\}^{++}$, where B is type of H.

Note that we are sloppy in (iv), we left out an iso: $\{\eta(c) \mid c \in C\} \subseteq B^G$,
hence $\{\eta(c) \mid c \in C\}^{++} \subseteq (B^G)^{++} \approx (B^G)$.

3.3 Definition. For each $f{:}A \to B$ in H we define a morphism $f^G{:}A^G \to B^G$ by a double induction to the structure of A and B.

(i) (a) $f^G{:}A^G \to B^G{:}S \mapsto [\{[\{fa \mid \eta a \in S\}]\}]$, when A and B are a sorts.

(b) $f{:}A^G \to (CxD)^G{:}S \mapsto ((\pi_1(f(S)))^G, (\pi_2(f(S)))^G)$, when A is a sort and B is of the form CxD.

(c) $f{:}A^G \to [\,]^G{:}S \mapsto \neg\neg\exists a \in A\ \eta a \in S \wedge f(a)$, when A is a sort and B is of the form $[\,]$.

(d) $f{:}A^G \to [C]^G{:}S \mapsto \eta U\{fa \mid \eta a \in S\}$, when A is a sort and B is of the form $[C]$.

(ii) $f^G{:}(CxD)^G \to B^G = f_1{}^G x f_2{}^G$, when A is of the form CxD and B an arbitrary type.

(iii) $f^G{:}[\,]^G \to B^G{:}\omega \mapsto \eta f(j\omega)$, when A is of the form $[\,]$ and B an arbitrary type.

(iv) $f^G{:}[C]^G \to B^G{:}D \mapsto \eta f(D)$, when A is of the form $[C]$ and B an arbitrary type.

3.4 Lemma. For each function $f{:}A \to B$ in H it holds that $\forall a \in A\ f^G(\eta_A(a)) = \eta_B(f(a))$.

□

3.5 Lemma. Let **[ ]** be an interpretation of L_H in a topos **E**. The interpretation $[\![\ ]\!]^G$ of L_H in the topos $sh_{\neg\neg}$**E**. suggested by $[\![A]\!]^G = [\![A^G]\!]$ and $[\![f]\!]^G = [\![f^G]\!]$ is well defined.

□

3.6 Definition. With induction to the complexity of terms and formulas we define a partial translation $(\ )^G{:}L_H \to L_H$:

(i) $(x{:}A)^G = x{:}A^G$

(ii) $(f{:}A \rightarrow B)^G = f^G{:}A^G \rightarrow B^G$

(iii) $(\top)^G = \top$

(iv) $(t{=}s)^G = t^G{=}s^G$

(v) $(t\epsilon s)^G = t^G \epsilon s^G$

(vi) $(\phi \wedge \psi)^G = \phi^G \wedge \psi^G$

(vii) $\{x{:}A \mid \phi\}^G = \{x{:}A^G \mid \phi^G\}$

This translation extends to the full language via the definability of the omitted connectives in the mentioned ones. A calculation shows:

3.7 Lemma.

(viii) $(\phi \rightarrow \psi)^G = \phi^G \rightarrow \psi^G$

(ix) $(\bot)^G = (\neg\neg\bot =)\ \bot$

(x) $(\neg\phi)^G = \neg(\phi^G)$

(xi) $(\forall x{:}A\ \phi)^G = \forall x{:}A^G\ \phi^G$

(xii) $(\phi \vee \psi)^G = \neg\neg(\phi^G \vee \psi^G)$

(xiii) $(\exists x{:}A\ \phi)^G = \neg\neg\exists x{:}A^G\ \phi^G$

Proof. To show where the $\neg\neg$ in front of the disjunction, falsum and the existential quantifier comes from, recall, for instance, that $\phi \vee \psi$ is defined by $\forall \omega \epsilon \Omega\ [\phi \rightarrow \omega \wedge \psi \rightarrow \omega] \rightarrow \omega$. Replace in this definition Ω by $\Omega_{\neg\neg} = \{\omega \epsilon \Omega \mid \omega = \neg\neg\omega\}$ and observe that

$$\forall \omega \epsilon \Omega_j [\phi \rightarrow \omega \wedge \psi \rightarrow \omega] \rightarrow \omega \ \leftrightarrow\ \forall \omega \epsilon \Omega\ [\phi \rightarrow \neg\neg\omega \wedge \psi \rightarrow \neg\neg\omega] \rightarrow \neg\neg\omega \ \leftrightarrow$$
$$\forall \omega \epsilon \Omega\ [\phi \rightarrow \omega \wedge \psi \rightarrow \omega] \rightarrow \neg\neg\omega \ \leftrightarrow^*\ \neg\neg\forall \omega \epsilon \Omega\ [\phi \rightarrow \omega \wedge \psi \rightarrow \omega] \rightarrow \omega \ \leftrightarrow\ \neg\neg(\phi \vee \psi).$$

$\leftrightarrow^*$ is one of the cases where double negation shift is intuitionistically acceptable. We prove $\rightarrow^*$ by substitution of the truth value $\forall \omega \epsilon \Omega\ [\phi \rightarrow \omega \wedge \psi \rightarrow \omega] \rightarrow \omega$! A similar proof works for $\exists$ and $\bot$.

□

This translation is related to the Gödel negative translations.
Note two differences with usual negative translations: the types are replaced by other types, and the atoms are not double negated, as their translations are already stable on the translated types.

3.8 Lemma. Let **[]** and $[]^G$ be as in lemma 3.5. Let ϕ be some formulas of H. Then $[\phi]^G = [\phi^G]$ and hence $sh_{\neg\neg}\mathbf{E} \models \phi \ \Leftrightarrow\ \mathbf{E} \models \phi^G$.

□

One should observe that up until now the whole argument remains valid for an arbitrary topology $j{:}\Omega \rightarrow \Omega$ instead of $\neg\neg{:}\Omega \rightarrow \Omega$. However in contrast to an arbitrary topology the double negation topology $\neg\neg$ has the pleasant property that $\neg\neg(\phi \rightarrow \psi) = \neg\neg\phi \rightarrow \neg\neg\psi$.

3.9 Theorem. Let H be some (intuitionistic) higher order logic such that $H \vdash H^G$. Let ϕ a formula of H. Then H+Excluded Middle $\vdash \phi \Leftrightarrow H \vdash \phi^G$.

□

3.10 Examples

(i) Geometric higher order logics H satisfy $H \vdash H^G$.

(ii) $HAH \vdash HAH^G$.

(iii) $IZF \vdash IZF^G$.

In case the type theory H has axioms assuring that its sorts are separated, one can translate further, replacing $[A]^G$ by the (hereditarily) stable subsets of A. In this way we have a uniform method to produce negative translations for a whole range of known situations. Details of this will be published elsewhere.

4 Topologies, sheaves and localisations for some algebraic theories

Various subsets of {Borceux, Kelly, Veit and Van den Bossche} have studied the relationships between localisations, universal closure operations, Gabriel-Grothendieck topologies and Lawvere-Tierney topologies in all sorts of categories (cf. [Borceux], [Borceux & Veit (1&2)], [Borceux & Kelly] and [Borceux & Van den Bossche]). Roughly summarizing, the typical results obtained are:

(i) universal closure operations are in 1-1 correspondence with both Gabriel-Grothendieck topologies and Lawvere-Tierney topologies

(ii) each localization induces a universal closure operation, and the 1-1 correspondence exists only if the category satisfies some condition

(iii) Lawvere-Tierney topologies, Gabriel-Grothendieck topologies and localisations each form a locale.

The techniques used provide a uniform approach to notions of localization in seemingly unrelated areas as topos theory and ring or module theory.

In the case of $\mathbf{E}_{\mathbf{T}}$, the subcategory of $\mathbf{T}$-algebras in an elementary topos E we can give short elementary proofs for these results, extending known results for $\mathsf{SETS}_{\mathbf{T}}$. In contrast to [Borceux and Veit (1)] we use a simple notion of Lawvere-Tierney topology. We give constructive proofs for $\mathsf{SETS}_{\mathbf{T}}$. Then the results for $\mathbf{E}_{\mathbf{T}}$ follow from the soundness theorem of toposes for constructive logic.

As in the external world, i.e. in ordinary classical mathematics, there are several equivalent ways to define a notion of (finitary, one sorted and purely algebraic) *algebra* in a topos. To do it internally requires either the introduction of some internally categorical notions for the Lawvere-algebras or a internal reconstruction of the necessary model theory for the universal algebras. This is all fairly straightforward, and it requires only some precision to rephrase all definitions in the language of type theory for toposes.

In order to be able to construct free algebras from now on we will assume the presence of a natural number object in toposes, i.e., we assume that the type theory we are working in includes **HHA**.

Let $\mathbf{T}$ be some algebraic theory in a topos $\mathbf{E}$. We will reason internally.

We will need an object of *algebraic* truth values to define the notions of topologies in this algebraic context. Recall that Ω is isomorphic with the object of subobjects of $\mathbf{1}$. We replace $\mathbf{1}$ by the free algebra $\mathsf{F}\mathbf{1}$ generated by $\mathbf{1}$ and consider subalgebras instead of subobjects.

Notation.

(i) Let $\mathsf{F}n$ denote the free algebra generated by n elements, e.g., $\{1,\ldots,n\}$.

(ii) With a word w in the free algebra $\mathsf{F}\mathbf{1}$ corresponds an *unary* algebraic operation $A \to A$: $a \mapsto w(*=a)$ on each algebra A. Similarly with a word w in $\mathsf{F}n$ corresponds an *n-ary* operation $A^n \to A:(a_1,\ldots,a_n)\mapsto w(1=a_1,\ldots,n=a_n)$. We will denote word and corresponding operation by the same letter.

(iii) Consider the canonical injections $\sigma_i: F\mathbf{1}\to Fn: w\mapsto w(*=i)$. Let $S_1, \ldots, S_n$ be subalgebras of $F\mathbf{1}$. Then $S_1\nabla\ldots\nabla S_n$ denotes the join (i.e., smallest subalgebra containing...) in Fn of the subobjects $\sigma_i{\upharpoonright}S_i: S_i \rightarrowtail Fn$ for $1\le i\le n$.

4.1 Definition

(i) $\Omega_{\mathbf{T}}=\{A\subseteq F\mathbf{1} \mid A \text{ subalgebra of } F\mathbf{1}\}$

(ii) A morphism $j:\Omega_{\mathbf{T}}\to\Omega_{\mathbf{T}}$ is called a *Lawvere-Tierney topology* if

(a) $jF\mathbf{1}=F\mathbf{1}$

(b) $\forall R,S\in\Omega_{\mathbf{T}}\ j(R\cap S)=jR\cap jS$

(c) $\forall R\in\Omega_{\mathbf{T}}\ jjR=jR$

(d) $\forall S_1\ldots S_n\in\Omega_{\mathbf{T}}\ \forall f:F\mathbf{1}\to Fn: jf^{-1}(S_1\nabla\ldots\nabla S_n)=f^{-1}(jS_1\nabla\ldots\nabla jS_n)$

(iii) A *Gabriel-Grothendieck topology* is a subset $J\subseteq\Omega_{\mathbf{T}}$ such that $jR=F\mathbf{1}$ for $R\in J$, i.e.,

(a) $F\mathbf{1}\in J$

(b) $\forall R\in J\ \forall f:F\mathbf{1}\to F\mathbf{1}\ f^{-1}(R)\in J$

(c) $\forall S_1\ldots S_n\in J\ \forall w\in Fn\ w^{-1}(S_1\nabla\ldots\nabla S_n)\in J$

(d) $\forall R\in J\ \forall S\in\Omega_{\mathbf{T}}\ [(\forall f:F\mathbf{1}\to R\ f^{-1}(R\cap S)\in J)\to S\in J]$

(e) $\forall R\in J\ \ \forall R\hookrightarrow S\ S\in J$ (these last two useful conditions follow from (a,b,c))

Hence to each LT-topology $j:\Omega_{\mathbf{T}}\to\Omega_{\mathbf{T}}$ corresponds a GG-topology $J=\{R\in\Omega_{\mathbf{T}}\mid jR=F\mathbf{1}\}$. The above definition of GG-topology is taken from [Borceux & Veit (1)]. The definition of LT-topology is inevitable if one wants to establish an 1-1 correspondence with the techniques of 1.3. Note that our definition is more simple to handle than the definition in [Borceux & Veit (1)].

4.2 Theorem.

(i) In **E** there exists a locale isomorphism between the locales $\{J\subseteq\Omega_{\mathbf{T}}\mid J \text{ is a GG-topology}\}$ and $\{j:\Omega_{\mathbf{T}}\to\Omega_{\mathbf{T}}\mid j \text{ is LT-topology}\}$.

(ii) There is a 1-1 correspondence between the family of global elements of $\{j:\Omega_{\mathbf{T}}\to\Omega_{\mathbf{T}}\mid j \text{ is LT-topology}\}$ in **E** and the universal closure operations on **E**.

Proof. (i) Straightforward, using the ideas of 1.3.

(ii) Let $j:\Omega_{\mathbf{T}}\to\Omega_{\mathbf{T}}$ be a LT-topology, and let $m:A\rightarrowtail B$ be a subalgebra of B. One can show directly (without trying to make use of supposed subobject classifying properties of $\Omega_{\mathbf{T}}$ (c.f. (4.4)) that $A=\ \{b\in B\mid \exists R\in J\ \forall w\in R\ \ w(b)\in mA\}$ is a subalgebra of B. This defines the universal closure operation corresponding to j. Note that this definition resembles the usual one in the case of toposes: there the closure of A in B can be rephrased internally as $\{b\in B\mid j\exists a\in A\ \ ma\in B\}$.

□

Of interest are *semi-commutative* algebras. Then the notions of LT-topology and GG-topology simplify, and, although itself not an algebra, $\Omega_{\mathbf{T}}$ becomes a subobject classifier for algebras. Moreover, semi-commutativity is a *sufficient* condition for a 1-1 correspondence between localisations of $\mathbf{E}_{\mathbf{T}}$ and the (global) LT-topologies on $\Omega_{\mathbf{T}}$.

4.3 Definition. Let $\mathbf{T}$ be an algebraic theory.
Then $\mathbf{T}$ is *semi-commutative* if unary algebraic operators commute with n-ary algebraic operators.

Consider rings. A semi-commutative ring satisfies for instance $(x*x)+(y*y)=(x+y)*(x+y)$. The notion of semi-commutativity is independent of the usual notion of commutativity for rings.

4.4 Definition. Let A be an algebra. A function $\phi: A \to \Omega_{\mathbf{T}}$ is called *characteristic* if and only if

(i) for all $n \in \mathbb{N}$, all $w \in Fn$ and all $a_1, ..., a_n \in A$ it holds that $\phi(a_1) \cap ... \cap \phi(a_n) \subseteq \phi(w(a_1, ..., a_n))$

(ii) for all $w \in F\mathbf{1}$ and all $a \in A$ it holds that $* \in \phi(w(a)) \leftrightarrow w \in \phi(a)$.

4.5 Theorem. (extending [Borceux & Van den Bossche]) Let $\mathbf{T}$ be a semi-commutative theory, and A an algebra. Then there is a bijection between subalgebras $m: B \rightarrowtail A$ and characteristic functions $\phi: A \mapsto \Omega_{\mathbf{T}}$ satisfying the usual pullback property.

□

4.6 Theorem. Let $\mathbf{T}$ be a semi-commutative theory.

(i) $j: \Omega_{\mathbf{T}} \to \Omega_{\mathbf{T}}$ is a LT-topology if j satisfies (4.1.ii): a,b,c together with:

(d') $\forall R \in \Omega_{\mathbf{T}}\ \forall f: F\mathbf{1} \to F\mathbf{1}\ jf^{-1}(R) = f^{-1}(jR)$.

(ii) $J \subseteq \Omega_{\mathbf{T}}$ is a GG-topology if J satisfies (4.1.iii): a,b together with:

(c') $\forall R \in J\ \forall S \in \Omega_{\mathbf{T}} [(\forall f: \Omega_{\mathbf{T}} \to \Omega_{\mathbf{T}}\ f^{-1}(R \cap S) \in J) \to S \in J]$

Proof. Similar as in [Borceux & Veit (1)]. □

We generalize the notion of j-singleton (2.2) to the present context. It goes smoother with GG-topologies.

4.7 Definition.

(i) A J-singleton is a morphism on an algebra A is a morphism $S \to A$, where $S \in J$.

(ii) Two singletons $f: R \to A$, $g: S \to A$ are equal (notation $f \approx g$) if $f{\restriction}(R \cap S) = g{\restriction}(R \cap S)$.

(iii) An algebra A is J-separated iff $\forall f,g: F\mathbf{1} \to A\ \forall R \in J\ [f{\restriction}R = g{\restriction}R \to f = g]$.

(iv) An algebra A is a J-sheaf iff $\forall R \in J\ \forall f: R \to A\ \exists! g: F\mathbf{1} \to A\ g{\restriction}R = f$.

Observe that if the algebraic theory $\mathbf{T}$ is empty, i.e, when there are no functions and axioms, then any object is an algebra, and we are again in the situation of the first two paragraphs. The construction of the associated sheaf à la Johnstone is verbatim the same as in (2.3):

4.8 Generalized Johnstone construction. Let A,B be algebras and $f:A\to B$ a morphism.

(i) $A^+=\{S\subseteq A \mid S \text{ is a singleton of } A\}/\approx$

(ii) $f^+=A^+\to B^+:[S]\mapsto[\{fa\mid a\in S\}]$

(iii) $L_J A=A^{++}$

(iv) $\gamma=A\to A^+:a\mapsto[\{a\}]$

(v) $\eta=\gamma^2:A\to L_J A$.

4.9 Lemma. If $\mathbf{T}$ is an semi-commutative algebraic theory and A an algebra then

(i) A^+ is an J-separated algebra,

(ii) $\gamma:A\to A^+$ is a monomorphism,

(iii) if A is J-separated, then A^+ is a sheaf,

(iv) if A is a J-sheaf, then A^+ is isomorphic with A,

(v) $(\ )^+:\mathbf{E_T}\to\mathbf{E_T}$ is a left exact functor.

□

Observe that we need the condition of semi-commutativity already to prove that A^+ is an algebra. It is clear that the simple calculations necessary to prove this fact need the semi-commutativity to go through.

If we restrict the codomain of L_J to $sh_J\mathbf{E_T}$, the category of J-sheaves, we get

4.10 Theorem In the case of a semi-commutative algebra $\mathbf{T}$ it holds that

(i) $L_J:\mathbf{E_T}\to sh_J\mathbf{E_T}$ is the associated sheaf functor corresponding to the topology J.

(ii) the localisations of $\mathbf{E_T}$ form a locale and there exists an (external) isomorphism with the locale of the global elements of $\{j:\Omega_\mathbf{T}\to\Omega_\mathbf{T} \mid j \text{ is a LT-topology}\}$

□

Our way of proving (4.9) and (4.10) seems to depend on the semi-commutativity of the algebraic theory involved and provides no clue for another sufficient condition mentioned in [Borceux & Veit (1)] for the bijection between the localisations and the global Lawvere-Tierney topologies.

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