EVALUATION, PROVABLY DEDUCTIVE EQUIVALENCE IN HEYTING'S ARITHMETIC OF SUBSTITUTION INSTANCES OF PROPOSITIONAL FORMULAS

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Logic Group
Preprint Series
No. 4

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November 1985.

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ABSTRACT

This paper contains the following results:

(i) a theorem of the form: if HA (Heyting's Arithmetic) proves some Σ_1^0 substitution instance of an intuitionistically non valid propositional formula then HA proves a substitution instance of a simpler intuitionistically non-valid formula - unless of course the original formula was - in some appropriate sense - already as simple as possible. The result is shown to be adequate.

(ii) a proof that De Jongh's Completeness Theorem for arithmetical interpretations of Intuitionistic Propositional Logic is verifiable in HA + con(HA).

(iii) a characterization of the closed fragment of the provability logic of HA - this is a solution of Friedman's 35th problem for the case of HA.

These results are instances of or corollaries to answers of a common kind of question, which we call the evaluation problem for a certain set of interpretations. A framework is developped to analyze this kind of question.

0. Introduction

0.0 Introductory Remarks

We start with some examples:

(i) Consider a Σ^0_1-sentence A. As is well known: $HA \vdash \Box(\neg\neg A) \leftrightarrow \Box A$, and $HA \vdash \Box(\neg\neg A \to A) \leftrightarrow \Box(A \vee \neg A)$

Here 'HA' stands for Heyting's Arithmetic and '$\Box$' for provability in HA. The relation $\sim_{HA}$ on sentences of the language of HA defined by: $C \sim_{HA} D :\Leftrightarrow HA \vdash (\Box C \leftrightarrow \Box D)$, is called the relation of provably deductive equivalence in HA. All our examples state provably deductive equivalences.

(ii) Consider an arbitrary sentence B of the language of HA. We have (by [4]): $HA \vdash \Box((\neg\neg B \to B) \to (B \vee \neg B)) \leftrightarrow \Box(\neg\neg B \vee \neg B)$

(iii) Consider a Σ^0_1-sentence Ω that is 'flexible' in the following sense: $HA \vdash \forall A \in \Sigma^0_1$-sentences $(\Box((\Omega \leftrightarrow A) \to \bot) \to \Box\bot)$
(One may show the existence of such an Ω by standard diagonal techniques, see [6] or [11])
We have: $HA \vdash \Box(\Omega \vee \neg\Omega) \leftrightarrow \Box\bot$

(iv) $HA \vdash \Box((\Box\Box\bot \to \Box\bot) \to (\neg\neg\Box\bot \to \Box\bot)) \leftrightarrow \Box\Box\Box\bot$

All these examples are of the form $HA \vdash \Box\varphi(\vec{C}) \leftrightarrow \Box\varphi^*(\vec{C})$. Here $\varphi(\vec{p})$ and $\varphi^*(\vec{p})$ are propositional formulas and $IP \vdash \varphi^* \to \varphi$.[(1)] ('IP' stands for Intuitionistic Propositional Logic). In every example the substitutions are from a prescribed class. In example (i) this is the class of Σ^0_1-sentences, in (iii) flexible Σ^0_1-sentences, in (iv) sentences of the form $\Box\Box\ldots\Box\bot$.

In this paper we will provide generalizations of each of (i)-(iv).[(2)] These generalizations are best viewed as answers to a certain kind of problem.

<u>The problems</u>: Let L be a language of propositional logic with a set of propositional variables P. P may be finite or infinite. Let G be a set of functions from P to the sentences of the language of HA. (E.g. in (i) the elements of G are the functions from P to Σ^0_1-sentences; in (iv) $G=\{g_0\}$ with $g_0(p_i)=\underbrace{\Box\ldots\Box}_{i+1}\bot$.

Define $(\ )^g$ by:

- $(\varphi)^g := g(\varphi)$ if $\varphi \in P$
- $(\bot)^g := \bot$ $\quad (\top)^g := (\underline{0}=\underline{0})$
- $(\ )^g$ commutes with $\wedge, \vee, \to$.

Define: $\varphi \sim_G \psi :\Leftrightarrow$ for all $g \in G$ $HA \vdash \Box(\varphi)^g \leftrightarrow \Box(\psi)^g$

The problem is to characterize $\sim_G$. We call this the *evaluation problem* for G.

In the cases we study in this paper answers to evaluation problems take a definite form.

The answers: Let U be a theory in L given by $U \vdash \varphi \Leftrightarrow$ for all $g \in G$ $HA \vdash (\varphi)^g$, and let $\underline{U}$ be the Lindenbaum algebra of U. A moment's reflection shows that $\sim_G$ can be viewed as an equivalence relation on $\underline{U}$. It turns out that in each case studied the equivalence classes have minimal elements in $\underline{U}$. (In each of the examples (i)-(iv) φ^* is a representative of the minimal element of the $\sim_G$ equivalence class of $[\varphi]_U$. Here $[\varphi]_U$ is the U-equivalence class of φ). To characterize $\sim_G$ it will be shown to be sufficient to specify the set of minimal elements. The set of minimal elements in its turn is given by a suitable set X of representatives in L.

Example: Let $\Omega_0, \Omega_1, \ldots$ be independent flexible Σ^0_1-sentences, i.e. for each n $HA \vdash \forall B_0 \in \Sigma^0_1$-sentences ... $\forall B_n \in \Sigma^0_1$-sentences $(\Box\neg \bigwedge\limits_{i=0}^{n} (\Omega_i \leftrightarrow B_i) \to \Box\bot)$

Our generalization of (iii) looks as follows:
$G: = \{G\}$, where $G(p_i): = \Omega_i$.
$U: = IP$
$X: = \{\top, \bot\}$

This generalization can be viewed as a proof that De Jongh's Completeness Theorem for Arithmetical Interpretations of Propositional Logic can be verified in HA+con(HA). An immediate consequence of the generalization of (iv) is a characterization of the closed fragment of the provability logic of HA. This solves the analogue for the case of HA of Friedman's 35th problem.

0.1 How to read this paper

The paper is divides into a propositional part (part 1) and an arithmetical part (part 2).

The minimal way of reading the paper is just to look at 1.0, 1.1, 2.0, 2.1, 2.2. The reader who is anxious to see arithmetic in action may very well start reading these anyway. The minimal packet can be extended by any of 2.3 or 1.2, 2.4 or 1.3, 2.5.

0.2 Prerequisites

Most of the materials presupposed in the paper are contained in [2] and [5].

0.3 Acknowledgements

First of all this paper is built on the work of Dick de Jongh. The central lemmas 2.1.1.1 and 2.1.3.7 were obtained by analyzing his proof of theorem 4.3 in [4]. I thank Johan van Benthem, Dick de Jongh, Carst Koijmans, Piet Rodenburg and Rick Statman for stimulating conversations. I am grateful to Dirk van Dalen for his continuing stimulation and interest.

1. Evaluation Heyting Algebras

1.0 Definitions and Elementary Facts

1.0.0 Definition

A Heyting Algebra is a structure $\langle H, \leqslant, \wedge, \vee, \rightarrow, \top, \bot \rangle$, where $\langle H, \leqslant, \wedge, \vee, \top, \bot \rangle$ is a lattice and $\rightarrow$ satisfies: $a \wedge b \leqslant c \Leftrightarrow a \leqslant b \rightarrow c$.

1.0.1 Fact

Any Heyting Algebra is distributive and satisfies: for every $a \in H, X \subseteq H$, if $\bigsqcup X$ exists, then $a \wedge \bigsqcup X$ is the supremum of $\{a \wedge x \mid x \in X\}$

For information on Heyting Algebras, see [3], [8].

1.0.2 Definition

One can think of three equally convenient definitions of *Evaluation Heyting Algebra* (EHA). We will use them interchangeably.

A) An EHA is a structure $\langle H, \leqslant, (\ )^*, \wedge, \vee, \rightarrow, \top, \bot \rangle$, where $\langle H, \leqslant, \wedge, \vee, \rightarrow, \top, \bot \rangle$ is a Heyting Algebra and $(\ )^*$ satisfies:

(i) $a \leqslant b \Rightarrow a^* \leqslant b^*$

(ii) $a^* \leqslant a$

(iii) $a^{**} = a^*$

(iv) $\top^* = \top$

B) An EHA is a structure $\langle H, H_0, \leqslant, \wedge, \vee, \rightarrow, \top, \bot \rangle$, where $\langle H, \leqslant, \wedge, \vee, \rightarrow, \top, \bot \rangle$ is a Heyting Algebra and $H_0 \subseteq H$, with: $\top \in H_0$ and for every $a \in H$ $\{b \in H_0 \mid b \leqslant a\}$ has a maximum.

C) An EHA is a structure $<H,\sim,\leq,\wedge,\vee,\rightarrow,T,\perp>$ where $<H,\leq,\wedge,\vee,\rightarrow,T,\perp>$ is a Heyting Algebra and:

(i) $\sim$ is an equivalence relation on H

(ii) each $\sim$ equivalence class has a minimum element

(iii) $a\sim a', b\sim b' \Rightarrow a\wedge b\sim a'\wedge b'$

(iv) $a\sim T\Rightarrow a=T$

We show that A,B,C are 'equivalent' definitions.

a) "A structures are B structures".

Define $H_0 := \{b\in H | b^*=b\}$. Clearly $T\in H_0$. We claim: a^* is the maximum of $\{b\in H_0 | b\leq a\}$. $a^*\leq a$ and $a^{**}=a^*$, hence $a^*\in\{b\in H_0 | b\leq a\}$. Moreover for $b\in H_0$, with $b\leq a$, we have $b=b^*\leq a^*$.

b) "B structures are A structures".

Define $a^* :=$ the maximum of $\{b\in H_0 | b\leq a\}$. It is trivial to check A(i)-(iv).

c) "A structures are C structures".

Define $a\sim b :\Leftrightarrow a^*=b^*$

Ad C (i) : clearly $\sim$ is an equivalence relation.

Ad C (ii) : a^* is the minimum of equivalence class [a] of a: $a^*\sim a$ and if $a\sim b$, then $a^*=b^*$, hence $a^*\leq b$.

Ad C (iii): We first show $(a\wedge b)^*=(a^*\wedge b^*)^*$. $a^*\wedge b^*\leq a\wedge b$, hence $(a^*\wedge b^*)^*\leq(a\wedge b)^*$. Further $a\wedge b\leq a$, hence $(a\wedge b)^*\leq a^*$. Similarly $(a\wedge b)^*\leq b^*$. Hence $(a\wedge b)^*\leq a^*\wedge b^*$. So $(a\wedge b)^*=(a\wedge b)^{**}\leq(a^*\wedge b^*)^*$.

Assume $a\sim a'$, $b\sim b'$, we have:
$(a\wedge b)^*=(a^*\wedge b^*)^*=(a'^*\wedge b'^*)^*=(a'\wedge b')^*$, hence $a\wedge b\sim a'\wedge b'$.

Ad C (iv) : Suppose $a\sim T$, i.e. $a^*=T^*$; then $T=T^*=a^*\leq a$. Hence $T=a$.

d) "C structures are A structures".

Define $a^* :=$ the minimum of [a].

A(ii)-A(iv) are trivial.

Ad A (i): suppose $a \leq b$ then $a \wedge b = a$. $a^* \sim a$, $b^* \sim b$, hence $a^* \wedge b^* \sim a \wedge b = a \sim a^*$. It follows $a^* \leq a^* \wedge b^*$, hence $a^* \leq b^*$.

We leave it to the reader to check that the transitions in a and b are inverses and similarly for c, d.

□

1.0.3 Fact

Let H be an EHA with $(\ )^*$, H_0, $\sim$ as above.

(i) H_0 is closed under $\vee$

(ii) $\perp \in H_0$

(iii) Define $\wedge_0$ on H_0 by $(a \wedge_0 b) := (a \wedge b)^*$ and $\rightarrow_0$ on H_0 by $(a \rightarrow_0 b) := (a \rightarrow b)^*$. Then $\mathcal{H}_0 := \langle H_0, \leq, \wedge_0, \vee, \rightarrow_0, \top, \perp \rangle$ is a Heyting Algebra. (Here $\leq := \leq \restriction H_0$, etc.)

Proof

(i) Suppose $a, b \in H_0$. $a \leq a \vee b$, $b \leq a \vee b$, hence $a = a^* \leq (a \vee b)^*$, $b = b^* \leq (a \vee b)^*$. Conclude $(a \vee b) \leq (a \vee b)^*$. Thus $a \vee b = (a \vee b)^*$.

(ii) Trivial

(iii) Clearly H_0 is closed under $\wedge_0, \vee, \rightarrow_0$. $\top, \perp$ are in H_0.
Suppose $a, b, c \in H_0$, we have:

$$\begin{aligned} c \leq a \text{ and } c \leq b &\Leftrightarrow c \leq a \wedge b \\ &\Leftrightarrow c = c^* \leq (a \wedge b)^* = a \wedge_0 b \end{aligned}$$

$$\begin{aligned} a \wedge_0 b \leq c &\Leftrightarrow a \wedge b \leq c \\ &\Leftrightarrow a \leq b \rightarrow c \\ &\Leftrightarrow a = a^* \leq (b \rightarrow c)^* = (b \rightarrow_0 c) \end{aligned}$$

□

1.0.4 Definition

An EHA H is called *conjunctive* if $(a \wedge b)^* = a^* \wedge b^*$, it is called *disjunctive* if $(a \vee b)^* = a^* \vee b^*$. We call it *full* if it is both conjunctive and disjunctive.

1.0.5 Fact

(i) An EHA H is conjunctive iff H_0 is closed under $\wedge$

(ii) An EHA H is disjunctive iff $a \sim a', b \sim b' \Rightarrow a \vee b \sim a' \vee b'$

Proof

(i) Suppose H is conjunctive. We have for $a, b \in H_0$:

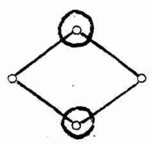
$(a\wedge b)^*=a^*\wedge b^*=a\wedge b$, so $a\wedge b\in H_0$. Conversely suppose H_0 is closed under $\wedge$. Consider c,d in H. We have $c\wedge d\leqslant c, c\wedge d\leqslant d$, hence $(c\wedge d)^*\leqslant c^*\wedge d^*$. But $c^*\wedge d^*\in H_0$ and $c^*\wedge d^*\leqslant c^*\leqslant c, c^*\wedge d^*\leqslant d^*\leqslant d$. So $c^*\wedge d^*\leqslant c\wedge d$. Conclude $c^*\wedge d^*=(c\wedge d)^*$.

(ii) Suppose H is disjunctive. If $a\sim a', b\sim b'$, then $a^*=a'^*, b^*=b'^*$, so $(a\vee b)^*=a^*\vee b^*=a'^*\vee b'^*=(a'\vee b')^*$, or $a\vee b\sim a'\vee b'$.

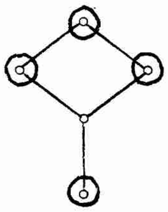
Conversely suppose: $a\sim a', b\sim b'\Rightarrow a\vee b\sim a'\vee b'$. Surely $a\sim a^*, b\sim b^*$ so $a\vee b\sim a^*\vee b^*$. Hence $(a\vee b)^*\leqslant a^*\vee b^*$. On the other hand $a\leqslant a\vee b$, so $a^*\leqslant(a\vee b)^*$. Similarly $b^*\leqslant(a\vee b)^*$, hence $(a^*\vee b^*)\leqslant(a\vee b)^*$

□

1.0.6 Examples



is a non disjunctive EHA
(The elements of H_0 are designated by ⊙)



is a non conjunctive EHA.

1.0.7 Remarks on the definition of EHA

(i) Perhaps the condition $T^*=T$ (or $T\in H_0$ or $a\sim T\Rightarrow a=T$) could be left out of the definitions of EHA; see also 1.0.8. In any case all the structures we will meet satisfy this condition.

(ii) All the important EHA's that are treated in the paper are full. There is one example of an EHA that is not disjunctive in connection with the closed fragment of the provability logic of Peano Arithmetic. Moreover the solution to the evaluation problem for substitutions of Π^0_1 sentences is probably an EHA and is definitely not conjunctive.

1.0.8 Excursion: EHA's viewed categorically

1.0.3 iii suggests that EHA's perhaps could be viewed as *subobjects* in some appropriate category. This is indeed possible, if one drops the condition $(T)^*=T$. We briefly outline the relevant facts. (This section is in no way essential for the rest of the paper.)

The objects of our category are Heyting Algebras. Define a morphism f between two Heyting Algebras H_0, H_1 as a pair $<f_*,f^*>$, which is an adjunction between H_0 and H_1 considered as partial orders. (The suggestion to take adjunctions as morphisms is due to Carst Koijmans.) To spell it out:

- f_* is monotonic from H_0 to H_1
- f^* is monotonic from H_1 to H_0
- $f_*(x)\leq y \Leftrightarrow x\leq f^*(y)$

We have:

(i) This category has products but not in general exponents.

(ii) f_* preserves arbitrary sups (if they exist); f^* preserves arbitrary infs (if they exist).

(iii) f is mono iff f_* is injective.

(iv) Let H be any Heyting Algebra and let α be a *coclosure operation* on H, i.e.

- α is monotonic
- α is idem potent, i.e. $\alpha\circ\alpha=\alpha$
- $\alpha(x)\leq x$.

Define $H_0:=\{x\in H\mid \alpha(x)=x\}$ and $H_0:=<H_0,\leq\restriction H_0,\ldots>$ and $f_0:=<emb_{H_0H},\alpha>$, then: $H_0 \overset{f_0}{\rightarrowtail} H$.

(v) Let $G\xrightarrow{f}H$. Define $\alpha:=f_*\circ f^*$. Then α is a coclosure operation. Define H_0 as in (iv). $f_1:=<f_*,f^*\restriction H_0>$.

We have:

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "f_1"'] & & H \\ & H_0 \arrow[ur, tail, "f_0"'] & \end{tikzcd}$$

Moreover G and H_0 are isomorphic in case f is mono; together with (iv) this tells us that subobjects are given by coclosure operations.

(vi) Let $G\overset{f}{\rightarrowtail}H$. This determines an EHA in the weak sense, which is conjunctive if f_* preserves $\wedge$, disjunctive if f^* preserves $\vee$.

Our main concern in the rest of this section is the problem how to show certain EHA's to be disjunctive. However it is perhaps a good idea first to look at a few more examples.

1.0.9 Conventions

a) Given a Heyting Algebra H, we write $<H,(\ )^*>$, $<H,H_0>$, $<H,\sim>$ for an EHA built on H.

b) Let L be a language for propositional logic, U a theory in L. Suppose $X\subseteq L$.

- $\equiv_U$ will be the relation of provable equivalence in U
- $[\varphi]_U$ will be the $\equiv_U$ equivalence class of φ in L.
- $X/U := \{[\varphi]_U \mid \varphi \in X\}$
- $\underline{U}$ will be the Lindenbaum Algebra of U

One of our main interests will be in EHA's of the form $<\underline{U}, X/U>$.

c) IP is intuitionistic propositional logic. Its language L_{IP} has propositional variables $p_0, p_1, \ldots$ and logical constants $T, \perp, \wedge, \vee, \rightarrow$. $\neg$ and $\leftrightarrow$ are introduced in the usual way.

d) IP^n is like IP, only its language L_{IP^n} has just the propositional variables $p_0, p_1, \ldots, p_{n-1}$.

1.0.10 Examples

I. Let H be a Heyting Algebra. Define:

$a] := \{x \in H \mid x \leqslant a\}$

$[b := \{x \in H \mid x \geqslant b\}$

$a][b := a] \cup [b$

We have:

i) $<H, a][b>$ is a conjunctive EHA.

ii) Suppose $a \lneq b$. Suppose further there is a c incomparable to b, then $<H, a][b>$ is *not* disjunctive.

iii) Suppose $a \neq T$. $<H, a][T>$ is disjunctive iff T has the disjunction property in H, i.e. $(a \vee b = T$ iff $a = T$ or $b = T)$.

Proof

i) $a][b$ is closed under $\wedge$. $T \in a][b$. Take:

$$(x)^* := \begin{cases} x & \text{if } b \leqslant x \\ a \wedge x & \text{otherwise} \end{cases}$$

ii) $(c \vee b)^* = c \vee b$, $c^* \vee b^* = (c \wedge a) \vee b = b$

iii) "$\Rightarrow$" Suppose $<H, a][T>$ is disjunctive and that $(c \vee d) = T$. Hence $c^* \vee d^* = T$. Suppose $c \neq T$, $d \neq T$, we find $(c \vee d) \wedge a = (c \wedge a) \vee (d \wedge a) = T$. So $a = T$. Quod non.

"$\Leftarrow$" Suppose H has the disjunction property. If one of c,d is T clearly $(c \vee d)^* = T = c^* \vee d^*$. If c nor d is T, also $c \vee d \neq T$, hence: $(c \vee d)^* = (c \vee d) \wedge a = (c \wedge a) \vee (d \wedge a) = c^* \vee d^*$

□

II. The following two EHA's play an inportant role in part 2

$\mathcal{D}:=\langle \underline{IP},\bot]\,[T\rangle$, and $\mathcal{J}:=\langle \underline{IP}^1,[\neg\neg p_0 \vee \neg\neg\neg p_0 \to p_0]_{IP^1}]\,[T\rangle$

We will have a closer look at $\mathcal{D}$ in 1.2. Clearly both $\mathcal{D}$ and $\mathcal{J}$ are full EHA's. We turn to the problems connected with disjunctivity.

1.0.11 Definitions

i) Let $\langle H,H_0\rangle$ be an EHA. Let $H_1\subseteq H_0$. H_1 is a *basis* for $\langle H,H_0\rangle$ if for every $h_0\in H_0$ $h_0=\sqcup\{h_1\in H_1 \mid h_1\leqslant h_0\}$.

ii) Let H be a Heyting Algebra. An element h of H *has the disjunction property* if for every x,y in H: $h\leqslant x\vee y \Rightarrow h\leqslant x$ or $h\leqslant y$.

1.0.12 Theorem

Let $\langle H,H_0\rangle$ be an EHA. Suppose H_1 is a basis for $\langle H,H_0\rangle$ such that every h_1 in H_1 has the disjunction property. Then $\langle H,H_0\rangle$ is disjunctive.

Proof

Suppose $h_0\in H_0$ and $h_0\leqslant x\vee y$. For any $h_1\in H_1$ with $h_1\leqslant h_0$ we have $h_1\leqslant x\vee y$ and hence $h_1\leqslant x$ or $h_1\leqslant y$. Thus $h_1\leqslant x^*$ or $h_1\leqslant y^*$. Conclude $h_0=\sqcup\{h_1\in H_1 \mid h_1\leqslant h_0\}\leqslant x^*\vee y^*$. It follows that $(x\vee y)^*\leqslant x^*\vee y^*$.

□

EHA's are often given as $\langle \underline{U},X/U\rangle$. The rest of this section is devoted to providing lemmas to apply 1.0.12 to such EHA's (or in some cases: to apply the reasoning of the proof of 1.0.12).
As a start we need a few facts about Kripke models.

1.0.13 Kripke models

We present a Kripke model $\underline{K}$ as a triple $\langle K,\leqslant,\Vdash\rangle$. Such a model need not have a bottom. If there is a bottom k_0 and this fact is relevant, we write: $\underline{K}=\langle K,k_0,\leqslant,\Vdash\rangle$.

Let $K'\subseteq K$. K' non empty. Define $\underline{K}[K']:=\langle K',\leqslant',\Vdash\rangle$. Where: $\leqslant':=\langle\restriction K'$ and $\Vdash'$ is given by: for $k'\in K'$ $k'\Vdash' p_i :\Leftrightarrow k'\Vdash p_i$.

For $k\in K$: $\check{k}:=\{k'\in K \mid k\leqslant k'\}$ and $\underline{K}(k):=\underline{K}[\check{k}]$. Because $\Vdash$ and $\Vdash'$ will in this case coincide on $\check{k}$ we write: $\underline{K}(k)=\langle\check{k},\leqslant,\Vdash\rangle$.

1.0.14 The relativized Henkin construction

We briefly review the relevant facts about the relativized Henkin construction. Let $X \subseteq \mathcal{L}_{IP}$ and let X be closed under subformulas. We say that a set Δ is X-*saturated* if:

a) $\Delta \subseteq X$

b) Δ is consistent in IP

c) $\varphi \in X$ and $\Delta \vdash_{IP} \varphi \Rightarrow \varphi \in \Delta$

d) $(\varphi \vee \psi) \in X$ and $\Delta \vdash_{IP} (\varphi \vee \psi) \Rightarrow \Delta \vdash_{IP} \varphi$ or $\Delta \vdash_{IP} \psi$

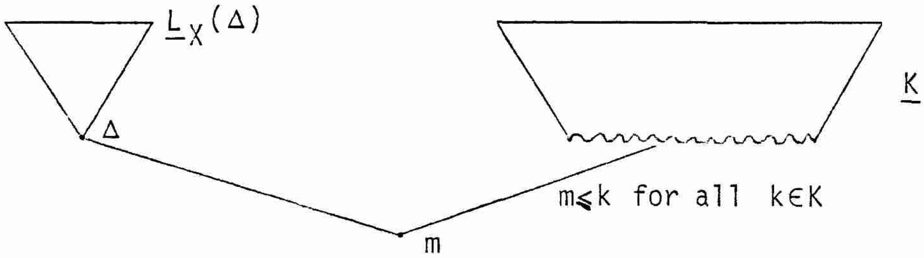
Define the Kripke model $\underline{L}_X = \langle L, \leqslant, \Vdash \rangle$ as follows:

- $L := \{\Delta \subseteq X \mid \Delta$ in X-saturated$\}$
- $\Delta \leqslant \Delta' :\Leftrightarrow \Delta \subseteq \Delta'$
- $\Delta \Vdash p_i :\Leftrightarrow p_i \in \Delta$

We have by a standard argument: for all $\varphi \in X$: $\Delta \Vdash \varphi \Leftrightarrow \varphi \in \Delta$.

1.0.15 Push Down Lemma

Let $X \subseteq \mathcal{L}_{IP}$ and let X be closed under subformulas. Suppose Δ is X-saturated. Let $\underline{K}$ be a Kripke model such that for every $k \in K$ $k \Vdash \Delta$. Construct a new model $\underline{M}$ as follows:



Where $m \Vdash p_i :\Leftrightarrow p_i \in \Delta$.

Then: $m \Vdash \Delta$.

<u>Proof</u>: the proof is a simple induction on the elements of Δ. E.g. suppose $(\psi \to \chi) \in \Delta$. We have: $\Delta \Vdash (\psi \to \chi)$ and: for all $k \in K$ $k \Vdash \psi \to \chi$. So in case $m \nVdash \psi$ we are done. Suppose $m \Vdash \psi$. It follows that $\Delta \Vdash \psi$. But $\psi \in X$, so $\psi \in \Delta$. Hence $\chi \in \Delta$ and by IH: $m \Vdash \chi$.

□

1.0.16 Definition

i) Let $\Gamma \subseteq L$. We say that Γ has the *disjunction property* if:
$\Gamma \vdash_{IP} \varphi \vee \chi \Rightarrow \Gamma \vdash_{IP} \varphi$ or $\Gamma \vdash_{IP} \chi$.

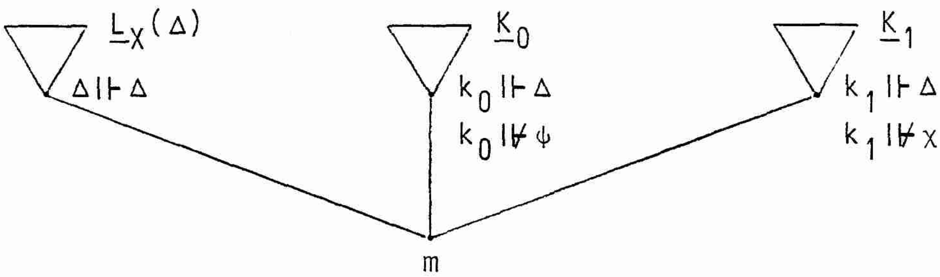
ii) Let $\varphi \in L$. We say that φ has the disjunction property if: $\{\varphi\}$ has the disjunction property.

1.0.17 Lemma

Let X be closed under subformulas. Suppose Δ is X-saturated. Then Δ has the disjunction property.

Proof:

Suppose $\Delta \nvdash_{IP} \varphi, \Delta \nvdash_{IP} \chi$. Then there are Kripke models $\underline{K}_0 = \langle K_0, k_0, \leq, \Vdash \rangle$, $\underline{K}_1 = \langle K_1, k_1, \leq, \Vdash \rangle$ such that $k_0 \Vdash \Delta, k_1 \Vdash \Delta, k_0 \nVdash \varphi, k_1 \nVdash \chi$. Construct a Kripke model $\underline{M}$ as follows:



Where $m \Vdash p_i :\Leftrightarrow p_i \in \Delta$.

By the push down lemma: $m \Vdash \Delta$, but $m \nVdash \varphi, m \nVdash \chi$. Hence $\Delta \nvdash_{IP} \varphi \vee \chi$.

□

1.0.18 Applications

i) Suppose φ does not contain disjunctions, then φ has the disjunction property.

ii) $\neg\varphi$ has the disjunction property.

Proof:

i) Let Sub(φ) be the set of subformulas of φ. Trivially φ is Sub(φ)-saturated.

ii) Under the negation sign one may substitute classical equivalents for each other preserving IP-equivalence. So $IP \vdash \neg\varphi \leftrightarrow \chi$. Where χ contains no disjunctions. Apply (i).

□

1.0.19 Disjunctive Normal forms for IP

Consider any formula φ. Let $\mathrm{Sub}(\varphi)$ be the set of subformulas of φ. Let $\underline{L}:=\underline{L}_{\mathrm{Sub}(\varphi)}$. Clearly $\underline{L}$ is finite. Let $\Delta_0,\ldots,\Delta_{n-1}$ be the minimal nodes of $\underline{L}$ such that $\Delta_i \Vdash \varphi$ (or: $\varphi \in \Delta_i$). Define: $\varphi_i := \bigwedge\{\psi \in \Delta_i \mid \psi$ is atom or implication$\}$. First we note that $\varphi_i \vdash_{IP} \Delta_i$. This is shown by a simple induction on the elements of Δ_i. It follows that $\varphi_i \vdash_{IP} \nu \Leftrightarrow \Delta_i \vdash_{IP} \nu$ for all $\nu \in L_{IP}$. Hence φ_i has the disjunction property. We have: $IP \vdash \varphi \leftrightarrow \bigvee_{i=0}^{n-1} \varphi_i$. For the "$\leftarrow$" direction note that $\varphi \in \Delta_i$ and hence $IP \vdash \varphi_i \rightarrow \varphi$. For "$\rightarrow$": consider any Kripke model $\underline{K}$ and any node $k \in K$ with $k \Vdash \varphi$. Let $\Delta := \{\psi \in \mathrm{Sub}(\varphi) \mid k \Vdash \psi\}$. Clearly Δ is $\mathrm{Sub}(\varphi)$-saturated, hence for some i $\Delta_i \subseteq \Delta$ and hence $k \Vdash \varphi_i$.

We call $\bigvee_{i=0}^{n-1} \varphi_i$ the disjunctive normal form for φ. There is a weak connection between this disjunctive normal form and the classical one: if φ is built up from atoms and negations of atoms using $\wedge$ and $\vee$ only one gets a - not fully efficient - version of the classical disjunctive normal form.

1.0.20 Theorem

Suppose $X \subseteq L_{IP}$, $\top \in X$, X is closed under taking subformulas, conjunction and disjunction. Suppose further that $X/_{IP}$ is finite. Then $\langle \underline{IP}, X/_{IP}\rangle$ is a full EHA.

Proof:

Take $a^* := \bigvee\{b \in X/_{IP} \mid b \leq a\}$. Clearly $a^* \in X/_{IP}$. Note $\top^* = \top$. Hence $\langle \underline{IP}, X/_{IP}\rangle$ is an EHA. $X/_{IP}$ is closed under $\wedge$, hence $\langle \underline{IP}, X/_{IP}\rangle$ is conjunctive.

Consider $\varphi \in X$. Let $\bigvee_{i=0}^{n-1} \varphi_i$ be the disjunctive normal form of φ. Each of the φ_i is a conjunction of subformulas of φ. Hence $\varphi_i \in X$. Let $X_1 := \{\psi \in X \mid \psi$ has the disjunction property$\}$. Clearly $X_1/_{IP}$ is a basis for $\langle \underline{IP}, X/_{IP}\rangle$ and every element of $X_1/_{IP}$ has the disjunction property.

□

1.0.21 Application

Let X_0 be a finite set of formulas. Let X be the closure of $X_0 \cup \{\top\}$ under taking subformulas, conjunction and disjunction. Then $\langle \underline{IP}, X/_{IP}\rangle$ is a full EHA.

Proof:

By the normal form theorem $X/_{IP}$ is finite.

□

1.0.22 Remark

Clearly 1.0.20 and 1.0.21 go through when we restrict ourselves to IP^n.

1.0.23 Open Problem

For $n>0$: is $<\underline{IP}, L_{IP^n/IP}>$ an EHA? (In case n=1 this reduces to the question, whether there is an element of $\underline{IP}$ between the top and the rest of the Rieger Nishimura Lattice.)

In sections 1.1, 1.2, 1.3 we introduce the main characters of this paper: the EHA's $\mathcal{NN}$, $\mathcal{D}$ and $\mathcal{UP}$.

1.1 $\mathcal{NN}$ and $\mathcal{NN}^n$

1.1.0 Definitions

a) NNIL, i.e. No Nestings of Implications to the Left, is the smallest set such that:
   - $T,\bot,p_0,p_1,\ldots$ are in NNIL
   - NNIL is closed under $\wedge,\vee$
   - if $\varphi\in$ NNIL then $(p_i\rightarrow\varphi)\in$ NNIL

b) $NNIL^n := NNIL \cap L_{IP^n}$

c) $\mathcal{NN} := <\underline{IP}, NNIL/_{IP}>$

d) $\mathcal{NN}^n := <\underline{IP}^n, NNIL^n/_{IP^n}>$

1.1.1 Theorem

$\mathcal{NN}^n$ is a full EHA.

Proof:

Let $(p_i\rightarrow\chi)\in NNIL^{m+1}$ and let ν be the result of substituting T for p_i in χ. Clearly $IP\vdash(p_i\rightarrow\chi)\leftrightarrow(p_i\rightarrow\nu)$. Moreover modulo a renaming of propositional variables ν is in $NNIL^m$. Using this observation plus the normal form theorem one sees by an easy induction on n that $NNIL^n/_{IP^n}$ is finite. Apply 1.0.20. □

Before turning to the problem of showing that $\mathcal{NN}$ is an EHA, we give two small facts about $\mathcal{NN}^n$ and $\mathcal{NN}$.

1.1.2 Fact

For each n $NNIL^n/_{IP^n}$ has a maximum element S_n *below* T.

Proof:

We present $\sigma_n \in S_n$:

$\sigma_0 := \perp$

$\sigma_1(p_0) := p_0 \vee \neg p_0$

$\sigma_{n+2}(p_0,\ldots,p_{n+1}) := \bigvee_{i=0}^{n+1} (p_i \to \sigma_{n+1}(p_0,\ldots,p_{i-1},p_{i+1},\ldots,p_{n+1}))$

We leave the proof that the σ_n are as desired as an excercise.

□

1.1.3 Fact

In NN/NN^n: $a \sim \perp \Rightarrow a = \perp$

Proof:

We leave the proof as an excercise to the reader.

□

1.1.4 Discussion

We now turn to the problem of showing that *NN* is an EHA.

Consider $\varphi \in L_{IP}$. We have to produce a $\varphi^* \in$ NNIL such that $IP \vdash \varphi^* \to \varphi$ and for all $\chi \in$ NNIL $IP \vdash \chi \to \varphi \Rightarrow IP \vdash \chi \to \varphi^*$. Clearly for some n: $\varphi \in L_{IP^n}$. An obvious conjecture is that we can find φ^* in L_{IP^n} and that we can take as φ^* in fact a $NNIL^n$ representative of $[\varphi]^*$ in NN^n. A moment's reflection shows that what we need to realize this idea is the following interpolation theorem:

1.1.4.0 NNIL Interpolation Theorem

Suppose $\varphi \in L, \chi \in$ NNIL and $IP \vdash \chi \to \varphi$. Then there is a χ' in NNIL containing only atoms occurring both in χ and φ such that $IP \vdash \chi \to \chi'$ and $IP \vdash \chi' \to \varphi$.

This theorem is true, but we were only able to prove it *from* the fact that there indeed is a φ^* having the desired properties and containing only atoms from φ. (The argument is as follows: suppose $IP \vdash \chi \to \varphi$. Let ν be an interpolant provided by the ordinary interpolation theorem. Take $\chi' := \nu^*$.)

Thus we state:

1.1.4.1 Open Problem

Give a direct proof of the NNIL Interpolation Theorem.(3)

We will prove the fact that *NN* is an EHA in a different way. We specify an algorithm *N*, that produces from a given φ a sequence of formulas: $\varphi =: \varphi^0, \varphi^1, \varphi^2, \ldots$. We show:

(i) This sequence terminates in an element of NNIL

(ii) $IP \vdash \varphi^{i+1} \to \varphi^i$

(iii) For all $\chi \in NNIL$ $\; IP \vdash \chi \to \varphi^i \Rightarrow IP \vdash \chi \to \varphi^{i+1}$

Clearly the element in which the sequence terminates is a φ^* as desired. The algorithm will be such that the atoms of φ^{i+1} are among those of φ^i. Thus the atoms of φ^* will be among those of φ.

The algorithm *N* will play an essential role in part 2 of the paper.

Before we can even describe the algorithm we have to take a closer look at the syntax and 'proof theory' of IP.

1.1.5 Definition

We define $(\cdot)\langle\cdot\rangle$ from $L_{IP} \times L_{IP}$ to L_{IP} by:

For φ atomic $\varphi\langle\chi\rangle := \begin{cases} T \text{ if } \varphi = \chi \\ \varphi \text{ otherwise} \end{cases}$

$$(\varphi\wedge\psi)\langle\chi\rangle := \begin{cases} T \text{ if } (\varphi\wedge\psi) = \chi \\ \varphi\langle\chi\rangle \text{ if } (\varphi\wedge\psi) \neq \chi \text{ and } \psi\langle\chi\rangle = T \\ \psi\langle\chi\rangle \text{ if } (\varphi\wedge\psi) \neq \chi \text{ and } \varphi\langle\chi\rangle = T \\ \varphi\langle\chi\rangle \wedge \psi\langle\chi\rangle \text{ otherwise} \end{cases}$$

$$(\varphi\vee\psi)\langle\chi\rangle := \begin{cases} T \text{ if } (\varphi\vee\psi) = \chi \text{ or } \varphi\langle\chi\rangle = T \text{ or } \psi\langle\chi\rangle = T \\ (\varphi\langle\chi\rangle \vee \psi\langle\chi\rangle) \text{ otherwise} \end{cases}$$

$$(\varphi\to\psi)\langle\chi\rangle := \begin{cases} T \text{ if } (\varphi\to\psi) = \chi \text{ or } \psi\langle\chi\rangle = T \\ \psi\langle\chi\rangle \text{ if } (\varphi\to\psi) \neq \chi \text{ and } \varphi\langle\chi\rangle = T \\ (\varphi\langle\chi\rangle \to \psi\langle\chi\rangle) \text{ otherwise} \end{cases}$$

1.1.6 Fact

a) $IP \vdash \chi \to (\varphi \leftrightarrow \varphi\langle\chi\rangle)$

b) $IP \vdash (\chi\wedge\varphi)\leftrightarrow(\chi\wedge\varphi<\chi>)$

c) $IP \vdash (\chi\rightarrow\varphi)\leftrightarrow(\chi\rightarrow\varphi<\chi>)$

Proof: trivial

□

1.1.7 Definition

A formula φ is *stable* if for all Kripke models $\underline{K}=<K,\leqslant,\Vdash>$ and for all $k'\in K$, for all $K'\subseteq K$ such that $k'\in K'$ we have: $k'\Vdash\varphi \Rightarrow k'\Vdash'\varphi$, where $\underline{K}[K']=<K',\leqslant',\Vdash'>$.

$STAB:=\{\varphi\in L_{IP}|\varphi \text{ is stable}\}$

1.1.8 Fact

$NNIL\subseteq STAB$

Proof: induction on φ in NNIL. E.g. if $\varphi=(p_m\rightarrow\psi)$: suppose $k'\Vdash p_m\rightarrow\psi$. Consider $k''\geqslant'k'$ with $k''\Vdash' p_m$, then $k''\geqslant k'$ and $k''\Vdash p_m$. Hence $k''\Vdash\psi$. By IH $k''\Vdash'\psi$.

□

Later we will see that in fact $NNIL/_{IP}=STAB/_{IP}$; in other words that every stable formula is provably equivalent to a NNIL formula.

1.1.9 Definition

Define $[\cdot](\cdot)$ from $L_{IP}xL_{IP}$ to L_{IP} by:

- $[\psi]\varphi:=\varphi$ if φ is an atom
- $[\psi](\cdot)$ commutes with $\wedge,\vee$
- $[\psi](\chi\rightarrow\rho):=((\chi\wedge\psi<\chi>)\rightarrow\rho<\chi>)$

1.1.10 Comments on 1.1.9

i) Note that by 1.1.6: $IP\vdash[\psi](\chi\rightarrow\rho)\leftrightarrow(\psi\rightarrow(\chi\rightarrow\rho))$. The reason that we chose the more complex form in our definition is that we want to simplify the formula we started with.

ii) Note that $IP\vdash[\psi]\varphi\rightarrow(\psi\rightarrow\varphi)$. Thus $[\cdot](\cdot)$ might be regarded as a kind of strengthening of implication.

iii) From the 'proof-theoretical' point of view the raison d'être of $[\cdot](\cdot)$ lies in theorem 1.1.11.

1.1.11 Theorem

Let $\psi:=\bigwedge_{i=0}^{n}(\nu_i\rightarrow\rho_i)$, $\varphi:=(\psi\rightarrow\chi)$, $\sigma:\in STAB$

Then: $IP \vdash \sigma \to \varphi \Rightarrow IP \vdash \sigma \to (\bigvee_{i=0}^{n} [\psi]\nu_i \vee [\psi]\chi)$

Proof:

Suppose $IP \nvdash \sigma \to (\bigvee_{i=0}^{n} [\psi]\nu_i \vee [\psi]\chi)$.

There is a Kripke model $\underline{K}=<K,k,\leqslant,\Vdash>$ such that $k\Vdash\sigma$ and $k\nVdash[\psi]\nu_i$ for $i=0,\ldots,n$ and $k\nVdash[\psi]\chi$. Let $K':=\{k\}\cup\{k'\in K \mid k'\Vdash\psi\}$ and $\underline{K}':=\underline{K}[K']=<K',\leqslant',\Vdash'>$. Clearly $k\Vdash'\sigma$. We claim: for any ε $k\nVdash[\psi]\varepsilon \Rightarrow k\nVdash'\varepsilon$.

Proof of the claim: by a simple induction on ε. We treat the case of implication. Suppose ε is an implication and $k\nVdash[\psi]\varepsilon$. It follows that $k\nVdash\psi\to\varepsilon$. There is a $k'\geqslant k$ such that $k'\Vdash\psi$ and $k'\nVdash\varepsilon$. From $k'\Vdash\psi$ it is seen that every $k''\geqslant k'$ is in K'. Hence on k' $\Vdash$ and $\Vdash'$ coincide. Conclude $k'\nVdash'\varepsilon$ and thus $k\nVdash'\varepsilon$.

□ Claim

From the claim we have $k\nVdash'\nu_i$ $(i=0,\ldots,n)$ and $k\nVdash'\chi$. Consider $k'\geqslant'k$ and suppose $k'\Vdash'\nu_i$, then $k'\neq k$ and thus $k'\Vdash'\psi$. Hence $k'\Vdash'(\nu_i\to\rho_i)$ and so $k'\Vdash'\rho_i$. Conclude $k\Vdash'(\nu_i\to\rho_i)$. Ergo $k\Vdash'\psi$. We have $k\Vdash'\psi$, $k\Vdash'\sigma, k\nVdash'\chi$, so $k\nVdash'\sigma\to(\psi\to\chi)$. So finally we see: $IP\nvdash\sigma\to(\psi\to\chi)$.

□

1.1.12 Example

Suppose σ is stable and $IP\vdash\sigma\to(\neg\neg p\to p)$. Then $IP\vdash\sigma\to([\neg\neg p]\neg p\vee[\neg\neg p]p)$.

$$[\neg\neg p]\neg p=(p\wedge(\neg\neg p)<p>)\to\bot$$
$$=\neg(p\wedge\neg\bot)$$

$$[\neg\neg p]p = p$$

Ergo $IP\vdash\sigma\to(p\vee\neg p)$. Note that $(p\vee\neg p)\,\varepsilon$ NNIL.

We need one more definition.

1.1.13 Definition

A formula φ has an *outer disjunction* if φ is of the form $(\psi\vee\chi)$ or if φ is of the form $(\psi\wedge\chi)$ and ψ has an outer disjunction or χ has an outer disjunction. A dual definition can be given for 'outer conjunction'.

1.1.14 Fact

If φ has an outer disjunction then there are φ_0, φ_1 such that $IP \vdash \varphi \leftrightarrow (\varphi_0 \vee \varphi_1)$ and both φ_0 and φ_1 are shorter than φ and $(\varphi_0 \vee \varphi_1)$ contains the same atoms and implications as φ. A dual fact holds when φ has an outer conjunction.

Proof: by a simple induction following the definition of outer disjunction/ conjunction.

$\square$

1.1.15 The algorithm N

Consider NN. We construct an algorithm to compute a φ^* in $a^* \cap NNIL$ from φ in a. This algorithm and the related algorithms T and Ω are the key to the arithmetical part of the paper: the arithmetical constructions follow the algorithms step by step.

Our specification of the algorithm is non deterministic; one can view it as producing a finitely branching tree. We assign to each formula ψ a certain ordinal $o(\psi)$. $o(\psi)$ will strictly decrease when going down in the tree. It follows that the tree is finite. The formulas in which the branches terminate are suitable elements of $a^* \cap NNIL$.

Our presentation plus termination proof of the algorithm can as well be seen as a proof by induction on $o(\varphi)$ of the existence of suitable φ^*.

1.1.15.0 Definition

a) $I(\varphi) := \{(\psi \to \chi) \mid (\psi \to \chi) \text{ is an subformula of } \varphi\}$

b) $i(\varphi) := \max\{|I(\psi \to \chi)| \mid (\psi \to \chi) \in I(\varphi)\}$

(Note that we count types not tokens. Also: $i(\psi \to \chi) = |I(\psi \to \chi)|$.)

c) $o(\varphi) := \omega . i(\varphi) + \ell(\varphi)$

(Here $\ell(\varphi)$ is the length of φ.)

We present the algorithm by cases. At each step we check:

(i) $\varphi^* \in NNIL$

(ii) $IP \vdash \varphi^* \to \varphi$

(iii) for all $\sigma \in NNIL$ if $IP \vdash \sigma \to \varphi$ then $IP \vdash \sigma \to \varphi^*$

case A φ is atomic

Set $\varphi^* := \varphi$

case B $\varphi=(\psi\wedge\chi)$

Set $\varphi^*:=(\psi^*\wedge\chi^*)$. Clearly $o(\psi)<o(\varphi)$, $o(\chi)<o(\varphi)$. (i), (ii), (iii) for φ^* are easily seen to follow from (i), (ii), (iii) for ψ^*,χ^*.

case C $\varphi=(\psi\vee\chi)$

Set $\varphi^*:=(\psi^*\vee\chi^*)$. Clearly $o(\psi)<o(\varphi)$, $o(\chi)<o(\varphi)$. (i) and (ii) are evident. For (iii) we reason as in 1.0.12, 1.0.20: Suppose $\sigma\in$NNIL and IP$\vdash\sigma\to(\psi\vee\chi)$.

Let $\bigvee_{i=0}^{p-1}\sigma_i$ be the disjunctive normal form of σ. Clearly the σ_i are in NNIL.

IP$\vdash\sigma_i\to(\psi\vee\chi)$, hence IP$\vdash\sigma_i\to\psi$ or IP$\vdash\sigma_i\to\chi$. Ergo by IH: IP$\vdash\sigma_i\to\psi^*$ or IP$\vdash\sigma_i\to\chi^*$. Conclude IP$\vdash\sigma_i\to(\psi^*\vee\chi^*)$. Thus: IP$\vdash\sigma\to(\psi^*\vee\chi^*)$.

case D $\varphi=(\psi\to\chi)$

case Do χ has an outer conjunction

By 1.1.14 there are χ_0,χ_1 such that IP$\vdash\chi\leftrightarrow(\chi_0\wedge\chi_1)$ and for i=0,1: $\ell(\chi_i)<\ell(\chi)$ and $I(\chi_i)\subseteq I(\chi)$. Clearly IP$\vdash(\psi\to\chi)\leftrightarrow((\psi\to\chi_0)\wedge(\psi\to\chi_1))$ and $o(\psi\to\chi_i)<o(\psi\to\chi)$ for i=0,1. Set $(\psi\to\chi)^*:=((\psi\to\chi_0)^*\wedge(\psi\to\chi_1)^*)$.

case D1 ψ has an outer disjunction

By 1.1.14 there are ψ_0,ψ_1 such that IP$\vdash\psi\leftrightarrow(\psi_0\vee\psi_1)$ and for i=0,1: $\ell(\psi_i)<\ell(\psi)$, $I(\psi_i)\subseteq I(\psi)$. Clearly IP$\vdash(\psi\to\chi)\leftrightarrow((\psi_0\to\chi)\wedge(\psi_1\to\chi))$ and $o(\psi_i\to\chi)<o(\psi\to\chi)$ for i=0,1. Set $(\psi\to\chi)^*:=((\psi_0\to\chi)^*\wedge(\psi_1\to\chi)^*)$

case D2 $\chi=(\nu\to\rho)$

We need a lemma about $(\cdot)\langle\cdot\rangle$ of 1.1.5.

1.1.15.1 Lemma

a) For every subformula τ of $\sigma\langle\eta\rangle$ there is a subformula τ' of σ such that $\tau=\tau'\langle\eta\rangle$

b) Consider a subformula τ of $\sigma\langle\eta\rangle$. Let τ' be *minimal* in the subformula ordering of σ such that $\tau=\tau'\langle\eta\rangle$. If $\tau\neq T$ we have that τ' has the same form as τ, i.e. if $\tau=\mu\wedge\lambda$, $\tau'=\mu'\wedge\lambda'$ etc.

c) $i(\sigma<\eta>)\leqslant i(\sigma)$ and if a subformula of the form $(\eta\rightarrow\lambda)$ occurs in σ, the inequality is strict.

d) $i((\eta\wedge\sigma<\eta>)\rightarrow\tau<\eta>)\leqslant i((\eta\wedge\sigma)\rightarrow\tau)$ and if a subformula of the form $(\eta\rightarrow\lambda)$, occurs in σ or τ, the inequality is strict.

e) $i((\eta\wedge\sigma<\eta>)\rightarrow\tau<\eta>)<i(\sigma\rightarrow(\eta\rightarrow\tau))$

Proof: a,b are left to the reader, c is like d.

d) Let $\vartheta:=((\eta\wedge\sigma<\eta>)\rightarrow\tau<\eta>)$, $\vartheta':=((\eta\wedge\sigma)\rightarrow\tau)$. Define $f:I(\vartheta)\rightarrow I(\vartheta')$ by:

$$f(\varepsilon):=\begin{cases}\vartheta' \text{ if } \varepsilon=\vartheta\\ \eta \text{ if } \varepsilon=\eta \quad \text{(and thus } \eta \text{ is an implication)}\\ \text{an } \varepsilon' \text{ minimal in the subformula ordering of } \vartheta' \text{ such that } \varepsilon=\varepsilon'<\eta>,\\ \text{otherwise}\end{cases}$$

One easily shows that f is a function from $I(\vartheta)$ to $I(\vartheta')$ and that f is injective. Finally no formula of the form $(\eta\rightarrow\lambda)$ is in the range of f.

e)
$$\begin{aligned} i((\eta\wedge\sigma<\eta>)\rightarrow\tau<\eta>) &= i((\eta\wedge\sigma<\eta>)\rightarrow(\eta\rightarrow\tau)<\eta>)\\ &< i((\eta\wedge\sigma)\rightarrow(\eta\rightarrow\tau))\\ &= i(\sigma\rightarrow(\eta\rightarrow\tau))\end{aligned}$$

□

Returning to D2 we have: $IP\vdash(\psi\rightarrow(\nu\rightarrow\rho))\leftrightarrow((\nu\wedge\psi<\nu>)\rightarrow\rho<\nu>)$ and $o((\nu\wedge\psi<\nu>)\rightarrow\rho<\nu>)<o(\psi\rightarrow(\nu\rightarrow\rho))$. Set $\varphi^*:=((\nu\wedge\psi<\nu>)\rightarrow\rho<\nu>)^*$.

case D3 ψ has no outer disjunction, so ψ is of the form $\bigwedge_{i=0}^{n}\psi_i$, where the ψ_i are implications or atoms.

case D3.0 Suppose one of the ψ_i is an implication and occurs more than once in φ. Say this is ψ_{i_0}.

We have
$$\begin{aligned} IP\vdash(\psi\rightarrow\chi) &\leftrightarrow((\psi_{i_0}\wedge\psi)\rightarrow\chi)\\ &\leftrightarrow((\psi_{i_0}\wedge\psi<\psi_{i_0}>)\rightarrow\chi<\psi_{i_0}>)\end{aligned}$$

$$\begin{aligned} i(\psi\rightarrow\chi) &= i((\psi_{i_0}\wedge\psi)\rightarrow\chi)\\ &\geqslant i((\psi_{i_0}\wedge\psi<\psi_{i_0}>)\rightarrow\chi<\psi_{i_0}>)\end{aligned}$$

Moreover it is easy to see that - except in the trivial case that $\ell(\psi_{i_0})=3$ and $\psi=\chi=\psi_{i_0}$ - $\ell(\psi\rightarrow\chi)>\ell((\psi_{i_0}\wedge\psi<\psi_{i_0}>)\rightarrow\chi<\psi_{i_0}>)$. Hence $o(\psi\rightarrow\chi)>o((\psi_{i_0}\wedge\psi<\psi_{i_0}>)\rightarrow\chi<\psi_{i_0}>)$. Set $\varphi^*:=((\psi_{i_0}\wedge\psi<\psi_{i_0}>)\rightarrow\chi<\psi_{i_0}>)^*$.

case D3.1 One of the ψ_i, say ψ_{i_0}, is an atom.

In case $\psi_{i_0}=\bot$: $\varphi^*:=\top$

In case $\psi_{i_0}=\top$: $\varphi^*:=(\varphi<\top>)^*$

Suppose $\psi_{i_0}=p_s$. $IP\vdash\varphi\leftrightarrow(p_s\rightarrow\varphi<p_s>)$ and $o(\varphi<p_s>)<o(\varphi)$. Set $\varphi^*:=(p_s\rightarrow(\varphi<p_s>)^*)$

One easily checks (i),(ii) for φ^*, as to (iii):

Suppose $\sigma\in$ NNIL and $IP\vdash\sigma\rightarrow\varphi$, then $IP\vdash((\sigma\wedge p_s)\rightarrow\varphi<p_s>)$. $(\sigma\wedge p_s)\in$ NNIL, hence by IH: $IP\vdash((\sigma\wedge p_s)\rightarrow(\varphi<p_s>)^*)$, so $IP\vdash(\sigma\rightarrow(p_s\rightarrow(\varphi<p_s>)^*))$.

case D3.2 All the ψ_i are implications and each ψ_i occurs only in φ.

Say $\psi_i=(\rho_i\rightarrow\nu_i)$, so $\varphi=(\bigwedge_{i=0}^{n}(\rho_i\rightarrow\nu_i)\rightarrow\chi)$.

We proceed in the following way: we construct $\varphi_0,\varphi_1,\varphi_2,\varphi_3$ with the following properties:

- $IP\vdash\varphi\leftrightarrow(\varphi_1\wedge\varphi_2)$
- $i(\varphi_1)\leqslant i(\varphi), i(\varphi_2)<i(\varphi)$
- $IP\vdash\varphi_3\rightarrow\varphi_1$
- For all $\sigma\in$NNIL $IP\vdash(\sigma\rightarrow\varphi_1)\Rightarrow IP\vdash(\sigma\rightarrow\varphi_3)$
- $i(\varphi_3)<i(\varphi_1)$
- $\varphi_0:=(\varphi_3\wedge\varphi_2)$

As is easily seen from these properties:

- $IP\vdash\varphi_0\rightarrow\varphi$
- For all $\sigma\in$NNIL $IP\vdash(\sigma\rightarrow\varphi)\Rightarrow IP\vdash(\sigma\rightarrow\varphi_0)$
- $i(\varphi_0)<i(\varphi)$, hence $o(\varphi_0)<o(\varphi)$

Set $\varphi^*:=(\varphi_0)^*$.

For the construction of φ_3 we need a lemma.

1.1.15.2 Lemma

Let $\eta=\bigwedge_{i=0}^{n}(\lambda_i\rightarrow\tau_i)$

a) Suppose $IP\vdash\tau_i\rightarrow\vartheta$ for $i=0,\dots,n$, then $IP\vdash(\bigvee_{i=0}^{n}[\eta]\lambda_i\vee[\eta]\vartheta)\rightarrow(\eta\rightarrow\vartheta)$ and hence by 1.1.11: for all $\sigma \in STAB$ and so for all $\sigma \in NNIL$:

$$IP \vdash \sigma \rightarrow (\eta \rightarrow \vartheta) \Leftrightarrow IP \vdash \sigma \rightarrow (\bigvee_{i=0}^{n} [\eta]\lambda_i \vee [\eta]\vartheta)$$

b) $i(\bigvee_{i=0}^{n} [\eta]\lambda_i \vee [\eta]\vartheta) < i(\eta \rightarrow \vartheta)$

Proof

a) We have: $IP \vdash [\eta]\vartheta \rightarrow (\eta \rightarrow \vartheta)$ and secondly $IP \vdash [\eta]\lambda_i \rightarrow (\eta \rightarrow \lambda_i)$, $IP \vdash \eta \rightarrow (\lambda_i \rightarrow \tau_i)$ and $IP \vdash \tau_i \rightarrow \vartheta$, hence $IP \vdash [\eta]\lambda_i \rightarrow (\eta \rightarrow \vartheta)$.

b) It is sufficient to show $i([\eta]\vartheta) < i(\eta \rightarrow \vartheta)$ and for $i=0,\dots,n$ $i([\eta]\lambda_i) < i(\eta \rightarrow \vartheta)$.

Consider for example $[\eta]\lambda_i$. An outer implication of $[\eta]\lambda_i$ has the form $((\gamma \wedge \eta\langle\gamma\rangle) \rightarrow \delta\langle\gamma\rangle)$.

We have:

$$\begin{aligned} i((\gamma \wedge \eta\langle\gamma\rangle) \rightarrow \delta\langle\gamma\rangle) &< i(\eta \rightarrow (\gamma \rightarrow \delta)) \\ &\leq i(\eta \rightarrow \vartheta) \end{aligned}$$

The last inequality is because $(\gamma \rightarrow \delta)$ occurs in λ_i and hence in η.

□

1.1.15.2 suggests that we should define φ_1 as: $\varphi_1 := (\bigwedge_{i=0}^{n} (\nu_i \rightarrow (\rho_i \wedge \chi)) \rightarrow \chi)$.
We construct φ_2. Define:

$$\psi_{1i}^{k} := \begin{cases} (\nu_i \rightarrow (\rho_i \wedge \chi)) & \text{if } i<k \\ (\nu_i \rightarrow \rho_i) & \text{if } i \geq k \end{cases}$$

$$\psi_{2i}^{k} := \begin{cases} (\nu_i \rightarrow (\rho_i \wedge \chi)) & \text{if } i<k \\ \rho_i & \text{if } i=k \\ \nu_i \rightarrow \rho_i & \text{if } i>k \end{cases}$$

$$\psi_1^{k} := \bigwedge_{i=0}^{n} \psi_{1i}^{k}, \quad \psi_2^{k} := \bigwedge_{i=0}^{n} \psi_{2i}^{k}.$$

Claim 1

$$IP \vdash (\psi_1^{k} \rightarrow \chi) \Leftrightarrow ((\psi_1^{k+1} \rightarrow \chi) \wedge (\psi_2^{k} \rightarrow \chi))$$

Proof of claim 1: immediate from:

$IP \vdash (((\sigma\to\tau)\wedge\eta)\to\varepsilon)\leftrightarrow((((\sigma\to(\tau\wedge\varepsilon))\wedge\eta)\to\varepsilon)\wedge((\tau\wedge\eta)\to\varepsilon))$

□ Claim 1

Claim 2

$$IP \vdash \varphi \leftrightarrow ((\psi_1^{n+1}\to\chi)\wedge \bigwedge_{k=0}^{n} (\psi_2^{k}\to\chi))$$

Proof of claim 2: immediate from claim 1.

□ Claim 2

Clearly $\varphi_1 = (\psi_1^{n+1}\to\chi)$. Put $\varphi_2 := \bigwedge_{k=0}^{n} (\psi_2^{k}\to\chi)$

Claim 3

$i(\varphi_1)\leqslant i(\varphi), i(\varphi_2)<i(\varphi)$

Proof of claim 3: we treat the second inequality the first is similar. It is clearly sufficient to show $i(\psi_2^k\to\chi)<i(\varphi)$. Define $f: I(\psi_2^k\to\chi)\to I(\varphi)$ as follows:

$$f(\varepsilon) := \begin{cases} \varphi & \text{if } \varepsilon=(\psi_2^k\to\chi) \\ \varepsilon & \text{if } \varepsilon \text{ is a subformula of } \nu_i, \rho_i \text{ or } \chi \\ \psi_j & \text{if } \varepsilon=\psi_{2j}^k \text{ and } \psi_{2j}^k \text{ is } \textit{not} \text{ a subformula of } \nu_i,\rho_i \text{ or } \chi \text{ and for no} \\ \ell<j & \psi_{2\ell}^k=\psi_{2j}^k \end{cases}$$

Clearly f is a function from $I(\psi_2^k\to\chi)$ to $I(\varphi)$. f is an injection. This follows from the fact that each ψ_i occurs only once in φ. Finally ψ_k is not in the range of f (this could only be the case by the third clause, so $\varepsilon=\psi_{2k}^k$, but $\psi_{2k}^k=\rho_k$, so the second clause applies).

□ Claim 3

Put $\varphi_3 := (\bigvee_{i=0}^{n} [\psi]\ \nu_i \vee [\psi]\chi)$. By lemma 1.1.15.2 the required properties follow.

END OF N

Evidently the propositional variables of φ^* are among those of φ.

1.1.16 Example

$((p\to q)\to r)^* = ((p\vee r)\wedge(q\to r))^*$

$\qquad = (p\vee r)^*\wedge(q\to r)^*$

$\qquad = (p\vee r)\wedge(q\to r)$

This is equivalent to $((p\wedge(q\to r))\vee r)$

1.1.17 Remark

φ^* as constructed is classically equivalent to φ.

1.1.18 STAB and NNIL

Every φ in STAB is provably equivalent to some ψ in NNIL. In other words $\mathrm{STAB}/_{\mathrm{IP}}=\mathrm{NNIL}/_{\mathrm{IP}}$. To show this it is evidently sufficient to prove that N computes a φ^* in a* from φ in a for $\langle \underline{\mathrm{IP}},\mathrm{STAB}/_{\mathrm{IP}}\rangle$. Thus if φ is stable N will compute a φ^* in NNIL which is *provably equivalent* to φ.

To see that N has this property we just have to run through cases A-D of 1.1.15 and check that they work also for STAB. A pleasant surprise is that A, B, D are trivial. We consider case C. We have to reproduce the reasoning of C for STAB. It is clearly sufficient to show:

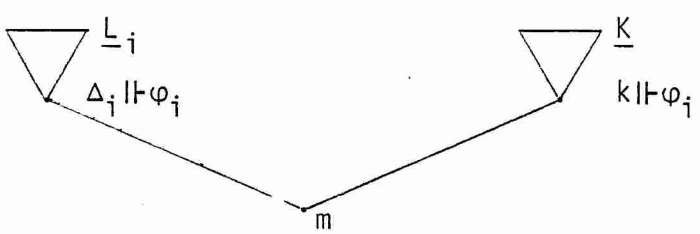
1.1.18.0 Lemma

Let $\varphi\in\mathrm{STAB}$ and let $\bigvee_{i=0}^{n-1}\varphi_i$ be the disjunctive normal form of φ. Then the φ_i are also in STAB.

Proof:

Let $\underline{K}=\langle K,k,\leq,\Vdash\rangle$ be a Kripke model such that $k\Vdash\varphi_i$. Let $K'\subseteq K$, $k\in K'$. Let $\underline{K}'=\langle K',k,\leq',\Vdash'\rangle:=\underline{K}[K']$. We show $k\Vdash'\varphi_i$.

Let $\Delta_0,\ldots,\Delta_{n-1}$ be as in 1.0.18. As is easily seen from the minimality condition: $\Delta_k\subseteq\Delta_s\Rightarrow k=s$. Consider the model $\underline{L}_i:=\langle L_i,\Delta_i,\leq,\Vdash\rangle:=\underline{L}_{\mathrm{Sub}(\varphi)}(\Delta_i)$. Construct a new model $\underline{M}=\langle M,m,\leq,\Vdash\rangle$ as follows:



Where $m\Vdash p_i:\Leftrightarrow p_i\in\Delta_i$. By the Push Down Lemma: $m\Vdash\varphi_i$, hence $m\Vdash\varphi$.

Let $M':=\{m\}\cup L_i\cup K'$ and let $\underline{M}'=\langle M',m,\leq',\Vdash\rangle:=\underline{M}[M']$. Note that on the elements of L_i $\Vdash$ of $\underline{L}_i$ and $\Vdash'$ of $\underline{M}'$ coincide and that on the elements of K' $\Vdash'$ of $\underline{K}'$ and $\Vdash'$ of $\underline{M}'$ coincide. φ is stable, hence $m\Vdash'\varphi$, thus for some j $m\Vdash'\varphi_j$. It follows that $\Delta_i\Vdash'\varphi_j$ so $\Delta_i\Vdash\varphi_j$, ergo $\Delta_i\vdash\Delta_j$. Conclude $\Delta_j\subseteq\Delta_i$ and thus i=j. So we find $m \Vdash' \varphi_i$ and finally $k \Vdash' \varphi_i$.

□

1.1.19 Acknowledgement

The first to conjecture the connection between NNIL and STAB was Johan van Benthem. He independently gave a proof. Van Benthem's proof is analogous to the usual proof that predicate logical formulas preserved under submodels are provably equivalent to universal formulas (but there are some extra details!) Van Benthem's proof also works for modal propositional logic.

1.1.20 Open questions

a) Estimate $|\mathrm{NNIL}^n/_{\mathrm{IP}^n}|$

b) How fast - or how slow - is N?

c) Find standard representatives of the elements of $\mathrm{NNIL}/_{\mathrm{IP}}$

1.2 $<\underline{\mathrm{IP}},\{\top,\bot\}>$

We take a closer look at $\mathcal{D}$, i.e. $<\underline{\mathrm{IP}},\{\top,\bot\}>$. We have already seen that $\mathcal{D}$ is a full EHA.

1.2.0 Fact

Consider $\mathcal{D}$. For $\varphi \in L_{\mathrm{IP}}$:

$[\varphi]_{\mathrm{IP}}{}^* = \top \Leftrightarrow \mathrm{IP} \vdash \varphi$

$[\varphi]_{\mathrm{IP}}{}^* = \bot \Leftrightarrow \mathrm{IP} \nvdash \varphi$

Proof: immediate.

□

We show how to compute the unique element $t(\varphi)$ of $[\varphi]^*_{\mathrm{IP}} \cap \{\top,\bot\}$ from φ. Clearly the algorithm is just a decision procedure for derivability in IP. Of course many such procedures are known. The point of this one is not its efficiency but its use in part 2.

1.2.1 The algorithm T

T is just a variation of N. We only elaborate on points where something different happens. We consider $\top,\bot$ to be ordered by $\bot<\top$.

case A φ is atomic

$$t(\varphi) := \begin{cases} \top \text{ if } \varphi=\top \\ \bot \text{ otherwise} \end{cases}$$

case B $\varphi=(\psi\wedge\chi)$

$t(\varphi):=\min(t(\psi),t(\chi))$

case C $\varphi=(\psi\vee\chi)$

$t(\varphi):=\max(t(\psi),t(\chi))$

Case D

cases D0, D1, D2, D3.0 are as in the case of N.

case D3.1 $\psi=\bigwedge\limits_{i=0}^{n}\psi_i$, the ψ_i are atoms or implications. One of the ψ_i, say ψ_{i_0}, is an atom.

If $\psi_{i_0}=\bot$ set $t(\varphi):=T$

If $\psi_{i_0}=T$ set $t(\varphi):=t(\varphi<T>)$

Suppose $\psi_{i_0}=p_s$, we have $IP\vdash\varphi\leftrightarrow(p_s\rightarrow\varphi<p_s>)$

1.2.1.0 Lemma

If p_i does not occur in σ, then $IP\vdash p_i\rightarrow\sigma\Leftrightarrow IP\vdash\sigma$.

Proof:

"$\Leftarrow$" trivial

"$\Rightarrow$" By a simple Kripke model argument or by the interpolation property for IP.

□

Set $t(\varphi):=t(\varphi<p_s>)$

case D3.2 $\psi=\bigwedge\limits_{i=0}^{n}\psi_i$; all the ψ_i are implications; each ψ_i occurs only once in φ.

Take φ_0 as in N. Clearly $IP\vdash\varphi\Leftrightarrow IP\vdash\varphi_0$. Set $t(\varphi):=t(\varphi_0)$

END OF T

1.3 $<UP,\omega+1/_{UP}>$

1.3.0 Definitions

We formulate UP in the language L_{UP}, which we consider as just a *notational variant* of L_{IP}.

L_{IP} is the smallest set such that:

- $0,1,2,\ldots,\omega$ are in L_{UP}
- L_{UP} is closed under $\wedge,\vee,\rightarrow$

The identification with L_{IP} is as follows:

$\widetilde{IP}(0):=\perp$

$\widetilde{IP}(n+1):=p_n$

$\widetilde{IP}(\omega):=\top$

$\widetilde{IP}$ commutes with $\wedge,\vee,\rightarrow$

We use 'm', 'n' as ranging over $0,1,2,\ldots$ and 'α', 'β' as ranging over $0,1,2,\ldots,\omega$.

UP is the theory axiomatized by:

- Intuitionistic Propositional Logic
- $(n\rightarrow n+1)$ $(n\in\omega)$

We are interested in $\mathcal{UP}:=\langle \underline{\mathrm{UP}}, \{0,1,\ldots,\omega\}/_{\mathrm{UP}}\rangle$

Set $n(\varphi):=$ the unique element of $[\varphi]_{\mathrm{UP}}^{*}\cap\{0,1,\ldots,\omega\}$. (We will further on prove the existence of $n(\varphi)$.)

1.3.1 Kripke Models for UP

$\underline{K}=\langle K,\leq,\Vdash\rangle$ is a Kripke Model for UP if K is a non empty set, $\leq$ a weak partial order on K and $\Vdash$ forcing relation satisfying:

- $k\nVdash 0$, $k\Vdash\omega$
- $k\leq k'$ and $k\Vdash m\Rightarrow k'\Vdash m$
- $m\leq n$ and $k\Vdash m\Rightarrow k\Vdash n$.

We have: $\mathrm{UP}\vdash\varphi\Leftrightarrow$ for all Kripke Models $\underline{K}=\langle K,\leq,\Vdash\rangle$ for UP, for all $k\in K$, $k\Vdash\varphi$

Proof:

"$\Leftarrow$" routine

"$\Rightarrow$" Suppose $\mathrm{UP}\nvdash\varphi$; let the atoms occuring in φ be among $0,1,\ldots,m,\omega$. Then $\mathrm{IP}+\{s\rightarrow s+1 \mid s=1,\ldots,m-1\}\nvdash\varphi$.

Let $\underline{K}_0=\langle K_0,\leq_0,\Vdash_0\rangle$ be an IP model with bottom node k_0 such that $k_0\Vdash_0 s\rightarrow s+1$ $(s=1,\ldots,m-1)$ and $k_0\nVdash_0\varphi$. Change $\underline{K}_0$ to a Kripke model $\underline{K}=\langle K_0,\leq_0,\Vdash\rangle$ for UP by postulating for all $k\in K_0$: $k\Vdash\alpha:\Leftrightarrow(\alpha\leq m$ and $k\Vdash_0\alpha)$ or $\alpha>m$. As is easily seen $k_0\nVdash\varphi$.

□

1.3.2 Fact

a) $UP \vdash \alpha\to(\varphi\vee\psi) \Leftrightarrow UP \vdash \alpha\to\varphi$ or $UP \vdash \alpha\to\psi$

b) Suppose m is bigger than all n occurring in φ, then: $UP \vdash m\to\varphi \Leftrightarrow UP \vdash \varphi$

Proof:

a) by a trivial variation of the usual proof of the disjunction property for IP.

b) "$\Leftarrow$" immediate

"$\Rightarrow$" Suppose $UP \nvdash \varphi$. Consider a Kripke Model $\underline{K}=\langle K,\leqslant,\Vdash\rangle$ for UP with mode $k_0\in K$ such that $k_0 \nVdash \varphi$. Change $\underline{K}$ to $\underline{K}'=\langle K,\leqslant,\Vdash'\rangle$ by postulating $k\Vdash' p :\Leftrightarrow ((k\Vdash p$ and $p<m)$ or $p\geqslant m)$. As is easily seen $\underline{K}'$ is a model for UP and $k_0 \nVdash' m\to\varphi$.

□

1.3.3 Theorem

UP is a full EHA.

Proof:

a) $T\in \omega+1/_{UP}$

b) By 1.3.2b the elements of $\omega+1$ implying a given φ are either $0,1,\dots,m$ for some m or all of $\omega+1$. In the first case $[\varphi]_{UP}{}^*=[m]_{UP}$, in the second $[\varphi]_{UP}{}^*=[\omega]_{UP}=T$.

c) $\omega+1/_{UP}$ is closed under $\wedge$, hence *UP* is conjunctive.

d) By 1.3.2a:

$UP \vdash \alpha\to(\varphi\vee\psi) \Leftrightarrow UP\vdash \alpha\to\varphi$ or $UP\vdash\alpha\to\psi$.

$\Leftrightarrow [\alpha]_{UP} \leqslant [\varphi]_{UP}$ or $[\alpha]_{UP}\leqslant[\psi]_{UP}$

$\Leftrightarrow [\alpha]_{UP} \leqslant [\varphi]_{UP}{}^*$ or $[\alpha]_{UP}\leqslant[\psi]_{UP}{}^*$

$\Leftrightarrow [\alpha]_{UP} \leqslant [\varphi]_{UP}{}^*\vee[\psi]_{UP}{}^*$

$\Leftrightarrow \alpha\leqslant \max(n(\varphi),n(\psi))$

Hence *UP* is disjunctive.

□

1.3.4 Remark

$UP_0=\langle\omega+1/_{UP},\leqslant,\wedge_0,\vee,\to_0,T,\perp\rangle$ is isomorphic to the complete Heyting Algebra $\langle\omega+1,\leqslant,\wedge,\vee,\to,T,\perp\rangle$, where

$\alpha\wedge\beta := \min(\alpha,\beta)$

$\alpha\vee\beta := \max(\alpha,\beta)$

$$\alpha\to\beta := \begin{cases}\omega & \text{if } \alpha\leqslant\beta\\ \beta & \text{otherwise}\end{cases}$$

1.3.5 Theorem

Let $\eta := \bigwedge\limits_{i=0}^{n} (\sigma_i\to\tau_i)$, then

$$UP \vdash \alpha \to (\bigwedge_{i=0}^{n} (\sigma_i \to \tau_i) \to \lambda) \Rightarrow UP \vdash \alpha \to (\bigvee_{i=0}^{n} [n]\, \sigma_i \vee [n]\, \lambda)$$

Proof: a trivial variation of the proof of 1.1.11.

□

1.3.6 The Algorithm Ω

We specify an algorithm to compute $n(\varphi)$ from φ. Ω is just a variant of N and T, so we only comment on points of divergence.

case A φ is atomic

Set $n(\varphi) := \varphi$

case B $\varphi = (\psi \wedge \chi)$

$n(\varphi) := \min(n(\psi), n(\chi))$

case C $\varphi = (\psi \vee \chi)$

$n(\varphi) := \max(n(\psi), n(\chi))$

cases D0, D1, D2, D3.0: as for N

case D3.1 $\psi = \bigwedge_{i=0}^{n} \psi_i$, the ψ_i are atoms or implications, one of the ψ_i, say ψ_{i_0}, is an atom, say α.

We have $UP \vdash \varphi \leftrightarrow (\alpha \to \varphi\langle\alpha\rangle)$

$$\text{Set } n(\varphi) := \begin{cases} \omega & \text{if } \alpha \leqslant n(\varphi\langle\alpha\rangle) \\ n(\varphi\langle\alpha\rangle) & \text{otherwise} \end{cases}$$

This is easily seen to be correct.

case D3.2 $\psi = \bigwedge_{i=0}^{n} \psi_i$, the ψ_i are implications and each ψ_i occurs only once in φ.

Let φ_0 be as in N D3.2. One easily shows using 1.3.5 that for all α in $\omega+1$ $UP \vdash \alpha \to \varphi \Leftrightarrow UP \vdash \alpha \to \varphi_0$. Set $n(\varphi) := n(\varphi_0)$

END OF Ω

2. Solutions of Evaluation Problems in Heyting's Arithmetic

2.0 Introduction

The precise choice of L_{HA}, the language of HA, is immaterial for this paper. However it is most convenient at least to *think* of L_{HA} as having symbols for the primitive recursive functions and bounded quantifiers '$\forall x<t$', '$\exists x<t$' (x is not free in t). We will assume the theory if formalized in a Natural Deduction System.

2.0.0 Convention

Suppose $A(x_0,\ldots,x_{n-1})$ is a formula of L_{HA}, where n-1 is the maximum of the indices of the free variables occurring in A. '$\Box A$' *in the context of* HA will stand for: $\mathrm{Prov}_{HA}(\mathrm{Sub}_n(\ulcorner A(v_0,\ldots,v_{n-1})\urcorner, x_0,\ldots,x_{n-1}))$, where Sub_n is a Gödel substitution function. E.g. we have $HA \vdash \mathrm{Sub}_1(\ulcorner A(v_0)\urcorner, \underline{m}) = \ulcorner A(\underline{m})\urcorner$

2.0.1 Definition

a) Let B and C be sentences of L_{HA}. Define: $B \sim_{HA} C :\Leftrightarrow HA \vdash \Box B \leftrightarrow \Box C$

$\sim_{HA}$ is the relation of *provably deductive equivalence*

b) Let B,C be sentences of L_{HA}.

$B \approx_{HA} C :\Leftrightarrow HA \vdash \forall A \in \Sigma_1^0$- sentences $(\Box(A \to B) \leftrightarrow \Box(A \to C))$

2.0.2 Fact

(i) $\sim_{HA}$ and $\approx_{HA}$ are equivalence relations.

(ii) $B \sim_{HA} B', C \sim_{HA} C' \Rightarrow B \wedge C \sim_{HA} B' \wedge C'$

$B \approx_{HA} B', C \approx_{HA} C' \Rightarrow B \wedge C \approx_{HA} B' \wedge C'$

(iii) $\approx_{HA} \subsetneq \sim_{HA}$

Proof: the only not fully trivial point is $\approx_{HA} \neq \sim_{HA}$. Let R be a Σ_1^0 Rosser sentence. We have $R \sim_{HA} \bot$. Suppose $R \approx_{HA} \bot$. It follows that $HA \vdash \Box_{HA}(R \to R) \leftrightarrow \Box_{HA}(R \to \bot)$. Quod non.

$\Box$

2.0.3 Remark

It will be shown that: $B \approx_{HA} B', C \approx_{HA} C' \Rightarrow B \vee C \approx_{HA} B' \vee C'$ (see 2.1.1.2).

A similar fact does not hold for $\sim_{HA}$: let R be a Σ_1^0 Rosser sentence and $S := \Box R < \Box \neg R$. We have: $R \sim_{HA} \bot, S \sim_{HA} \bot, R \vee S \sim_{HA} \Box \bot$.

For the moment fix a propositional language L, with set of propositional variables P. Let G be some non empty set of functions from P to the sentences of L_{HA}.

2.0.4 Definition

a) Let $g: P \to$ the sentences of L_{HA}. Define $(\ )^g: L \to$ the sentences of L_{HA}, by:

- $(\varphi)^g := g(\varphi)$ if $\varphi \in P$
- $(\top)^g := (\underline{0}=\underline{0})$, $(\bot)^g := \bot$
- $(\cdot)^g$ commutes with $\wedge, \vee, \rightarrow$

b) Let U be a logic (in L) such that for all $\varphi \in L$: $U \vdash \varphi \Leftrightarrow$ for all $g \in G$ $HA \vdash (\varphi)^g$.
Define $\sim_G$ and $\approx_G$ on $L/_U$ by:
$a \sim_G b :\Leftrightarrow$ for some $\varphi \in a, \psi \in b$, for all $g \in G$ $(\varphi)^g \sim_{HA} (\psi)^g$
$a \approx_G b :\Leftrightarrow$ for some $\varphi \in a, \psi \in b$, for all $g \in G$ $(\varphi)^g \approx_{HA} (\psi)^g$

The definition is independent of the choice of φ, ψ, e.g.:
$\varphi, \varphi' \in a \Rightarrow U \vdash \varphi \leftrightarrow \varphi'$
$\Rightarrow$ for all $g \in G$ $U \vdash (\varphi)^g \leftrightarrow (\varphi')^g$
$\Rightarrow$ for all $g \in G$ $HA \vdash \Box(\varphi)^g \leftrightarrow \Box(\varphi')^g$

2.0.5 Fact

i) $\sim_G, \approx_G$ are equivalence relations

ii) $a \sim_G a', b \sim_G b' \Rightarrow a \wedge b \sim_G a' \wedge b'$
$a \approx_G a', b \approx_G b' \Rightarrow a \wedge b \approx_G a' \wedge b'$

We call the problem to characterize $\sim_G$: *the evaluation problem for G* (with respect to HA). Our solution of this problem for certain concrete G takes the following form: we specify a pair $\langle U, X \rangle$, where U is a theory in L and $X \subseteq L$ such that:

a) $\langle \underline{U}, X/_U \rangle$ is an EHA, with equivalence relation - say - $\sim_X$.
b) $U \vdash \varphi \Leftrightarrow$ for all $g \in G$ $HA \vdash (\varphi)^g$
c) $\sim_G = \sim_X$

We claim that to prove a,b,c it is sufficient to show:

(i) $\langle \underline{U}, X/_U \rangle$ is an EHA
(ii) $U \vdash \varphi \Rightarrow$ for all $g \in G$ $HA \vdash (\varphi)^g$
(iii) For $\varphi^* \in [\varphi]_U^* \cap X$ and $g \in G$: $(\varphi)^g \sim_{HA} (\varphi^*)^g$
(iv) For $\chi, \nu \in X$
$U \nvdash \chi \rightarrow \nu \Rightarrow$ there is a $g \in G$ such that $HA \nvdash \Box(\chi)^g \rightarrow \Box(\nu)^g$

Proof:

- (a)=(i)
- For (b) we show: (for all $g \in G$ $HA \vdash (\varphi)^g) \Rightarrow U \vdash \varphi$

Suppose $U \nvdash \varphi$, $\varphi^* \in [\varphi]_U^* \cap X$. Clearly $U \nvdash \varphi^*$ or $U \nvdash \varphi^* \leftrightarrow \top$. By (iv): there is a

$g \in G$ such that $HA \nvdash \Box(\varphi^*)^g \leftrightarrow \Box(\underline{0}=\underline{0})$, so $HA \nvdash \Box(\varphi^*)^g$. By (iii) $HA \nvdash \Box(\varphi)^g$. Hence by Σ-completeness $HA \nvdash (\varphi)^g$.

- For (c): note that (iii) tells us that $a \sim_G a^*$ and (iv) that $a^* \neq b^* \Rightarrow a^* \not\sim_G b^*$, hence:

$$
\begin{aligned}
a \sim_G b &\Leftrightarrow a^* \sim_G b^* \\
&\Leftrightarrow a^* = b^* \\
&\Leftrightarrow a \sim_X b
\end{aligned}
$$

□

Before we present solutions to some concrete evaluation problems, we interpolate a section with arithmetical preliminairies.

2.1 Translations and Derived Rules in HA

2.1.0 q-realizability

For every formula A of L_{HA} we define a formula xqA of L_{HA} as follows:

- $xqP := P$ for P atomic
- $xq(A \wedge B) := ((x)_0 qA \wedge (x)_1 qB)$
- $xq(A \vee B) := (((x)_0 = \underline{0} \rightarrow (x)_1 qA) \wedge ((x)_0 \neq \underline{0} \rightarrow (x)_1 qB))$
- $xq(A \rightarrow B) := (\forall y(yqA \rightarrow \exists z(\{x\} y \simeq z \wedge zqB)) \wedge (A \rightarrow B))$
- $xq(\exists y A(y)) := (x)_1 qA((x)_0)$
- $xq(\forall y A(y)) := \forall y(\exists z \{x\} y \simeq z \wedge zqA(y))$

2.1.1.0 Fact

a) $HA \vdash xqA \rightarrow A$

b) For every Σ_1^0 formula A with free variables $y_0, \ldots, y_{n-1}$ there is an e such that $HA \vdash A \rightarrow \exists z(\{\underline{e}\}(y_0, \ldots, y_{N-1}) \simeq z \wedge zqA)$

c) Let A be a Σ_1^0-formula. Suppose $A, B_0, \ldots, B_{n-1}, C$ have free variables among $y_0, \ldots, y_{N-1}$. Let $x_0, \ldots, x_{n-1}$ be free variables distinct from the y_i. Then: $B_0, \ldots, B_{n-1} \vdash_{HA+A} C \Rightarrow$ There is an e such that $x_0 qB_0, \ldots, x_{n-1} qB_{n-1} \vdash_{HA+A} \exists z(\{\underline{e}\}(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{N-1}) \simeq z \wedge zqC)$

d) (a),(b),(c) are verifiable in HA

Proof:

a) induction on A

b) idem

c) induction on the proof of $B_0, \ldots, B_{n-1} \vdash_{HA+A} C$ using (b). For details see [10] pp. 188-202.

d) All the proofs are simple inductions.

2.1.1.1 Theorem

Let B,C,D be sentences of L_{HA}, then:

$HA\vdash(\forall A\in\Sigma^0_1\text{-sentences }((\Box(A\to B)\vee\Box(A\to C))\to\Box(A\to D)))$
$\quad\to(\forall A\in\Sigma^0_1\text{-sentences }(\Box(A\to(B\vee C))\to\Box(A\to D))$

Proof: (in HA)

Suppose for every Σ^0_1-sentence A $((\Box(A\to B)\vee\Box(A\to C))\to\Box(A\to D))$. Suppose further that for Σ^0_1-sentence A_0 $\Box(A_0\to(B\vee C))$. It follows that there is an e such that $\Box(A_0\to eq(B\vee C))$ and thus $\Box(A_0\to\{e\}\downarrow)$, $\Box(A_0\to((\{e\})_0=\underline{0}\to(\{e\})_1\, qB))$, $\Box(A_0\to((\{e\})_0\neq\underline{0}\to(\{e\})_1\, qC))$. Hence we have $\Box(A_0\to\{e\}\downarrow)$, $\Box((A_0\wedge(\{e\})_0=\underline{0})\to B)$, $\Box((A_0\wedge(\{e\}_0)\neq\underline{0})\to C)$. $\ulcorner(\{e\})_0=\underline{0}\urcorner$ and $\ulcorner(\{e\})_0\neq\underline{0}\urcorner$ are Σ^0_1-sentences (remember convention 2.0.0), so by assumption: $\Box((A_0\wedge(\{e\})_0=\underline{0})\to D)$, $\Box((A_0\wedge(\{e\})_0\neq\underline{0})\to D)$. Combining this with $\Box(A_0\to\{e\}\downarrow)$ we find $\Box(A_0\to D)$.

□

2.1.1.2 Corollary

$B\approx_{HA}B', C\approx_{HA}C' \Rightarrow B\vee C\approx_{HA}B'\vee C'$.
Proof: straightforward.

□

2.1.1.3 Application: Leivant's Principle

$B\vee C\sim_{HA}(B\wedge\Box B)\vee(C\wedge\Box C)$

Proof:

$B\approx_{HA}(B\wedge\Box B), C\approx_{HA}(C\wedge\Box C)$, hence $B\vee C\approx_{HA}(B\wedge\Box B)\vee(C\wedge\Box C)$. Conclude $B\vee C\sim_{HA}(B\wedge\Box B)\vee(C\wedge\Box C)$.

□

2.1.1.4 Credit

The principle 2.1.1.1 is implicit in the proof of De Jongh's theorem on Formulas of One Propositional Variable, see [4].

2.1.2 The Friedman Translation

Consider a formula B of L_{HA}, we define $(\cdot)^B$: $L_{HA}\to L_{HA}$ as follows:

- $(P)^B:=P\vee B$ for P atomic
- $(\cdot)$ commutes with $\wedge,\vee,\to,\forall,\exists$.

(In case free variables of B would be bound in the process of translating, one should rename bound variables so that the free variables of B remain free.)

2.1.2.0 Fact

a) For $A\in\Sigma^0_1$ $HA\vdash(A)^B\leftrightarrow(A\vee B)$

b) Let A be a Σ^0_1-formula, then $B_0,\dots,B_{n-1}\vdash_{HA+A}C\Rightarrow(B_0)^D,\dots,(B_{n-1})^D\vdash_{HA+A}(C)^D$

c) $HA\vdash B\rightarrow(A)^B$

d) (a),(b),(c) can be verified in HA.

Proof: by simple inductions, see e.g. [13].

□

2.1.3 De Jongh's Translation, a variant

To simplify the presentation we assume that L_{HA} contains symbols for the primitive recursive functions and bounded quantifiers '$\forall x<t$', '$\exists x<t$' - we make it into a *syntactical* constraint that x does not occur free in t. To eliminate these extra assumptions on L_{HA} we only have to be a bit more careful in our formulations.

2.1.3.0 Definition

We employ an ideosyncratic definition of Σ^0_1.
Σ^0_1 is the smallest set such that:
- if P is an atom, $P\in\Sigma^0_1,(\neg P)\in\Sigma^0_1$
- Σ^0_1 is closed under $\wedge,\vee,\exists,\exists x<t,\forall x<t$

One easily shows in HA the equivalence of this definition with the usual one.

2.1.3.1 Definition

a) the *complexity* c(A) of a formula A of L_{HA} is defined as follows:
- $c(P):=0$ for P atomic
- $c(B\wedge C):=c(B\vee C):=\max(c(B),c(C))$
- $c(B\rightarrow C):=\begin{cases}c(C) & \text{if } B \text{ is atomic}\\ \max(c(B),c(C))+1 & \text{otherwise}\end{cases}$
- $c(\exists xB):=c(B)$
- $c(\exists x<tB):=c(B)$
- $c(\forall x<tB):=c(B)$
- $c(\forall xB):=\begin{cases}c(\forall yC) & \text{if } B=(\forall yC)\\ c(B)+1 & \text{otherwise}\end{cases}$

b) $B_0,\dots,B_{n-1}\vdash_m C:\Leftrightarrow$ there is HA proof Π that $B_0,\dots,B_{n-1}\vdash_{HA}C$ and Π contains only formulas of complexity $\leqslant m$.

c) $B_0,\dots,B_{n-1}\,\square\!\!-_m C$ is the HA-formalization of $B_0,\dots,B_{n-1}\vdash_m C$. $\square_m A:=\square\!\!-_m A$.
We extend convention 2.0.0 to $\square\!\!-_m$.

2.1.3.2 Fact

a) for $A\in\Sigma^0_1$ $c(A)=0$

b) for every n there is a formula $Tr_n(x,y)$ of complexity n in L_{HA} such that for every formula $A(v_0,\ldots,v_{k-1})$ of complexity $\leq n$
$HA\vdash\forall y_0,\ldots,y_{k-1}\ Tr_n(\ulcorner A(v_0,\ldots,v_{k-1})\urcorner,<y_0,\ldots,y_{k-1}>)\leftrightarrow A(y_0,\ldots,y_{k-1})$
$\underline{\qquad\qquad n-1}$

c) $HA\vdash(B_0,\ldots,B_{n-1}\Box\!-_m C)\to\forall\vec{x}(\bigwedge_{i=0}^{n-1} B_i\to C)$ and hence $HA\vdash(\Box_m A\to A)$

d) for $A\in\Sigma^0_1$ $HA\vdash A\to\Box_2 A$

e) the proofs of (a)-(d) can be formalized in HA, so for example we have:
$HA\vdash\forall A\forall x\ \Box(\Box_x A\to A)$, and: $HA\vdash\forall A\in\Sigma^0_1\ \Box(A\to\Box_2 A)$.

Proof (sketch)

(a),(b) are more or less routine.

(c) Suppose '$Proof_m(p)$' is a Δ^0_1-formula that formalizes: p is a code of a HA-proof involving only formulas of complexity $\leq$ m. Suppose further that '$Ass(p)=b$' stands for: b is a code of the conjunction of the assumptions of p. '$Conc(p)=a$' stands for: a is a code of the conclusion of p.

We assume we have a bijective coding of finite sequences, such that $(x)_y=0$ in case $y\geq length(x)$. (In other words we treat finite sequences as eventually 0 functions.)
Show by induction on p:
$\forall p\ Proof_m(p)\to\forall x(Tr_m(Ass(p),x)\to Tr_m(Conc(p),x))$

(d) Inspection of the usual proof of Σ^0_1-completeness for HA shows that we only need the non-induction axioms, which are in Π^0_1-form and sentences of the form $\forall x<\underline{p}\ (x=\underline{0}\vee\exists u<x\ \ x=Su)$. These sentences can be proved by inductions of complexity 2.

□

2.1.3.3 De Jongh's Translation (A variant)

Given formulas E,A of L_{HA}, we define EpmA as follows: let $k:=m+c(E)+2$, then
- $EpmP:=P$ for P atomic
- $Epm(\cdot)$ commutes with $\wedge,\vee,\exists,\exists x<t,\forall x<t$
- $Epm(B\to C):=((EpmB\to EpmC)\wedge\Box_k(E\to(B\to C)))$
- $Epm(\forall xB):=(\forall xEpmB\wedge\Box_k(E\to\forall xB))$

2.1.3.4 Theorem

a) for $A\in\Sigma^0_1$ $HA\vdash A\to EpmA$

b) $HA\vdash EpmA\to\Box_k(E\to A)$

c) Suppose x does not occur free in E, when $HA \vdash (EpmA)[t/x] \leftrightarrow Epm(A[t/x])$

d) $B_0,\dots,B_{n-1} \vdash_m C \Rightarrow EpmB_0,\dots,EpmB_{n-1} \vdash_{HA} EpmC$

e) (a)-(d) can be formalized in HA

Proof:

a) induction on A, use 2.1.3.2 d

b) induction on A

c) induction on A; the crucial lemma is:

$HA \vdash (\Box_k B)[t/x] \leftrightarrow \Box_k(B[t/x])$

This is derived as follows:

$$
\begin{aligned}
HA \vdash (\Box_k B)[t/x] &\leftrightarrow \exists x(t=x \wedge \Box_k B)\\
&\leftrightarrow \exists x(\Box_k(t=x) \wedge \Box_k B)\\
&\leftrightarrow \Box_k(B[t/x])
\end{aligned}
$$

d) induction on the proof of $B_0,\dots,B_{n-1} \vdash_m C$. We can arrange it so that no variables of E are used in $\forall I$ or $\exists E$ in these proofs.

Two examples:

(I) $B_0,\dots,B_{n-1} \vdash_m \forall x A(x) \Rightarrow B_0,\dots,B_{n-1} \vdash_m A(t)$

We have:

$$
\begin{aligned}
EpmB_0,\dots,EpmB_{n-1} &\vdash_{HA} Epm\forall x A(x)\\
&\vdash_{HA} \forall x EpmA(x)\\
&\vdash_{HA} (EpmA(x))[t/x]\\
&\vdash_{HA} EpmA(t)
\end{aligned}
$$

(II) $B_0,\dots,B_{n-1},C \vdash_m D \Rightarrow B_0,\dots,B_{n-1} \vdash_m C \to D$

We have:

$EpmB_0,\dots,EpmB_{n-1},EpmC \vdash_{HA} EpmD$, hence:

$EpmB_0,\dots,EpmB_{n-1} \vdash_{HA} EpmC \to EpmD$ (*)

Moreover: $EpmB_0,\dots,EpmB_{n-1} \vdash_{HA} \Box_k(E \to B_0),\dots,\Box_k(E \to B_{n-1})$

From $B_0,\dots,B_{n-1},C \vdash_m D$, it follows: $E \to B_0,\dots,E \to B_{n-1} \vdash_{m+c(E)+2} E \to (C \to D)$,

hence $EpmB_0,\dots,EpmB_{n-1} \vdash_{HA} \Box_{m+c(E)+2}(E \to (C \to D))$ (**)

Combining (*) and (**) we are done.

□

2.1.3.5 Definition

a) Define $[E]mA$ as follows: let $k := m+c(E)+2$

- $[E]mP := P$ for P atomic
- $[E]m(\cdot)$ commutes with $\wedge, \vee, \exists, \exists x<t, \forall x<t$
- $[E]m(B \to C) := \begin{cases} \Box_k(E \to (B \to C)) & \text{if } B \text{ is not atomic}\\ \neg B \vee [E]mC & \text{if } B \text{ is atomic}\end{cases}$
- $[E]m(\forall x B) := \Box_k(E \to \forall x B)$

b) Define $[E]A$ as follows:

- $[E]P := P$ for P atomic
- $[E](\cdot)$ commutes with $\wedge, \vee, \exists, \exists x<t, \forall x<t$
- $[E](B\rightarrow C) := \begin{cases} E\rightarrow(B\rightarrow C) & \text{if } B \text{ is not atomic} \\ \neg B \vee [E]C & \text{if } B \text{ is atomic} \end{cases}$
- $[E](\forall xB) := (E\rightarrow\forall xB)$

2.1.3.6 Fact

a) $[E]mA$ is Σ_1^0

b) $HA\vdash EpmA\rightarrow[E]mA$,

$HA\vdash[E]mA\rightarrow[E]A$,

$HA\vdash[E]A\rightarrow(E\rightarrow A)$

c) For $A\in\Sigma_1^0$: $HA\vdash EpmA\leftrightarrow[E]mA$

$HA\vdash[E]mA\leftrightarrow[E]A$

$HA\vdash[E]A\leftrightarrow A$

d) (a)-(c) can be formalized in HA.

Proof: routine

□

2.1.3.7 Theorem

Let $E := \bigwedge_{i=0}^{n}(B_i\rightarrow C_i)$, then:

$$HA\vdash\forall A\in\Sigma_1^0\text{-sentences }\left(\Box\left(A\rightarrow\left(\bigwedge_{i=0}^{n}(B_i\rightarrow C_i)\rightarrow D\right)\right)\rightarrow\Box\left(A\rightarrow\left(\bigvee_{i=0}^{n}[E]B_i \vee [E]D\right)\right)\right)$$

Proof (in HA):

Assume A is a Σ_1^0-sentence and $\Box(A\rightarrow(E\rightarrow D))$. Clearly for some x $\Box_x(A\rightarrow(E\rightarrow D))$. It follows that $\Box Epx(A\rightarrow(E\rightarrow D))$. Ergo: $\Box(A\rightarrow(\bigwedge_{i=0}^{n} Epx(B_i\rightarrow C_i)\rightarrow EpxD))$.

Now $\ulcorner Epx(B_i\rightarrow C_i)\urcorner$ is $\ulcorner(EpxB_i\rightarrow EpxC_i\wedge\Box_{x+c(F)+2}(E\rightarrow(B_i\rightarrow C_i)))\urcorner$. Surely $\Box\Box_{x+c(E)+2}(E\rightarrow(B_i\rightarrow C_i))$, thus: $\Box(A\rightarrow(\bigwedge_{i=0}^{n}(EpxB_i\rightarrow EpxC_i)\rightarrow EpxD))$.

The $\ulcorner EpxB_i\urcorner$ and $\ulcorner EpxD\urcorner$ occur on positive places, so $\Box(A\rightarrow(\bigwedge_{i=0}^{n}([E]xB_i\rightarrow EpxC_i)\rightarrow [E]xD))$.

By taking the Friedman Translation based on $F := \bigvee_{i=0}^{n}[E]xB_i$, using the fact that $[E]xB_i$ and $[E]xD$ are Σ_1^0, we find:

$$\Box((A \vee F) \to ((\bigwedge_{i=0}^{n} (([E]xB_i \vee F) \to (EpxC_i)^F)) \to ([E]xD \vee F)))$$

Clearly $\Box(([E]xB_i \vee F) \to F)$ and $\Box(F \to (EpxC_i)^F)$, hence: $\Box(A \to ([E]xD \vee F))$.

Conclude: $\Box(A \to (\bigvee_{i=0}^{n} [E]B_i \vee [E]D))$.

□

2.1.3.8 Corollary

Let $E := \bigwedge_{i=0}^{n} (B_i \to C_i)$, suppose for $i=0,\ldots,n$ $HA \vdash C_i \to D$, then:

$(E \to D) \approx_{HA} \bigvee_{i=0}^{n} [E]B_i \vee [E]D$.

Proof: by the reasoning of 1.1.15.2 a.

□

2.1.3.9 Remarks

a) 2.1.3.7 is implicit in the proof of De Jongh's theorem on Formulas of One Propositional Variable (see [4]).

b) The reader may amuse himself by proving the following strengthening of 2.1.3.7 (derived from a suggestion of Rick Statman):

$$HA \vdash \forall A \in \Pi_2^0 \ \Box((A \wedge \bigwedge_{i=0}^{n} \forall x(B_i \to C_i)) \to D) \to \Box(A \to (\bigvee_{i=0}^{n} \exists x[E]B_i \vee [E]D))$$

where $E = \bigwedge_{i=0}^{n} \forall x(B_i \to C_i)$.

c) Open question: can 2.1.3.7 be strengthened to A that are Σ_1^0-substitution instances of formulas of NNIL?

2.1.3.10 Consequences

All kinds of traditional theorem follow from 2.1.3.7. For example:

a) for all $A \in \Sigma_1^0$ $HA \vdash \Box\neg\neg A \to \Box A$ (closure under Markov's Rule)

b) for all $A \in \Sigma_1^0$ $HA \vdash \Box(\neg\neg A \to A) \to \Box(A \vee \neg A)$

c) for x not free in B: $HA \vdash \Box(\neg B \to \exists xC) \to \Box\exists x(\neg B \to C)$ (Closure under the Independence of Premis Rule)

Proof:

a) $HA \vdash \Box(\neg\neg A) \to \Box([\neg A]A \vee [\neg A]\bot)$

$\to \Box A$

b) $HA \vdash \Box(\neg\neg A \to A) \to \Box([\neg\neg A]\neg A \vee [\neg\neg A]A)$

$\to \Box((\neg\neg A \to \neg A) \vee A)$

$\to \Box(A \vee \neg A)$

c) $HA \vdash \Box(\neg B \to \exists x C) \to \Box([\neg B]\exists x C \vee [\neg B]B)$

$\to \Box(\exists x[\neg B]C \vee (\neg B \to B))$

$\to \Box(\exists x(\neg B \to C) \vee \neg\neg B)$

$\to \Box\exists x(\neg B \to C)$

$\Box$

2.2 The Evaluation Problem for $I\Sigma$

Let $I\Sigma$ be $\{g \mid g: P_{IP} \to \Sigma^0_1\text{-sentences}\}$. We show that <IP,NNIL> solves the evaluation problem for $I\Sigma$.

(i) Yes, <IP,NNIL/$_{IP}$> is a (full) EHA.

(ii) It is routine to show that if $IP \vdash \varphi$ then for all g in $I\Sigma$ $HA \vdash (\varphi)^g$.

(iii) We show that for $g \in I\Sigma$ $(\varphi)^g \sim_{HA} (\varphi^*)^g$, where φ^* is the formula computed from φ by N. To do this we prove the stronger $(\varphi)^g \approx_{HA} (\varphi^*)^g$ by induction on $o(\varphi)$ following the stages of N.

For the moment let's fix a g in $I\Sigma$ and simply write φ for $(\varphi)^g$ in the context of HA.

case A φ is atomic

trivial.

case B $\varphi = (\psi \wedge \chi)$

By IH $\psi \approx_{HA} \psi^*, \chi \approx_{HA} \chi^*$, hence $\psi \wedge \chi \approx_{HA} \psi^* \wedge \chi^*$.

case C $\varphi = (\psi \vee \chi)$

By IH $\psi \approx_{HA} \psi^*, \chi \approx_{HA} \chi^*$, hence $\psi \vee \chi \approx_{HA} \psi^* \vee \chi^*$.

Cases D0, D1 are like B; D2, D3.0 are reductions to an IP equivalent and thus are simple.

case D3.1 $\psi = \bigwedge_{i=0}^{n} \psi_i$, the ψ_i are atoms or implications, one of the ψ_i, say ψ_{i_0}, is an atom.

If $\psi_{i_0}=\top$ or $\psi_{i_0}=\bot$ this is easy. Suppose $\psi_{i_0}=p_s$. We have:

$$
\begin{aligned}
HA\vdash\forall A\in\Sigma_1^0\text{-sentences }\ (\Box(A\to\varphi)&\leftrightarrow\Box(A\to(p_s\to\varphi\langle p_s\rangle))\\
&\leftrightarrow\Box((A\wedge p_s)\to\varphi\langle p_s\rangle)\\
&\overset{IH}{\leftrightarrow}\Box((A\wedge p_s)\to(\varphi\langle p_s\rangle)^*)\\
&\leftrightarrow\Box(A\to(p_s\to(\varphi\langle p_s\rangle)^*))\\
&\leftrightarrow\Box(A\to\varphi^*))
\end{aligned}
$$

Note that in the middle equivalence we use that in the context of HA $(A\wedge p_s)$ stands for a Σ_1^0-sentence.

<u>case D3.2</u> $\psi=\bigwedge\!\!\!\bigwedge_{i=0}^{n}\psi_i$, the ψ_i are implications, each ψ_i occurs only once in φ.

Let $\eta:=\bigwedge\!\!\!\bigwedge_{j=0}^{m}(\sigma_j\to\tau_j)$ and suppose $IP\vdash\tau_j\to\lambda$ for $j=0,\ldots,m$. An immediate consequence of 2.1.3.8 is: $(\eta\to\lambda)\approx_{HA}\bigvee\!\!\!\bigvee_{j=0}^{m}[\eta]\sigma_i\vee[\eta]\lambda$.

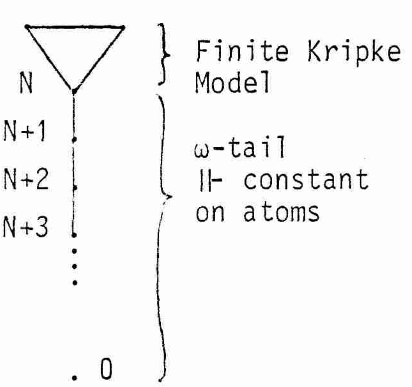
From this and from the construction of φ_0 we have: $\varphi\approx_{HA}\varphi_0$. Hence by IH $\varphi\approx_{HA}(\varphi_0)^*$.

We postpone point (iv) till after Excursion 2.2.0 on the Evaluation Problem for $I\Sigma$ with respect to HA^*.

2.2.0 <u>Excursion: The Evaluation Problem for $I\Sigma$ with respect to HA^*</u>

In [13] we introduced the theory HA^*. $HA^*=HA+CP^*$. Here CP^* is the scheme $(A\to\Box^*A)$. '$\Box^*A$' stands for $\Box_{HA^*}A$ in the context of HA or HA^*. We show in [13] that CP^* is interderivable (over HA) with SLP^*, i.e. the scheme: $((\Box^*A\to A)\to A)$.

In [12], part 6, we draw a consequence of Solovay's method in the proof of the completeness theorem for arithmetical interpretations of modal logic. This consequence will be needed for the problem at hand. Because [12] is - perhaps - not easily accessible we reproduce the result and the argument here.



A *monotonic* tail model is a structure for the language of Modal Propositional Logic (i.e. the language with propositional variables $p_0,p_1,\ldots$ and logical constants $\bot,\top,\Box,\wedge,\vee,\to$). The structure has the form $\langle\omega,\prec,\Vdash\rangle$ where:

- $\prec$ is on irreflexive partial order
- if $m\neq 0$ $0\prec m$

- if $n\neq 0, n<m$ then $n>m$
- for some $N\neq 0$:
  - for every n,m, if $n\geq N$ and $n>m>0$ then $n<m$
  - for every $n\leqslant N$ and every i: $n\Vdash p_i \Leftrightarrow N\Vdash p_i$

  Such an N will be called: tail element.
- if $m\Vdash p_i$ and $n\geqslant m$ then $n\Vdash p_i$
- $m\Vdash \varphi\to\psi :\Leftrightarrow$ for all $n\geqslant m$ $n\Vdash\varphi \Rightarrow n\Vdash\psi$
- $m\Vdash \Box\varphi$: for all $n>m$ $n\Vdash\varphi$.

One may show (Tail Lemma)

$0\Vdash\varphi \Leftrightarrow$ for some M, for all $n>M$ $n\Vdash\varphi$

$0\nVdash\varphi \Leftrightarrow$ for some M, for all $n>M$ $n\nVdash\varphi$

Consider a tail model $\underline{K}=\langle\omega,<,\Vdash\rangle$. Define a primitive recursive h as follows:

- $h(0):=0$
- $h(k+1):=\begin{cases} n \text{ if for some } n>h(k)\ \mathrm{Proof}_{HA*}(k, \ulcorner\exists x\, hx\nless\underline{n}\urcorner) \\ h(k) \text{ otherwise}\end{cases}$

Clearly: $HA\vdash$"h is weakly monotonic in $<$"

Define:

$$[\varphi]:=\begin{cases} \mathbb{W}\{\exists x\, hx=\underline{i}\mid i\Vdash\varphi\} & \text{if there are only finitely many } i \text{ such that } i\Vdash\varphi \\ \underline{0}=\underline{0} & \text{if for all } i \quad i\Vdash\varphi\end{cases}$$

The Tail Lemma inplies that always one of these cases applies. Define $\langle\varphi\rangle$ by:

- $\langle p_i\rangle:=[p_i]$
- $\langle T\rangle:=(\underline{0}=\underline{0}), \langle\bot\rangle:=\bot$
- $\langle\cdot\rangle$ commutes with $\wedge,\vee,\to$
- $\langle\Box\varphi\rangle:=\Box_{HA*}\langle\varphi\rangle$

2.2.0.0 Theorem

$HA*\vdash\langle\varphi\rangle\leftrightarrow[\varphi]$

Proof: induction on φ.

- atoms and $\vee$ are more or less trivial
- $\varphi=(\psi\wedge\chi)$. We treat the case that $\{i\mid i\Vdash\psi\}$ and $\{j\mid j\Vdash\chi\}$ are both finite

  $HA*\vdash\langle\psi\wedge\chi\rangle\leftrightarrow\langle\psi\rangle\wedge\langle\chi\rangle$

  $$\overset{IH}{\leftrightarrow}\mathbb{W}\{\exists x\, hx=\underline{i}\mid i\Vdash\psi\}\wedge\mathbb{W}\{\exists x\, hx=\underline{j}\mid j\Vdash\chi\}$$

$\leftrightarrow \mathsf{W}\{\exists x h x=\underline{i} \wedge \exists x h x=\underline{j} \mid i \Vdash \psi, j \Vdash \chi\}$

$\overset{*}{\leftrightarrow} \mathsf{W}\{\exists x h x=\underline{k} \mid k \Vdash \psi \wedge \chi\}$

The "$\leftarrow$" side of "$\overset{*}{\leftrightarrow}$" is trivial. For the "$\leftarrow$" side, note that if $i \not\preccurlyeq j$ and $j \not\succcurlyeq i$ $HA^* \vdash \neg(\exists x h x=\underline{i} \wedge \exists x h x=\underline{j})$ and that if e.g. $i \preccurlyeq j$, we have $j \Vdash \psi \wedge \chi$ by monotonicity.

- $\varphi=(\psi \rightarrow \chi)$. By IH it is sufficient to show: $HA^* \vdash ([\psi] \rightarrow [\chi]) \leftrightarrow [\psi \rightarrow \chi]$

The "$\leftarrow$" side is more or less trivial.

"$\rightarrow$" In case for all i $i \Vdash (\psi \rightarrow \chi)$ it is easy, so suppose that only for finitely i $i \Vdash (\psi \rightarrow \chi)$. Let $j_0, \ldots, j_s$ be the maximal elements such that $j_k \nVdash (\psi \rightarrow \chi)$. Note that $j_k \Vdash \psi, j_k \nVdash \chi$. *Reason in* HA*:

Suppose $[\psi] \rightarrow [\chi]$ and $\Box^*[\psi \rightarrow \chi]$. We have $\Box^*(\exists x h x \not\preccurlyeq \underline{j}_k)$. Assume $Proof_{HA^*}(p, \ulcorner \exists x h x \not\preccurlyeq \underline{j}_k \urcorner)$ and $h(p)=y$. In case $y \preccurlyeq \underline{j}_k$, $h(p+1)=\underline{j}_k$ and so $[\psi]$, hence $[\chi]$. From $h(p+1)=\underline{j}_k$ and $[\chi]$ we have $\exists x h x > \underline{j}_k$ (for: $j_k \nVdash \chi$). In case $y \not\preccurlyeq \underline{j}_k$ $h(p) \not\preccurlyeq \underline{j}_k$, hence $\exists x h x \not\preccurlyeq \underline{j}_k$. Conclude $\mathsf{M}\{\exists x h x \not\preccurlyeq \underline{j}_k \mid k=0,\ldots,s\}$, so by the monotonicity of h $\exists x \mathsf{M}\{h x \not\preccurlyeq \underline{j}_k \mid k=0,\ldots,s\}$

From this: $\mathsf{W}\{\exists x h x=\underline{i} \mid i \Vdash (\psi \rightarrow \chi)\}$, i.e. $[\psi \rightarrow \chi]$. By the SLP* we may conclude $[\psi \rightarrow \chi]$ without assuming $\Box^*[\psi \rightarrow \chi]$.

- $\varphi=\Box\psi$

In case for all i $i \Vdash \psi$ this is easy, so assume only for finitely many i $i \Vdash \psi$. Let $l_0, \ldots, l_t$ be all the elements such that $l_k \Vdash \Box\psi, l_k \nVdash \psi$. Note that for i with $i \nVdash \Box\psi$ there is an l_k with $i < l_k$.

By the IH it is sufficient to show $HA^* \vdash \Box^*[\psi] \leftrightarrow [\Box\psi]$

Reason in HA*

"$\rightarrow$" Suppose $\Box^*[\psi]$. We have $\Box^*(\exists x h x \not\preccurlyeq \underline{l}_k)$ by the definition of $[\psi]$ and the fact that $\underline{l}_k \nVdash \psi$. Suppose $Proof_{HA^*}(p, \ulcorner \exists x h x \not\preccurlyeq \underline{l}_k \urcorner)$ and $h(p)=y$. In case $y \preccurlyeq \underline{l}_k$ we have $h(p+1)=\underline{l}_k$. In case $y \not\preccurlyeq \underline{l}_k$ we have $\exists x h x \not\preccurlyeq \underline{l}_k$. Conclude $\mathsf{M}\{\exists x h x \not\preccurlyeq \underline{l}_k \mid k=0,\ldots,t\}$. Hence by the monotonicity of h: $\exists x \mathsf{M}\{h x \not\preccurlyeq \underline{l}_k \mid k=0,\ldots,t\}$

So $\mathsf{W}\{\exists x h x=\underline{i} \mid i \Vdash \Box\psi\}$

"$\leftarrow$" Suppose $\exists x h x=\underline{i}$ for an $i \Vdash \Box\psi$. By the definition of h and the fact that $i \neq 0$: $\Box^*(\exists x h x \not\preccurlyeq \underline{i})$ (How else could h move up to i?) Then from $\exists x h x=\underline{i}$: $\Box^*(\exists x h x=\underline{i})$.

Combining: $\Box^*(\exists x h x \geq \underline{i})$, or $\Box^* \mathsf{W}\{\exists x h x=\underline{j} \mid j \geq i\}$. Hence $\Box^*[\psi]$.

□

We are now in the position to solve the Evaluation Problem for $I\Sigma$ with respect to HA*. The solution is $\langle IP, L\rangle$, this means that $(\ )^*$ on $\langle \underline{IP}, L/_{IP}\rangle$ is just the identity.

i) $\langle \underline{IP}, L/_{IP} \rangle$ is a full EHA. This is, of course, trivial.

ii) If $IP \vdash \varphi$ then for all $g \in I\Sigma$ $HA^* \vdash (\varphi)^g$. This is routine.

iii) Clearly: $\varphi^* \in [\varphi]^*_{IP}$ iff $IP \vdash \varphi \leftrightarrow \varphi^*$. So trivially: $\varphi \sim_{HA^*} \varphi^*$.

iv) Suppose $IP \nvdash \varphi \leftrightarrow \psi$, e.g. $IP \nvdash \varphi \rightarrow \psi$. There is a finite Kripke model for IP $\underline{K}_0 = \langle K_0, \preccurlyeq, \Vdash \rangle$ with bottom node k_0 such that $k_0 \Vdash \varphi, k_0 \nVdash \psi$. Without loss of generality we may assume:

- $K_0 = \{1,2,3,\ldots,N\}$, where $m<n \Rightarrow n<m$.
- if $m \neq N$ $m \Vdash \psi$.

Now we hang a tail $N+1, N+2, \ldots, 0$ under $\underline{K}_0$; to obtain a tail model $\underline{K}$. We postulate: $(n=0 \text{ or } n \geq N) \Rightarrow (n \Vdash p_i \Leftrightarrow N \Vdash p_i)$.

We find $N+1 \Vdash \Box\varphi$, $N+1 \nVdash \Box\psi$ and for $n<N+1, n \neq 0$: $n \Vdash \Box\psi$. It follows that there is a d such that for every n $n \Vdash (\Box\varphi \leftrightarrow \Box\psi) \rightarrow \Box^d \bot$.

Hence $HA^* \vdash \langle (\Box\varphi \leftrightarrow \Box\psi) \rightarrow \Box^d \bot \rangle \leftrightarrow [(\Box\varphi \rightarrow \Box\psi) \rightarrow \Box^d \bot]$

$$\leftrightarrow \underline{0=0}$$

Ergo $HA^* \vdash (\Box^* \langle \varphi \rangle \leftrightarrow \Box^* \langle \psi \rangle) \rightarrow \Box^{*d} \bot$. Let $g_0(p_i) := [p_i]$. Clearly $g_0 \in I\Sigma$ and $\langle \tau \rangle = (\tau)^{g_0}$. It follows that if $HA^* \vdash \Box^*(\varphi)^{g_0} \leftrightarrow \Box^*(\psi)^{g_0}$ then $HA^* \vdash \Box^{*d} \bot$, quod non (see [13]).

END OF 2.2.0

We return to the Evaluation Problem for $I\Sigma$ with respect to HA. We still had to check point (iv).

(iv) Let $\chi, \nu \in$ NNIL and suppose $IP \nvdash \chi \leftrightarrow \nu$.

In [13] we introduced the class A by: A is the smallest class such that:

- $P \in A$ if P is an atom
- A is closed under $\wedge, \vee, \forall, \exists$
- $A \in \Sigma^0_1, B \in A \Rightarrow (A \rightarrow B) \in A$

In [13] we show: for $A \in A$ $HA \vdash A \Leftrightarrow HA^* \vdash A$ and $HA \vdash \Box A \leftrightarrow \Box^* A$ (Theorem 5.4 of [13]). Clearly if $g \in I\Sigma$ and $\tau \in$ NNIL then $(\tau)^g \in A$.

By 2.2.0 there is a g_0 in $I\Sigma$ such that $HA^* \nvdash \Box^*(\chi)^{g_0} \leftrightarrow \Box^*(\nu)^{g_0}$, so clearly $HA \nvdash \Box^*(\chi)^{g_0} \leftrightarrow \Box^*(\nu)^{g_0}$. By the above remarks: $HA \nvdash \Box(\chi)^{g_0} \leftrightarrow \Box(\nu)^{g_0}$.

(Inspecting the argument of 2.2.0, giving χ the role of φ, we see that for all n $n \Vdash \chi$ in the tail model. Ergo $HA^* \vdash \langle \chi \rangle$, i.e. $HA^* \vdash (\chi)^{g_0}$, hence $HA \vdash (\chi)^{g_0}$.)

2.2.1 <u>Corollary</u>

Suppose $g \in I\Sigma$, the propositional variables of φ are $p_0, \ldots, p_{k-1}$, $IP \nvdash \varphi$, then

$HA \vdash \Box(\varphi)^g \to \Box(\sigma_k(p_0,\ldots,p_{k-1}))^g$. (The σ_k are introduced in 1.1.2.) For example we have in case the propositional variables of φ are p_0, p_1 and $g(p_0)=A, g(p_1)=B$: $HA \vdash \Box(\varphi)^g \to \Box((A\to(B\vee\neg B))\vee(B\to(A\vee\neg A)))$.

Proof: obvious

□

2.2.2 Remark

Using a uniformization argument as in [7] or in [12], part 0, 2.2.11 we can show: there is a g_0 in $I\Sigma$ such that for all χ,ν in NNIL: $IP \nvdash (\chi\leftrightarrow\nu) \Rightarrow HA \nvdash \Box(\chi)^{g_0} \leftrightarrow \Box(\nu)^{g_0}$.

2.3 De Jongh on formulas of one propositional variable revisited

Let $I^1 := \{g \mid g : \{p_0\} \to \text{sentences of } L_{HA}\}$

We want to recast the theorem of De Jongh on formulas of one propositional variable (see [4]) for sentences in the form of the solution of the evaluation problem for I^1.

We show: $\langle IP^1, \{\bot, p_0, \neg p_0, \neg\neg p_0, p_0\vee\neg p_0, \neg\neg p_0\vee\neg p_0, \neg\neg p_0\to p_0, \neg\neg p_0\vee(\neg\neg p_0\to p_0), T\}\rangle$ solves the evaluation problem for I^1.

(i) As we have seen $J=\langle \underline{IP}^1, [\neg\neg p_0\vee(\neg\neg p_0\to p_0)]_{IP^1}][T\rangle$ is a full EHA.

(ii) It is routine to show that: $IP^1 \vdash \varphi \Rightarrow$ for all $g\in I^1$ $HA \vdash (\varphi)^g$.

(iii) Let φ be an element of the Rieger Nishimura Lattice (RNL) and let $\tilde{\varphi}$ be the representative in the RNL of $[\varphi]^*_{IP^1}$. It is clearly sufficient to show for any g in $I1$: $(\varphi)^g \sim_{HA} (\tilde{\varphi})^g$. The proof is essentially De Jongh's. We only give a sketch. Fix g. We write in the context of HA 'φ' for '$(\varphi)^g$'.

We show $\varphi \approx_{HA} \tilde{\varphi}$. In case $\varphi = T$, this is easy. The further proof is by reducing the 'height' of φ in the RNL. The forms of the relevant elements of the RNL are $(\psi\vee\chi)$ or $(\nu\to\rho)\to(\psi\vee\chi)$.

case 1 $\varphi=(\psi\vee\chi)$

This is treated as case C of 2.2.

case 2 $\varphi=((\nu\to\rho)\to(\psi\vee\chi))$

Note that 2.1.3.7 implies (using the property of the RNL that $IP^1 \vdash \rho \to (\psi \vee \chi)$):
$(\nu \to \rho) \to (\psi \vee \chi) \approx_{HA} ((\nu \to \rho) \to \psi) \vee ((\nu \to \rho) \to \chi) \vee ((\nu \to \rho) \to \nu)$

Reduction of the rhs to an element of the RNL shows that this element is below φ.

(iv) For $g(p_0)=A$, A a Σ_1^0-sentence, we have by 2.2(iii): $HA \vdash \Box(\neg\neg p_0)^g \leftrightarrow \Box(p_0)^g$. So we cannot choose in general a Σ_1^0-sentence for our counterexample. Perhaps a Boolean combination of Σ_1^0-sentences will work.
We carry out the following plan. Start with χ in the RNL equal to T or below or equal to $\neg\neg p_0 \vee (\neg\neg p_0 \to p_0)$. Substitute $(p_0 \vee \neg p_1)$ for p_0. Rework a bit in IP to find χ'. Apply the algorithm N to χ'. Rework the result a bit in IP to find χ''. If the χ'' so found are pairwise inequivalent in IP, our result follows by 2.2(iv).

χ	χ'	χ''
$\bot$	$\bot$	$\bot$
p_0	$p_0 \vee \neg p_1$	$p_0 \vee \neg p_1$
$\neg p_0$	$\neg p_0 \wedge \neg\neg p_1$	$\neg p_0 \wedge p_1$
$\neg\neg p_0$	$p_1 \to \neg\neg p_0$	$p_1 \to p_0$
$p_0 \vee \neg p_0$	$p_0 \vee \neg p_1 \vee (\neg p_0 \wedge \neg\neg p_1)$	$p_0 \vee \neg p_1 \vee (\neg p_0 \wedge p_1)$
$\neg\neg p_0 \vee \neg p_0$	$(p_1 \to \neg\neg p_0) \vee (\neg p_0 \wedge \neg\neg p_1)$	$(p_1 \to p_0) \vee (\neg p_0 \wedge p_1)$
$\neg\neg p_0 \to p_0$	$\neg\neg(p_0 \vee \neg p_1) \to (p_0 \vee \neg p_1)$	$p_0 \vee \neg(p_0 \wedge p_1)$
$\neg\neg p_0 \vee (\neg\neg p_0 \to p_0)$	$(p_1 \to \neg\neg p_0) \vee \neg\neg(p_0 \vee \neg p_1) \to (p_0 \vee \neg p_1)$	$(p_1 \to p_0) \vee \neg(p_0 \wedge p_1)$
T	T	T

We leave it to the reader to check the pairwise in equivalence of the χ''.

□

2.4 A sharp version of De Jongh's Completeness Theorem for arithmetical interpretations of IP

Implicit in the work of 2.2 there is a proof of De Jongh's Completeness Theorem for arithmetical interpretations of IP. Inspection of the proof shows that it can be formalized in $HA + \forall x \neg \Box^x \bot$. In this section we show that De Jongh's Completeness Theorem can be proved in $HA + \neg\Box\bot$.

There is a primitive recursive sequence of Σ_1^0-sentences $\Omega_0, \Omega_1, \Omega_2, \ldots$ such that for any n:

$$HA \vdash \forall B_0 \in \Sigma_1^0\text{-sentences} \ldots \forall B_{n-1} \in \Sigma_1^0\text{-sentences } (\Box(\bigwedge_{i=0}^{n-1} (\Omega_i \leftrightarrow B_i) \to \bot) \to \Box\bot).$$

The Ω_i are examples of Kripke's Flexible Sentences. For a proof of their existence, see e.g. [6] or [11], 3.6. It is easy to check that the argument can be formalized in HA.

Let G be given by $G(p_i) := \Omega_i$. 'φ' in the context of HA will mean in this section: $(\varphi)^G$.

We show that $\langle IP, \{T, \bot\}\rangle$ solves the evaluation problem for $\{G\}$. Clearly this means nothing but:

$IP \nvdash \varphi \Rightarrow HA \vdash \Box\varphi \leftrightarrow \Box\bot$, which implies:

$IP \nvdash \varphi \Rightarrow HA + \neg\Box\bot \vdash \neg\Box\varphi$, or:

De Jongh's Completeness result can be verified in HA+con(HA).

(i) As we have seen $\langle \underline{IP}, \bot][T\rangle$ is a full EHA.

(ii) Trivially: $IP \vdash \varphi \Rightarrow HA \vdash \varphi$.

(iii) We show: $\varphi \sim_{HA} t(\varphi)$. Note that in case $t(\varphi)=T$ this is trivial, so we restrict ourselves to the case that $t(\varphi)=\bot$.

As in 2.2 we would like to show $\varphi \approx_{HA} t(\varphi)$, but that won't work, because e.g. for the case that $\varphi = p_0$ $HA \nvdash \Box(\Omega_0 \to p_0) \leftrightarrow \Box(\Omega_0 \to \bot)$.
(Note that $HA \vdash \Box(\Omega_0 \to p_0)$, $HA \vdash (\Box(\neg\Omega_0) \leftrightarrow \Box\bot)$.)

To get around the difficulty we prove the following. Let the propositional variables of φ be $p_{i_0}, \ldots, p_{i_{k-1}}$ and suppose $IP \nvdash \varphi$, then:

$$HA \vdash \forall A \in \Sigma_1^0\text{-sentences } \Box(A \to \varphi) \to \exists B_0, \ldots, B_{k-1} \in$$
$$\Sigma_1^0\text{-sentences } \Box((A \wedge \bigwedge_{j=0}^{k-1} (\Omega_{i_j} \leftrightarrow B_j)) \to \bot)$$

Clearly from this it follows that:

$$HA \vdash \Box\varphi \to \exists B_0, \ldots, B_{k-1} \in \Sigma_1^0\text{- sentences } \Box(\bigwedge_{j=0}^{k-1} (\Omega_{i_j} \leftrightarrow B_j) \to \bot)$$

Hence by the flexibility of the Ω_{i_j}:

$HA \vdash \Box\varphi \to \Box\bot$, ergo $\varphi \sim_{HA} \bot$.

The proof is by induction on $o(\varphi)$ following T.

<u>case A</u> φ is atomic

In case $\varphi = \perp$ this is trivial. In case $\varphi = p_s$, set $B_0 := \perp$.

<u>case B</u> $\varphi = (\psi \wedge \chi)$

Clearly $IP \nvdash \psi$ or $IP \nvdash \chi$. Suppose e.g. $IP \nvdash \psi$. Let $p_{s_0}, \ldots, p_{s_{l-1}}$ be the propositional variables of ψ; $p_{i_0}, \ldots, p_{i_{k-1}}$ those of φ.

Reason in HA

Suppose A is a Σ_1^0-sentence and $\Box(A \to \varphi)$, it follows that $\Box(A \to \psi)$, hence (by IH) there are Σ_1^0-sentences $C_0, \ldots, C_{l-1}$ such that:

$\Box((A \wedge \bigwedge_{j=0}^{l-1} (\Omega_{s_j} \leftrightarrow C_j)) \to \perp)$. Take:

$$B_j := \begin{cases} C_{j'} & \text{if } s_{j'} = i_j \text{ for some } j' \\ \Omega_{i_j} & \text{otherwise} \end{cases}$$

Then:

$\Box((A \wedge \bigwedge_{j=0}^{k-1} (\Omega_{i_j} \leftrightarrow B_j)) \to \perp)$.

<u>case C</u> $\varphi = (\psi \vee \chi)$

We have $IP \nvdash \psi$, $IP \nvdash \chi$. Let $p_{s_0}, \ldots, p_{s_{l-1}}$ be the propositional variables of ψ; $p_{t_0}, \ldots, p_{t_{m-1}}$ those of χ; $p_{i_0}, \ldots, p_{i_{k-1}}$ those of φ.

Reason in HA:

Suppose A is a Σ_1^0-sentence and $\Box(A \to (\psi \vee \chi))$. There is an e such that $\Box(A \to \{e\}\downarrow)$, $\Box((A \wedge (\{e\})_0 = \underline{0}) \to \psi)$, $\Box((A \wedge (\{e\})_0 \neq \underline{0}) \to \chi)$. Clearly by our conventions $\ulcorner((\{e\})_0 = \underline{0})\urcorner$ and $\ulcorner((\{e\})_0 \neq \underline{0})\urcorner$ are Σ_1^0-sentences, hence there are Σ_1^0-sentences $C_0, \ldots, C_{l-1}$ and $D_0, \ldots, D_{m-1}$ such that:

$\Box((A \wedge (\{e\})_0 = \underline{0} \wedge \bigwedge_{j=0}^{l-1} (\Omega_{s_j} \leftrightarrow C_j)) \to \perp)$

and

$\Box((A \wedge (\{e\})_0 \neq \underline{0} \wedge \bigwedge_{j=0}^{m-1} (\Omega_{t_j} \leftrightarrow D_j)) \to \perp)$

Take for $j = 0, \ldots, k-1$:

$$E_j := \begin{cases} C_{j'} & \text{if } s_{j'}=i_j \text{ for some } j' \\ \Omega_{i_j} & \text{otherwise} \end{cases}$$

$$F_j := \begin{cases} D_{j'} & \text{if } t_{j'}=i_j \text{ for some } j' \\ \Omega_{i_j} & \text{otherwise} \end{cases}$$

$B_j := (((\{e\})_0=\underline{0} \wedge E_j) \vee ((\{e\}_0 \neq \underline{0} \wedge F_j))$

Claim: $\Box((A \wedge \bigwedge\limits_{j=0}^{k-1} (\Omega_{i_j} \leftrightarrow B_j)) \to \bot)$.

Reason in $\Box$:

Assume $A \wedge \bigwedge\limits_{j=0}^{k-1} (\Omega_{i_j} \to B_j)$. From A we have $\{e\}\downarrow$, hence $(\{e\})_0=\underline{0}$ or $(\{e\})_0 \neq \underline{0}$. Suppose $(\{e\})_0=\underline{0}$. It follows that $B_j \leftrightarrow E_j$, hence $A \wedge \bigwedge\limits_{j=0}^{k-1} (\Omega_{i_j} \leftrightarrow E_j)$, thus $A \wedge \bigwedge\limits_{j=0}^{l-1} (\Omega_{s_j} \leftrightarrow C_j)$. Ergo $\bot$. Similarly we can derive $\bot$ from $(\{e\})_0 \neq \underline{0}$. By $\vee$E: $\bot$.

<u>case D</u> $\varphi = (\psi \to \chi)$

cases D0, D1, D2, D3.0 all employ IP equivalences; say $IP \vdash \varphi \leftrightarrow \varphi'$, where $o(\varphi') < o(\varphi)$. Suppose the propositional variables of φ' are $p_{s_0},\ldots,p_{s_{l-1}}$, those of φ $p_{i_0},\ldots,p_{i_{k-1}}$. Here the s_j are among the $i_{j'}$.

Reason in HA:

Suppose A is a Σ^0_1-sentence and $\Box(A\to\varphi)$, then $\Box(A\to\varphi')$. Hence there are Σ^0_1-sentences $C_0,\ldots,C_{l-1}$ such that:

$\Box((A \wedge \bigwedge\limits_{j=0}^{l-1} (\Omega_{s_j} \leftrightarrow C_j)) \to \bot)$.

Take:

$$B_j := \begin{cases} C_{j'} & \text{if } s_{j'}=i_j \text{ for some } j' \\ \Omega_{i_j} & \text{otherwise} \end{cases}$$

Then: $\Box((A \wedge \bigwedge\limits_{j=0}^{k-1} (\Omega_{i_j} \leftrightarrow B_j)) \to \bot)$

<u>case D3.1</u> $\psi = \bigwedge\limits_{r=0}^{m} \psi_r$, the ψ_r are atoms or implications. One of the ψ_r, say ψ_{r_0}, is an atom.

By the assumption that $IP \nvdash \varphi{:}\psi_{r_0} \neq \bot$. In case ψ_{r_0} =T the case reduces to $\varphi\langle T\rangle$ and we can reason as in D0. Suppose $\psi_{r_0} = p_q$, and that all the propositional variables of $\varphi\langle p_q\rangle$ are $p_{s_0},\ldots,p_{s_{l-1}}$, those of φ: $p_{i_0},\ldots,p_{i_{k-1}}$.

Reason in HA:

Suppose A is a Σ_1^0-sentence and $\Box(A\to\varphi)$, then $\Box((A\wedge\Omega_q)\to\varphi\langle p_q\rangle)$.
By IH there are Σ_1^0-sentences $C_0,\ldots,C_{l-1}$ such that:

$$\Box(((A\wedge\Omega_q)\wedge\bigwedge_{j=0}^{l-1}(\Omega_{s_j}\leftrightarrow C_j))\to\bot).$$

Note that q is not among the s_j. Take:

$$B_j := \begin{cases} (\underline{0}=\underline{0}) & \text{if } q=i_j \\ C_{j'} & \text{if } s_{j'}=i_j \text{ for some } j' \\ \Omega_{i_j} & \text{otherwise} \end{cases}$$

<u>case D3.2</u> $\psi = \bigwedge\limits_{i=0}^{n} \psi_i$, the ψ_i are all implications, each ψ_i occurs only once in φ.
We have: $HA \vdash \forall A \in \Sigma_1^0$-sentences $\Box(A\to\varphi)\leftrightarrow\Box(A\to\varphi_0)$. Proceed as in D0.

(iv) Clearly $T \nvdash_{HA} \bot$.

2.5 The closed fragment of the provability logic of HA

Let C be the smallest subset of the language of modal propositional logic such that:

- $\bot\in C, T\in C$
- C is closed under $\wedge,\vee,\to,\Box$

Define the interpretation $(\ )^a$ of C in L_{HA} as follows:

- $(\bot)^a := \bot, (T)^a := (\underline{0}=\underline{0})$
- $(\ )^a$ commutes with $\wedge,\vee,\to$

- $(\Box\varphi)^a := \Box_{HA}(\varphi)^a$

We adopt the convention to drop $(\ )^a$ in the context of HA. Thus we write '$HA \vdash \varphi$' for $HA \vdash (\varphi)^a$.

The problem of the closed fragment for HA is roughly to characterize those φ in C such that $HA \vdash \varphi$. This involves at least to give an algorithm to decide whether $HA \vdash \varphi$ or not. (The problem of the closed fragment for Peano Arithmetic is Friedman's 35th problem. This was independently solved by van Benthem, Boolos, Bernardi & Montagna about 1975. Note that Friedman's formulation in terms of consistency rather then provability makes no difference in the classical case, but is 'weaker' in the constructivistic case).

The crucial lemma for solving the problem of the closed fragment of the provability logic of HA is the solution to an Evaluation Problem.

Define: $\Box^0\bot := \bot$
$\Box^{n+1}\bot := \Box\Box^n\bot$
$\Box^\omega\bot := \top$

Let H: $\omega+1 \to$ the sentences of L_{HA}, be given by: $H(\alpha) := \Box^\alpha_{HA}\bot := (\Box^\alpha\bot)^a$.

We claim: the solution of the evaluation problem for $\{H\}$ is $\langle UP, \omega+1\rangle$.

In the context of HA we will drop the $(\ )^H$.

(i) As we have seen $\langle \underline{UP}, \omega+1/_{UP}\rangle$ is a full EHA.
(ii) Clearly $UP \vdash \varphi \Rightarrow HA \vdash \varphi$
(iii) We want to show $\varphi \sim_{HA} n(\varphi)$, to do this we prove $\varphi \approx_{HA} n(\varphi)$.
The proof is by induction on $o(\varphi)$ following algorithm Ω. The proof is merely a variation on the proof in 2.2, so we only indicate the differences.
Cases A, B, C, D0, D1, D2. D3.0 D3.2 are all like the corresponding cases in 2.2.

<u>case D3.1</u> $\psi = \bigwedge\limits_{i=0}^{n} \psi_i$, where the ψ_i are either implications or atoms; one of the ψ_i, say ψ_{i_0}, is an atom, say α.

Remember: $UP \vdash \varphi \leftrightarrow (\alpha \to \varphi\langle\alpha\rangle)$, and

$$n(\varphi)=\begin{cases}\omega \text{ if } & \alpha\leqslant n(\varphi<\alpha>)\\ n(\varphi<\alpha>) & \text{otherwise}\end{cases}$$

If $n(\varphi)=\omega$ we are done, so assume $n(\varphi<\alpha>)<\alpha$. Say $n(\varphi<\alpha>)=m$. By IH: $\varphi<\alpha>\approx_{HA}m$.

The following facts about $\approx_{HA}$ are easily verified:

2.5.0 Lemma

Let B be a Σ_1^0-sentence and let C,D be arbitrary arithmetical sentences:

a) $(\Box C\to C)\approx_{HA}C$

b) $(C\wedge\Box C)\approx_{HA}C$

c) $C\approx_{HA}D\Rightarrow(B\to C)\approx_{HA}(B\to D)$

It follows using (a) and (c):

$(\alpha\to\varphi<\alpha>)\ \approx_{HA}(\alpha\to m)$

$\qquad\qquad\approx_{HA}m$

(iv) Trivially $\alpha\neq\beta\Rightarrow\alpha\not\approx_{HA}\beta$.

Now we are in the position to solve the problem of the closed fragment for HA. Define $(\ )^b$: $C\to L_{UP}$ as follows:

- $(\bot)^b:=0,(T)^b:=\omega$
- $(\ )^b$ commutes with $\wedge,\vee,\to$
- $(\Box\varphi)^b:=1+n((\varphi)^b)$

We have for φ in C:

a) $HA\vdash\varphi\leftrightarrow(\varphi)^b$

b) $HA\vdash\varphi\Leftrightarrow n((\varphi)^b)=\omega$

Proof:

a) By induction on φ. E.g. if $\varphi=\Box\psi$: By IH $HA\vdash\psi\leftrightarrow(\psi)^b$, hence

$HA\vdash\Box\psi\leftrightarrow\Box(\psi)^b$

$\qquad\leftrightarrow\Box n((\psi)^b)$

Moreover in the context of HA: $\Box n((\psi)^b)=1+n((\psi)^b)$

b) By the 'unformalization' of $HA\vdash\Box\varphi\leftrightarrow 1+n((\varphi)^b)$ we have:

$HA\vdash\varphi\Leftrightarrow HA\vdash n((\varphi)^b)$

$\qquad\Leftrightarrow n((\varphi)^b)=\omega$

□

2.4.1 Excursion: the closed fragment of the provability logic of PA

The solution of Friedman's 35th problem can be cast in precisely the same form as our solution of the problem of the closed fragment for HA.

Let UP^c:=UP+Classical Logic. $<\underline{UP}^c, \omega+1/_{UP^c}>$ is a conjunctive EHA.

Let $n^c(\varphi)$ stand for the unique element of $[\varphi]^* \wedge \omega+1$. We have (because UP^c extends UP): $n(\varphi) \leqslant n^c(\varphi)$, but e.g. $n(\mathsf{I} \vee \neg \mathsf{I}) = \mathsf{I}$ and $n^c(\mathsf{I} \vee \neg \mathsf{I}) = \omega$.

Define the following functions:
$K:\omega+1 \to$ the sentences of L_{HA}; by $K(\alpha) := \Box^{\alpha}_{PA}\bot$.

$(\ )^c : C \to$ the sentences of L_{HA}, by:
- $(\bot)^c := \bot, (\top)^c := (\underline{0}=\underline{0})$
- $(\ )^c$ commutes with $\wedge, \vee, \to$
- $(\Box\varphi)^c := \Box_{PA}(\varphi)^c$

$(\ )^d : C \to L_{UP}$, by:
- $(\bot)^d := 0, (\top)^d := \omega$
- $(\ )^d$ commutes with $\wedge, \vee, \to$
- $(\Box\varphi)^d := 1 + n^c((\varphi)^d)$

Now write: '$\Box^c$' for $\Box_{PA}$, 'φ' for $(\varphi)^K$ if φ is in L_{UP}, 'φ' for $(\varphi)^c$ if φ is in C, *in the context of* PA.

We find for φ in C:

a) $PA \vdash \varphi \leftrightarrow (\varphi)^d$
b) $PA \vdash \varphi \Leftrightarrow n^c((\varphi)^d) = \omega$

2.4.2 Excursion: the provability logic of the closed fragment of HA*

Let $UP^* := UP + \{((m+1 \to m) \to m) \mid m \in \omega\}$
Define $m: L_{UP} \to \omega+1$, by:
- $m(\alpha) := \alpha$
- $m(\varphi \wedge \psi) := \min(m(\varphi), m(\psi))$
- $m(\varphi \vee \psi) := \max(m(\varphi), m(\psi))$
- $m((\varphi \to \psi)) := \begin{cases} \omega & \text{if } m(\varphi) \leqslant m(\psi) \\ m(\psi) & \text{otherwise} \end{cases}$

We find: $UP^* \vdash \varphi \leftrightarrow m(\varphi)$

Because UP^* extends UP, we have: $n(\varphi) \leqslant m(\varphi)$, but e.g. $n(\neg\neg 1)=1$, $m(\neg\neg 1)=\omega$.

Clearly $\underline{UP^*}$ is isomorphic to the complete Heyting Algebra $\langle \omega+1, \wedge, \vee, \rightarrow, T, \perp \rangle$ of 1.3.4.

Define the following functions:

$L:\omega+1 \rightarrow$ the sentences of L_{HA}, by $L(\alpha):=\Box^{\alpha}_{HA^*}\perp$

$(\ )^{e}:C \rightarrow$ the sentences of L_{HA}, by:

- $(\perp)^{e}:=\perp, (T)^{e}:=(\underline{0}=\underline{0})$
- $(\ )^{e}$ commutes with $\wedge, \vee, \rightarrow$.
- $(\Box\varphi)^{e} := \Box_{HA^*}(\varphi)^{e}$

$(\ )^{\delta}:C \rightarrow \omega+1$, by:

- $(T)^{\delta}:=\omega, (\perp)^{\delta}:=0$
- $(\varphi \circ \psi)^{\delta}:=m((\varphi)^{\delta} \circ (\psi)^{\delta})$ for $\circ=\wedge, \vee, \rightarrow$
- $(\Box\varphi)^{\delta}:=1+(\varphi)^{\delta}$

We write '$\Box^*$' for $\Box_{HA^*}$ *in the context of* HA^*, and 'φ' for $(\varphi)^{L}$ if φ is in L_{UP} and 'φ' for $(\varphi)^{e}$ if φ is in C.

We find for φ in C:

a) $HA^* \vdash \varphi \leftrightarrow (\varphi)^{\delta}$

b) $HA^* \vdash \varphi \Leftrightarrow (\varphi)^{\delta}=\omega$

2.4.2.0 An alternative proof of (iii)

Let φ^* be the result of applying N to φ in L_{UP}.

Suppose the non-ω atoms of φ are among $0,1,\dots,m$.

We have:

$$UP \vdash \alpha \rightarrow \varphi \Leftrightarrow IP \vdash \left(\bigwedge_{i=0}^{m-1} (i \rightarrow i+1) \wedge \alpha\right) \rightarrow \varphi$$

$$\Leftrightarrow IP \vdash \left(\bigwedge_{i=0}^{m-1} (i \rightarrow i+1) \wedge \alpha\right) \rightarrow \varphi^*$$

$$\Leftrightarrow UP \vdash \alpha \rightarrow \varphi^*$$

It follows that $n(\varphi)=n(\varphi^*)$. Moreover for χ is NNIL one easily shows: $n(\chi)=m(\chi')$. Hence: $n(\varphi)=m(\varphi^*)$.

Note further that $HA \vdash (\Box^{\alpha}\bot \leftrightarrow \Box^{*\alpha}\bot)$ and hence for $\varphi \in L^{UP}$: $HA \vdash (\varphi)^H \leftrightarrow (\varphi)^L$. Ergo (under the convention that 'φ' of L_{UP} means $(\varphi)^H$ in the context of HA):

$$
\begin{aligned}
HA \vdash \Box\varphi &\leftrightarrow \Box\varphi^* \\
&\leftrightarrow \Box^*\varphi^* \\
&\leftrightarrow \Box^* m(\varphi^*) \\
&\leftrightarrow \Box n(\varphi)
\end{aligned}
$$

□

2.4.3 Excursion: Intuitionistic Löb's Logic

Let IL be like Löb's logic, only with intuitionistic instead of classical logic. The provability logic of HA turns out to be quite different from IL.

In [5] K.A. Kirov proves a Kripke Model Completeness Theorem for IL: a Kripke Model for IL is a structure $\underline{K} = \langle K, \leqslant, R, \Vdash \rangle$, where:

- K is a non empty set
- $\leqslant$ is a weak partial order
- $R, \leqslant$ satisfy the following interpolation property: $k_1 \leqslant k_2 R k_3 \leqslant k_4 \Rightarrow k_1 R k_4$
- $\Vdash$ is a forcing relation on K, satisfying:
  - $k_1 \leqslant k_2$ and $k_1 \Vdash p_i \Rightarrow k_2 \Vdash p_i$
  - $k_1 \Vdash \Box\varphi :\Leftrightarrow$ for all k_2 $k_1 R k_2 \Rightarrow k_2 \Vdash \varphi$
  - $k_1 \Vdash \varphi \to \psi :\Leftrightarrow$ for all $k_2 \geqslant k_1$ $k_2 \Vdash \varphi \Rightarrow k_2 \Vdash \psi$
  - the clauses for $\bot, \top, \wedge, \vee$ are as usual

We have:

$IL \vdash \varphi \Leftrightarrow$ for all Kripke Models $\underline{K}$ with R transitive and upwards wellfounded, for all k in K $k \Vdash \varphi$

Using the Completeness Theorem, one shows: for no α $IL \vdash \Box\neg\neg\Box\bot \leftrightarrow \Box^{\alpha}\bot$. One may also prove:

$IL \vdash (\Box\varphi \to \Box^{k+1}\bot) \Rightarrow$ for some $l \leqslant k$ $IL \vdash \varphi \to (\Box^{k+1}\bot \to \Box^{l}\bot)$

All this shows that the closed fragment of IL is far removed from that of the provability logic of HA.

[1] G. Boolos, On deciding the truth of certain statements involving the notion of consistency, JSL 41, (1976), 779-781.

[2] G. Boolos, The unprovability of consistency, CUP, London, 1979.

[3] R.J. Grayson, Heyting-valued semantics in: G. Lolli, G. Longo and A. Marcja, eds., Logic Colloquium '82, North Holland, Amsterdam, 1984.

[4] D.H.J. de Jongh, Formulas of one propositional variable in intuitionistic arithmetic, in: A.S. Troelstra and D. van Dalen, eds., The L.E.J. Brouwer Centenary Symposium, North Holland, Amsterdam, 1981.

[5] K.A. Kirov, An intuitionistic analogue of the Modal Logic of Provability in Peano's Arithmetic, in: Mathematical Logic, Proceedings of the Conference on Mathematical Logic, Dedicated to the Memory of A.A. Markov (1903-1979), Sofia, september 22-23, 1980; Sofia, Bulgarian Academy of Sciences, 1984.

[6] S. Kripke, "Flexible" predicates in formal number theory, Proc. AMS 13, (1963), 647-650.

[7] F. Montagna, Relatively Precomplete Numerations and Arithmetic, Journal of Philosophical Logic 11, (1982), 419-430.

[8] H. Rasiowa, An Algebraic Approach to Non Classical Logics, North Holland, Amsterdam, 1974.

[9] C.A. Smorýnski, Applications of Kripke Models, in [10].

[10] A.S. Troelstra (ed.), Metamathematical Investigation of Intuitionistic Arithmetic and Analysis, Springer Lecture notes 344, Springer Verlag, Berlin, 1973.

[11] A. Visser, Numarations, λ-calculus & Arithmetic, in: J.P. Seldin and J.R. Hindly, eds., To H.B. Curry: Essays on Combinatory Logic. Lambda Calculus and Formalism, Academic Press, London, 1980.

[12] A. Visser, Aspects of Diagonalization & Provability, Ph.D. Thesis, Univ. of Utrecht, Utrecht, The Netherlands, 1981.

[13] A. Visser, On the Completeness Principle, Annals of Mathematical Logic 22, (1982), 263-295.

NOTES

1) As we will see this is misleading in example (iv).
2) The generalization of (ii) is due to De Jongh.
3) In the meantime this problem has been solved by Gerard Renardel.