From weak to strong L_1-convergence by an oscillation restriction criterion of BMO type

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June 11, 1997

Abstract. Recently, Girardi gave a characterization of relative strong $L^1_{\mathbb{R}}$-compactness in terms of relative weak $L^1_{\mathbb{R}}$-compactness and the Bocce criterion [18]. Here this result is generalized and extended by presenting a less stringent oscillation restriction condition (ORC) which enforces the transcendence of weak into an appropiately reformulated form of strong convergence in $\mathcal{L}^1_E$, for E a separable reflexive Banach space. The proof has a very simple, transparent structure, because it relies on basic, well-known facts from Young measure theory; this brings the result in line with the current literature.

1 Introduction

Let (Ω, Σ, μ) be a probability space. Let $(E, \|\cdot\|)$ be a separable reflexive Banach space, and let $\mathcal{L}^1_E$ stand for the *prequotient* $\mathcal{L}_1$-space of Bochner-integrable functions from Ω into E. In section 2 the space E will be finite-dimensional, but to understand what goes on in section 3 the reader will have to be familiar with some basic aspects of Bochner integrable functions [15]. [1]

It is well-known that weak convergence in $\mathcal{L}^1_E$ can result in strong $\mathcal{L}_1$-seminorm convergence under sufficient restriction of oscillations towards the limit function, at least when E is finite dimensional. For instance, Theorem II.26 in [14] forms an elementary scalar result of this kind. In recent years the possibility of such transcendence has gained considerable attention in the area of nonlinear partial differential equations, where the algebraic and geometric features of dynamical systems have been studied for their ability to extinguish oscillations, most notably and systematically by DiPerna [16, 17]. The well-known compensated compactness approach of Murat and Tartar also belongs to this domain [13, 19]. Another possibility to extinguish

[1] The prequotient setting is preferred for the use of Young measure theory; of course all results transfer directly to the usual L_1-quotient setting.

oscillations is formed by pointwise extremal conditions, following Tartar [22] and Visintin [25]; the latter author also gave an application to a Stefan-type problem. See [4, 8, 23, 24] for more on this. Yet another, more global approach of extremal type is due to Olech [21]; see [1, 9] for recent developments in this direction.

Here a different, more abstract device is employed to restrict oscillatory behavior: In section 2, for a sequence (or net) of integrable functions taking values in a finite dimensional Banach space E, the BMO-type oscillation restriction criterion (ORC) is formulated. This is an improvement over the *Bocce* criterion, introduced by Girardi in [18]. Later, in section 3, (ORC) is extended to the case where E is an infinite-dimensional (separable, reflexive) Banach space.

The first main result, Theorem 2.1, states that in the finite-dimensional case (ORC) and weak convergence of the sequence are equivalent to strong L_1-convergence in $\mathcal{L}^1_E$. This improves Girardi's main result in [18, Theorem 2.1], where the same equivalence is obtained, with the Bocce criterion replacing (ORC). Her proof is rather indirect and it involves another property (*small Bocce oscillation*).

For the infinite-dimensional case, considered in section 3, the appropriate form of strong convergence in $\mathcal{L}^1_E$ turns out to be *limited convergence*, as introduced in [4]. (Limited convergence in $\mathcal{L}^1_E$ is stronger than weak convergence, and it coincides with strong convergence in case E is finite-dimensional.) In terms of this concept I give in Theorem 3.1, the second main result, an extension of Theorem 2.1 to infinite dimensions.

As in [4, 8, 13, 16], the proof depends on the use of some basic facts from Young measure theory [2, 3, 6, 8, 10] (see [23] for a survey of the theory). By capturing essential features of oscillations, Young measure theory makes the ideas behind the proof very simple and lucid. The basic pattern of thought is the same as in the literature mentioned above.

An interesting open question, which arises naturally from the similarity of the respective proofs, is to seek for an (ORC)-like oscillation restriction criterion that can also deal with the pointwise extremal situation cited above.

2 Main Result: finite dimensions

In this entire section I shall assume that E is a *finite-dimensional* vector space. The Euclidean norm on E will still be denoted by $\|\cdot\|$, and the usual inner product on E by $<\cdot,\cdot>$.

Let Σ^+ denote the collection of all nonnull sets in Σ. A sequence (f_n) in the space L^1_E of E-valued integrable vector-functions is said to satisfy the *oscillation restriction criterion* (ORC) if the following is true: for every $\epsilon > 0, B \in \Sigma^+$ and for every subsequence (f_{n_j}) of (f_n) there exists $C \in \Sigma^+, C \subset B$ with

$$\liminf_{j\to\infty} \int_C \|f_{n_j}(\omega) - m_C(f_{n_j})\|\mu(d\omega) \leq \epsilon\mu(C).$$

Here

$$m_C(f) := \frac{1}{\mu(C)} \int_C f \, d\mu$$

denotes the conditional expectation of a function $f \in \mathcal{L}^1_E$, relative to C.

This criterion is a somewhat less stringent version of a similar one introduced by Girardi [18]. In terms of the sequential setup adopted here [2] Girardi's *Bocce criterion* requires the subset C, corresponding to the subsequence (f_{n_j}), to come from a *finite* collection of subsets, allowed to depend on B and ϵ. While the existence of such a finite collection follows easily *ex post facto* from strong L_1-convergence, its inclusion *a priori* in the criterion turns out to be redundant.

To appreciate the operational value of this improvement, the reader should compare the usefulness of Theorem 2.1 and Girardi's characterization in recovering the classical result about the equivalence between strong convergence on the one hand and weak convergence and convergence in measure on the other [14, II.21].

The main result of this section – dealing only with finite-dimensional E – is as follows:

Theorem 2.1 *For any sequence $(f_n)_0^\infty$ in $\mathcal{L}^1_E$ the following are equivalent:*

(i) (f_n) converges weakly to f_0 and satisfies (ORC),

(ii) (f_n) converges strongly to f_0.

The implication (ii) $\Rightarrow$ (i) is quite simple to prove. The implication (i) $\Rightarrow$ (ii) will be proven by the use of Young measure theory. The proof simply rests on transferring the restriction of oscillations, as embodied in (ORC), to any 'generalized limit' of (f_n), in the shape of a Young measure. The validity of (ORC) forces any such Young measure to be 'equivalent' to the weak limit function f_0 (formulated more precisely, it is the *relaxation* of f_0). By the nature of the 'generalized convergence' of (f_n) to the Young measure in question, now seen to be 'equivalent' to f_0, it follows that (f_n) itself converges strongly in $\mathcal{L}^1_E$ to f_0.

Let S be some completely regular Suslin space. This may seem to conflict with the finite-dimensional character of this section. Let me therefore point out that in this paper one only has to work with $S := E$, so in this section S can be supposed finite-dimensional for all practical purposes. Recall that a *Young measure* from Ω into S is a transition probability with respect to (Ω, Σ) and $(S, \mathcal{B}(S))$ [20]. Thus, a Young measure δ is a function from Ω into the set $\mathcal{P}(S)$ of all probability measures on S (the latter equipped with the Borel σ-algebra $\mathcal{B}(S)$) with the following measurability property: for every $F \in \mathcal{B}(S)$ the function $\omega \mapsto \delta(\omega)(F)$ is Σ-measurable. Note that any 'ordinary' measurable function $f : \Omega \to E$ has as its *relaxation* the Young measure ϵ_f from Ω into E given by

$$\epsilon_f(\omega) := \text{Dirac measure at } f(\omega).$$

Recall also that a *[sequentially] normal integrand* on $\Omega \times S$ is an extended real-valued function $g' : \Omega \times S \to (-\infty, +\infty]$ which is $\Sigma \times \mathcal{B}(S)$-measurable and such that for every $\omega \in \Omega$ the function $g'(\omega, \cdot)$ is [sequentially] lower semicontinuous on S.

[2]Everything said here automatically transposes to the nonsequential setup by replacing '(sub)sequence' by '(sub)net'.

The only tool from Young measure theory needed in this paper is the following Prohorov-type theorem. For S metrizable Lusin this was given in [2, Theorem I] and for S completely regular Suslin in [6]; see [7, Theorem 5.1] and [8, Theorem A.5] for even stronger versions, proven by a completely different method. Note that below the theorem has actually been stated for a sequence of *relaxations* (ϵ_{v_k}); this suffices for the present purposes and saves notation.

Theorem 2.2 (Prohorov's theorem for Young measures) *Suppose that* (v_k) *is a sequence of measurable functions from* Ω *into* S *satisfying the following* tightness *condition:*

$$\sup_k \int_\Omega \int_S h(\omega, v_k(\omega))\mu(d\omega) < +\infty \tag{1}$$

for some nonnegative normal integrand h *on* $\Omega \times S$ *for which* $h(\omega, \cdot)$ *is inf-compact on* S *for every* $\omega \in \Omega$. *Then there exist a subsequence* (v_{k_j}) *of* (v_k) *and a Young measure* δ_* *from* Ω *into* S *such that*

$$\liminf_{j\to\infty} \int_\Omega g'(\omega, v_{k_j}(\omega))\mu(d\omega) \geq \int_\Omega [\int_S g'(\omega, s)\delta^*(\omega)(ds)]\mu(d\omega) \tag{2}$$

for all normal integrands g' *on* $\Omega \times S$ *for which*

$$(\min(0, g'(\cdot, v_{k_j}(\cdot)))) \textit{ is uniformly integrable.} \tag{3}$$

Specializing this result to $S := E$ and $v_k := f_k, k \in \mathbb{N}$, gives:

Corollary 2.3 *Let* (f_k) *be any subsequence of* (f_n), *satisfying*

$$(f_k) \textit{ converges weakly to } f_0.$$

Then there exist a subsequence (f_{k_j}) *of* (f_k) *and a Young measure* δ_* *from* Ω *into* E *such that the following properties hold:*

$$\text{bar } \delta_*(\omega) = f_0(\omega) \textit{ a.e.}, \tag{4}$$

and for every $C \in \Sigma^+$

$$\liminf_{j\to\infty} \int_C \|f_{k_j}(\omega) - m_C(f_{k_j})\|\mu(d\omega) \geq \int_C [\int_E \|x - m_C(f_0)\|\delta_*(\omega)(dx)]\mu(d\omega) \tag{5}$$

and

$$\limsup_{j\to\infty} \int_\Omega \|f_{k_j}(\omega) - f_0(\omega)\|\mu(d\omega) \leq \int_\Omega [\int_E \|x - f_0(\omega)\|\delta_*(\omega)(dx)]\mu(d\omega). \tag{6}$$

Proof. Define $h(\omega, x) := \|x\|$. This is a normal integrand on $\Omega \times E$ which obviously has the required inf-compactness. By weak convergence it follows that

$$\sigma := \sup_k \int_\Omega \|f_k(\omega)\|\mu(d\omega) < +\infty, \tag{7}$$

so condition (1) has been shown to hold. Therefore, by Theorem 2.2 there exist a subsequence (f_{k_j}) of (f_k) and an associated Young measure δ_* from Ω into E such that (2) holds. When (2) is applied to the normal integrand $g' := h$, defined above, this gives

$$\int_\Omega [\int_E \|x\| \delta_*(\omega)(dx)] \mu(d\omega) \leq \sigma < +\infty,$$

so for a.e. ω the barycenter bar $\delta_*(\omega)$ of the probability measure $\delta_*(\omega)$ exists. Let (x_i^*) be a countable dense subset of E. Taking in (2) $g'(\omega, x) := 1_B(\omega) < x, x_i^* >$, $B \in \Sigma$, $i \in \mathbb{N}$, gives, by weak convergence to f_0 of (f_{k_j}),

$$\int_B < f_0(\omega), x_i^* > \mu(d\omega) \geq \int_B < \text{bar } \delta_*(\omega), x_i^* > \mu(d\omega).$$

Repeating this for $-g'$ gives for all i

$$< \text{bar } \delta_*(\omega), x_i^* > = < f_0(\omega), x_i^* > \text{ a.e.},$$

so evidently (4) follows.

Let $C \in \Sigma^+$ be arbitrary. Note first that weak convergence causes (m_j) to converge to m_0 in E; here I abbreviate by $m_j := m_C(f_{k_j})$. For $g'(\omega, x) := 1_C(\omega)\|x - m_0\|$ application of (2) gives

$$\liminf_j \int_C \|f_{k_j}(\omega) - m_0\| \mu(d\omega) \geq \int_C [\int_E \|x - m_0\| \delta_*(\omega)(dx)] \mu(d\omega).$$

To obtain (5), it remains to employ the triangle inequality

$$\|f_{k_j}(\omega) - m_j\| \geq \|f_{k_j}(\omega) - m_0\| - \|m_j - m_0\|$$

in an elementary limit argument. Of course, (6) follows directly by substituting $g'(\omega, x) := -\|x - f_0(\omega)\|$ in (2) [note that (3) holds by the weak convergence hypothesis]. QED

For the proof of Theorem 2.1 the following elementary fact will also be needed:

Lemma 2.4 *Let $\phi : \Omega \to [0, +\infty]$ be a measurable function with the following property: for every $\epsilon > 0, B \in \Sigma^+$ there exists $C \in \Sigma^+, C \subset B$, such that*

$$\int_C \phi(\omega) \mu(d\omega) \leq \epsilon \mu(C).$$

Then $\phi(\omega) = 0$ a.e.

Proof. For arbitrary $\epsilon > 0$ let B be the set of all $\omega \in \Omega$ with $\phi(\omega) \geq 2\epsilon$. If one had $B \in \Sigma^+$ the corresponding $C \in \Sigma^+, C \subset B$, would give $2\epsilon\mu(C) \leq \epsilon\mu(C)$, which is absurd. So B must be a null set. QED

Proof of Theorem 2.1.
(ii) $\Rightarrow$ (i): Of course, this implication is elementary, for under the hypothesis $\int_C \|f_n -$

$m_C(f_n)\|d\mu \to \int_C \|f_0 - m_C(f_0)\|d\mu$, uniformly in $C \in \Sigma^+$. Hence (ORC) follows directly from $\mathcal{L}_1$-approximating f_0 by means of simple functions.

(i) $\Rightarrow$ (ii): It will be enough to show that an arbitrary subsequence (f_k) of (f_n) has a further subsequence (f_{k_j}) such that

$$(f_{k_j}) \text{ converges strongly to } f_0. \tag{8}$$

By Corollary 2.3 there correspond to (f_k) a subsequence (f_{k_j}) and a Young measure δ_* such that (4) and (5) hold. Concatenation of (ORC) and (5) gives that for every $\epsilon > 0, B \in \Sigma^+$ there exists $C \in \Sigma^+$, $C \subset B$, such that $\epsilon\mu(C) \geq \int_C \phi'(\omega)\mu(d\omega)$, where

$$\phi'(\omega) := \int_E \|x - m_C(f_0)\|\delta_*(\omega)(dx).$$

By (4) one also has

$$\phi'(\omega) \geq \|f_0(\omega) - m_C(f_0)\|,$$

so by adding up and using the triangle inequality one obtains

$$\int_C \phi(\omega)\mu(d\omega) \leq 2\epsilon\mu(C),$$

where $\phi(\omega) := \int_E \|x - f_0(\omega)\|\delta_*(\omega)(dx)$. By Lemma 2.4 it follows that for a.e. ω, $\phi(\omega) = 0$, which implies that for a.e. ω, $\delta_*(\omega)$ is the Dirac measure at $f_0(\omega)$. Therefore, the desired (8) follows directly from (6). QED

3 Main result: infinite dimensions

In this section I shall state an infinite-dimensional generalization of Theorem 2.1. Let E be a separable reflexive Banach space; in *all* that follows the topology on E will be the weak topology $\sigma(E, E^*)$. Here E^* stands for the topological dual of E; the duality between E and E^* will be denoted by $< \cdot, \cdot >$. It is important to observe that $(E, \sigma(E, E^*))$ forms a Suslin locally convex space (for $(E, \|\cdot\|)$ is Polish).

Of course, there is no direct way in which Theorem 2.1 can be extended to infinite dimensions: in fact, even Girardi's stronger Bocce criterion holds for the sequence of constant functions $f_n \equiv e_n$, e_n being the n-th unit vector in $E := \ell^2$. And (f_n) converges weakly to the null function, but not strongly. A way out of this apparent impasse is provided by the notion of *limited* convergence, given in [4]: limited convergence is stronger than weak convergence in L^1_E, and it coincides with strong convergence when E happens to be finite-dimensional.

Recall from [4] that a sequence (f_n) in L^1_E is said to converge *limitedly* to $f_0 \in L^1_E$ if

$$\lim_{n\to\infty} \int_\Omega g(\omega, f_n(\omega) - f_0(\omega))\mu(d\omega) = 0$$

for every $\Sigma \times \mathcal{B}(E)$-measurable function $g : \Omega \times E \to \mathbb{R}$ satisfying the following three conditions: [3]

[3] It may be reassuring to realize that $\mathcal{B}(E, \|\cdot\|) = \mathcal{B}(E, \sigma(E, E^*))$.

(i) $g(\omega, 0) = 0$,

(ii) $g(\omega, \cdot)$ is sequentially continuous on E

(iii) $|g(\omega, x)| \leq \gamma \|x\| + \psi(\omega)$ on $\Omega \times E$ for some $\gamma \geq 0, \psi \in L^1_{\mathbb{R}}$.

Taking $g(\omega, x) :=< x, \ell(\omega) >, \ell \in \mathcal{L}^\infty_{E^*}$, (the dual of $\mathcal{L}^1_E$ [15]) shows that limited convergence always implies weak convergence in $\mathcal{L}^1_E$. Moreover, when E is finite-dimensional one can also substitute $g(\omega, x) := \|x\|$, so as to end up with strong convergence in $\mathcal{L}^1_E$ (in infinite dimensions the same substitution is obviously prohibited, since then $x \mapsto \|x\|$ is only lower semicontinuous in the weak topology used on E).

Correspondingly, let me redefine (ORC) for infinite dimensions in the following way: A sequence (f_n) in $\mathcal{L}^1_E$ is said to satisfy the *oscillation restriction criterion* (ORC) if for every $\epsilon > 0, B \in \Sigma^+, x^* \in E^*$ and for every subsequence (f_{n_j}) of (f_n) there exists $C \in \Sigma^+, C \subset B$, with

$$\liminf_{j \to \infty} \int_C | < f_{n_j}(\omega) - m_C(f_{n_j}), x^* > | \mu(d\omega) \leq \epsilon \mu(C).$$

Clearly, this criterion is equivalent to the one of the previous section: when E is finite-dimensional, take in finitely many steps the successive unit vectors as x^*, and take each time B to be the set C obtained in the previous step; finally, add over the coordinates and use the equivalence of the L_1-norm and the Euclidean norm on such E.

The main result of this paper, which generalizes Theorem 2.1 – and *a fortiori* the main result of [18] – to infinite dimensions, is as follows:

Theorem 3.1 *For any sequence $(f_n)_0^\infty$ in $\mathcal{L}^1_E$ the following are equivalent:*

(i) (f_n) converges weakly to f_0 and satisfies (ORC),

(ii) (f_n) converges limitedly to f_0.

I shall discuss the proof of this theorem rather briefly, for, apart from some details, it is quite similar to the proof of Theorem 2.1. To begin with, the proof of (ii) $\Rightarrow$ (i) contains nothing really new. As for the implication (i) $\Rightarrow$ (ii), note first that the analog of Corollary 2.3 now runs as follows:

Corollary 3.2 *Let (f_k) be any subsequence of (f_n), satisfying*

$$(f_n) \text{ converges weakly to } f_0.$$

Then there exist a subsequence (f_{k_j}) of (f_k) and a Young measure δ_* from Ω into E such that the following properties hold:

$$\text{bar } \delta_*(\omega) = f_0(\omega) \text{ a.e.}$$

and for every $C \in \Sigma^+, x^* \in E^*$

$$\liminf_{j\to\infty} \int_C | < f_{k_j} - m(f_{k_j}), x^* > d\mu \geq \int_C [\int_E | < x - m(f_0), x^* > |\delta_*(\omega)(dx)]\mu(d\omega)$$

and

$$\liminf_{j\to\infty} \int_\Omega g'(\omega, f_{k_j}(\omega))\mu(d\omega) \geq \int_\Omega [\int_E g'(\omega, x)\delta_*(\omega)(dx)]\mu(d\omega) \quad (9)$$

for all sequential normal integrands g' on $\Omega \times E$ satisfying (3).

Proof. Compared to the proof of Corollary 2.3, the only points that need some attention are the fulfilment of the tightness condition (1), the role of (x_i^*) in the infinite-dimensional setup, and the additional (9). As for tightness, $h(\omega, x) := \|x\|$ still does the job, thanks to reflexivity of E. Further, since E was already observed to be a locally convex Suslin space, it follows by [12, III.32] that there exists a countable subset (x_i^*) of E^* which separates the points of E. With this in mind the proof of (4) can be repeated completely. As for (9), note that if g' is an ordinary (nonsequential) normal integrand it follows directly by (2). The present sequential case can be dealt with as follows: let σ be as given by (7). Let g' be as in (9); suppose first in addition that g' is nonnegative. Define $g_\epsilon(\omega, x) := g'(\omega, x) + \epsilon\|x\|$, $\epsilon > 0$. Then g_ϵ is clearly $\Sigma \times \mathcal{B}(E)$-measurable. Also, $g_\epsilon(\omega, \cdot)$ is easily seen to be sequentially inf-compact; so by the Eberlein-Šmulian theorem it is also inf-compact. This shows that g_ϵ is an ordinary normal integrand. Therefore, (2) applies, which gives (thanks in part to $g_\epsilon \geq g'$):

$$\epsilon\sigma + \liminf_j \int_\Omega g'(\omega, f_{k_j}(\omega))\mu(d\omega) \geq \int_\Omega [\int_E g'(\omega, x)\delta_*(\omega)(dx)]\mu(d\omega).$$

Letting ϵ go to zero gives the desired inequality. Finally, by a standard argument (originally due to A.D. Ioffe) the nonnegativity hypothesis can be discarded in favor of (3) [2, 3, 23]. QED

Now the proof of (i) $\Rightarrow$ (ii) in Theorem 3.1 can be sketched, based on the one given in the previous section.

Of course, all that is needed, is that an arbitrary subsequence (f_k) of (f_n) has a subsequence which converges limitedly to f_0. This subsequence is provided by Corollary 3.2, which also yields an associated Young measure δ_*. Observe that the *only* time in the previous section when the fact that $\| \cdot \|$ is a norm – and not just a seminorm – on E played a role occurred when $\phi = 0$ a.e. was argued to imply $\delta_* = \epsilon_{f_0}$ a.e. But the entire argument which led up to $\phi = 0$ a.e. can still be mimicked here. Thus, one gets, for each fixed $i \in \mathbb{N}$, that $\phi_i(\omega) = 0$ a.e., where

$$\phi_i(\omega) := \int_E | < x - f_0(\omega), x_i^* > |\delta_*(\omega)(dx).$$

Here (x_i^*) is the point-separating sequence in E^* encountered in the proof of Corollary 3.2. The conclusion is that a.e. the probability measure $\delta_*(\omega)$ is carried by the intersection of all annihilator sets of $x \mapsto | < x - f_0(\omega), x_i^* > |$, $i \in \mathbb{N}$, i.e., the singleton $\{f_0(\omega)\}$. So the statement $\delta_* = \epsilon_{f_0}$ a.e. still obtains in this section.. The rest of the proof is easy: apply (9) to both g' and $-g'$ for $g'(\omega, x) := g(\omega, x - f_0(\omega))$, where g is as in the definition of limited convergence. Again (3) holds by virtue of the uniform integrability of $(\|f_n\|)$ [11]. The desired limited convergence of (f_{k_j}) to f_0 is thereby established.

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