Harmonic analysis on reductive symmetric spaces

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Abstract. We give a relatively non-technical survey of some recent advances in the Fourier theory for semisimple symmetric spaces. There are three major results: An inversion formula for the Fourier transform, a Paley-Wiener theorem, which describes the Fourier image of the space of compactly supported smooth functions, and the Plancherel theorem, which describes the decomposition into irreducibles of the regular representation.

1. Introduction

The beautiful theory of Fourier series has many generalizations that match its beauty. The majority of these generalizations concern the decomposition of functions on homogeneous spaces of a Lie group, such as, for example, the n-sphere S^n, which is a homogeneous space for the rotation group $O(n+1)$. The harmonic analysis on S^n is the theory of expansions in spherical harmonics Y_l^m, and the Plancherel theorem for S^n is the statement that these functions form an orthonormal basis for $L^2(S^n)$ (with respect to the rotation invariant surface measure). This theory of harmonic analysis on S^n was generalized to the compact Riemannian symmetric spaces by Cartan, [25]. These are homogeneous spaces G/K, where G is a connected semisimple compact Lie group, and K is the subgroup consisting of all points fixed by a given involution σ of G.

The key to Cartan's result is representation theory. The irreducible representations $(\pi_\lambda, \mathcal{H}_\lambda)$ of G are determined by a discrete parameter λ, the highest weight. For some (explicitly known, see [39], p. 535) values of λ there exist nontrivial K-fixed vectors in $\mathcal{H}_\lambda$. If that is the case, the space $\mathcal{H}_\lambda^K$ of such vectors is one-dimensional. The spherical harmonics Y_l^m generalize as the functions (matrix elements) $x \mapsto \langle \pi_\lambda(x)u, v\rangle$, where $u \in \mathcal{H}_\lambda^K$ is fixed and v runs through an orthonormal basis for $\mathcal{H}_\lambda$. The matrix element is a function on G/K, since u is K-fixed. The Plancherel theorem for G/K asserts that these functions, appropriately normalized, form an orthonormal basis for $L^2(G/K)$ (with respect to an invariant measure), and it gives the explicit decomposition of the regular representation of G on this space, as the direct sum of those $\mathcal{H}_\lambda$ for which $\mathcal{H}_\lambda^K \neq 0$.

The non-compact Riemannian symmetric spaces also carry a beautiful theory of harmonic analysis. These spaces are realized similarly as G/K, with G a connected semisimple Lie group of the non-compact type, and K the subgroup of fixed points of an involution, such that K is compact (in fact, K is then a maximal compact subgroup of G). In this theory, which is due to Harish-Chandra, [32], and Helgason, the Fourier transform of a function on G/K is given by integration against (generalized) spherical functions with respect to invariant measure. The parameter set for the spherical functions is basically a Euclidean space, and the Fourier expansion of a function on G/K is an integral, as in the ordinary Fourier theory on $\mathbb{R}$. The measure, with respect to which the integral is taken, is determined through Harish-Chandra's c-function, which is explicitly known through the formula of Gindikin-Karpelevic (see [39] for details).

Again, the harmonic analysis just described is best understood through representation theory, but since G is non-compact, the representations will necessarily be infinite dimensional, as there exists no finite dimensional non-trivial unitary representation of G. As before, the space $\mathcal{H}^K$ of K-fixed vectors in an irreducible representation $(\pi, \mathcal{H})$ of G is one-dimensional if it is not trivial, and the spherical functions are obtained as matrix elements $\langle \pi(x)u, v\rangle$ with $u \in \mathcal{H}^K$, $v \in \mathcal{H}$.

In a different, but nevertheless closely related, direction, a Fourier theory exists for functions on G, where G is a semisimple Lie group. This theory is the formidable achievement of Harish-Chandra, [33], [34]-[36]. For a general semisimple Lie group, the Fourier decomposition of a function is a mixture of sums and integrals. The generalized spherical functions that serve as the building blocks in the expansion of a function on G are matrix elements of irreducible representations of G. From the point of view of representation theory, the harmonic analysis is the decomposition into irreducibles of the (left or right) regular representation of G on $L^2(G)$. See [42], [49] or [50] for details.

In the present survey, we discuss a theory of harmonic analysis, which generalizes both the theory for G/K and that for G. This is the theory of harmonic analysis on semisimple (or reductive) symmetric spaces. Again, the analysis takes place on a homogeneous space G/H with G a connected semisimple Lie group and H the subgroup of fixed points for an involution, but neither G nor H are assumed to be compact. That this is a generalization of the theory for G/K is obvious. To see that it also generalizes the theory for G, we note that G is a homogeneous space for $G \times G$ (which is semisimple, when G is) through the left times right action. In this fashion, G is identified with the quotient space of $G \times G$ by the diagonal subgroup, which is the set of fixed point for the involution $(x, y) \mapsto (y, x)$. Examples of semisimple symmetric spaces that are not of the form G/K, and not of the form G, are the hyperbolic spaces

$$\{x \in \mathbb{R}^{p+q} \mid x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 = 1\}$$

which are homogeneous spaces for the pseudo-orthogonal groups $O(p, q)$; only the special cases where $q = 0$, $p = 1$ or $p = q = 2$ are of the previously discussed types. There exists a classification of the semisimple symmetric spaces (up to local isomorphism), see [20]. All semisimple symmetric spaces carry an invariant measure, which is unique up to normalization (both G and H are unimodular groups).

The study of harmonic analysis on semisimple symmetric spaces in general was initiated by M. Flensted-Jensen, [30], who constructed a general family of representations in the discrete series. By definition this series consists of the irreducible representations of G that enter as discrete summands in the decomposition of the regular representation on $L^2(G/H)$, which means simply that they are subrepresentations of the regular representation. The study was continued by T. Oshima and T. Matsuki, [45], who gave a quite complete description of the discrete series, which generalizes Harish-Chandra's description for the group case. In particular, a necessary and sufficient condition on G/H for the existence of discrete series representations is given in [30] and [45]. For further details we refer to the survey paper [10] and the references given there.

From a representation theoretic point of view, the representations of the discrete series are probably the most interesting components in $L^2(G/H)$, since they are in some sense most singular. The Langlands parameters of these representations are known, see [47], [31]. However, for the purpose of harmonic analysis, the decomposition of the remainder of $L^2(G/H)$, which by definition is not discrete, is of equal importance. This will be our main focus in the present survey.

The complete decomposition of the non-discrete part of $L^2(G/H)$, the Plancherel theorem for G/H, has been accomplished more recently, through work of P. Delorme, [29], and of the authors, [19]. The various representation components in the decomposition of $L^2(G/H)$ are naturally grouped into a finite number of 'series', according to their spectral properties for the algebra $\mathbb{D}(G/H)$ of invariant differential operators on G/H. One of these series is the discrete series, when it exists. Another comprise the so-called most continuous part. The decomposition of the corresponding subspace $L^2_{\mathrm{mc}}(G/H) \subset L^2(G/H)$ was determined in [15]. Some details will be presented in Section 2 below. Each of the series carry both a discrete and a continuous parameter (in this sense, they resemble a mixture of the compact and non-compact Riemannian cases G/K discussed at the outset). The continuous parameter runs in a Euclidean space, and the most continuous series is distinguished by having that parameter space of maximal dimension. The series that are not discrete and not most continuous are called intermediate series. The representations that enter in the the most continuous series, as well as in the intermediate series, are induced representations. The subgroup from which induction takes place is a parabolic subgroup, which is characteristic for the series.

We would like to mention also a closely related problem, which is the determination of the Paley-Wiener space for G/H. Recall that the Paley-Wiener theorem for $\mathbb{R}$ describes the Fourier image of the space of compactly supported L^2 functions. A variant of the theorem is the Paley-Wiener-Schwartz theorem, which describes the Fourier image of the space $C_c^\infty(\mathbb{R})$ of compactly supported smooth functions as the space of entire functions of exponential type (see for example [41], Thm. 7.3.1). This theorem was generalized to non-compact Riemannian symmetric spaces G/K by Helgason and Gangolli, see [39], Sect. IV.7, and to semisimple Lie groups G by Arthur, [2], and it has been further generalized to all reductive symmetric spaces G/H by the authors. The Fourier transform is the one associated with the most continuous series. It is a remarkable fact, established in [15], that the orthogonal projection of $C_c^\infty(G/H)$ onto $L^2_{\mathrm{mc}}(G/H)$ is injective, hence the Fourier transform associated with the most continuous series is injective on $C_c^\infty(G/H)$. The Paley-Wiener theorem for G/H describes the image under this Fourier transform of $C_c^\infty(G/H)$ (in fact, more precisely of its dense subspace of K-finite functions).

Both the Plancherel theorem and the Paley-Wiener theorem for G/H were announced in the seminar at the Mittag-Leffler Institute, Stockholm, in the fall of 1995. The Plancherel theorem was announced by Delorme; the proof has appeared in [28]-[29] (previously, Oshima (see [44], p. 604) had announced a Plancherel formula, but the details have not appeared.) The Paley-Wiener theorem was announced by the authors. Together with that the authors also announced that their proof implies the Plancherel formula under the hypothesis that certain identities, the so-called Maass-Selberg relations, are valid for G/H. The validity of these relations, which also play a main role in Delorme's work, as well as in Harish-Chandra's work for the group case, has been established for $L^2_{\mathrm{mc}}(G/H)$ in [7], and for the general case by Carmona and Delorme in [24]. Some details of the work of the authors have appeared in [16] and [17], the rest will appear in [18] and [19]. In the latter paper we will also include an independent proof, found later, of the Maass-Selberg relations. The methods of the two approaches to the Plancherel decomposition are in many respects different, though they both rely on the Maass-Selberg relations. For example, an important ingredient in Delorme's work, which is not used in the other approach, is an a priori characterization of the support of the Plancherel measure (cf. [23], Appendix C), which in turn is derived from a result of Bernstein [21]. On the other hand, in the work of the authors, an inversion formula for the Fourier transform on $C_c^\infty(G/H)$ plays a crucial role (see [17]). The proof of this formula is based on a theory of multivariable residue calculus, developed for the occasion, [16], using ideas from Langlands [43] and Heckman-Opdam [37].

The present paper is meant to be a relatively non-technical survey. For more elaborate expositions, we refer to [38], Part II, and to [10].

2. The Fourier transform

In this section we will briefly describe the Fourier transform associated with the most continuous part of $L^2(G/H)$. It is constructed out of the so-called *Eisenstein integrals* on G/H. We will also discuss the Plancherel theorem for $L^2_{\mathrm{mc}}(G/H)$. The main references are [6], [7], [13] and [15].

For the sake of completeness, let us first give the precise definition of the concept of a semisimple symmetric space, since it is actually slightly more general than what was described above. We assume that G is a connected semisimple Lie group with finite center, and that σ is an involution of G. The set G^σ of σ-fixed points in G is in general not connected, but its connected components are finite in number. The natural generality of spaces G/H is obtained by requiring that H is a subgroup of G^σ and that it contains the identity component of G^σ. For the purpose of the proofs of our main results, it would actually be more convenient to work at the outset with the more general class of reductive symmetric spaces, where the group G is allowed to be reductive (of so-called Harish-Chandra class). This is the generality of spaces used for example in [29] and [19], but in the present survey we will not go into this.

In order to describe the Fourier transform, the central issue is to determine the appropriate spherical functions. The point of departure is the matrix element formula $\langle \pi(x)u, v\rangle$ of the Riemannian case G/K. Recall that $(\pi, \mathcal{H})$ is an irreducible unitary representation of G, and that $u \in \mathcal{H}$ is K-fixed. Imitating this formula, we must look for the irreducible representations with a non-trivial H-fixed vector. Unfortunately, in general no such representations exist, unless the notion of representation vectors is extended. This complication appears already in the case of harmonic analysis on the group G, viewed as the symmetric space $G \times G/\mathrm{diag}(G)$. The irreducible unitary representations of $G \times G$ are of the form $(\pi_1 \otimes \pi_2, \mathcal{H}_1 \otimes \mathcal{H}_2)$, where the tensor product is taken in the category of Hilbert spaces. It can be seen (from Schur's lemma) that this representation can carry a $\mathrm{diag}(G)$-fixed vector only if π_1 is equivalent with the contragredient representation π_2^* of π_2. Then $\mathcal{H}_1 \otimes \mathcal{H}_2$ is identified with the space of Hilbert-Schmidt operators on $\mathcal{H}_1$. Again by Schur's lemma, the only $\mathrm{diag}(G)$-fixed operators on $\mathcal{H}_1$ are the constant multiples of the identity, which are not Hilbert-Schmidt if the representation is not finite-dimensional. Thus, in general there exists no non-trivial $\mathrm{diag}(G)$-fixed vector in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Returning to the general case, we must therefore allow representations that carry H-fixed vectors in a suitable generalized sense. For Lie groups there exists a notion of generalized representation vectors, based on the notion of smooth representation vectors. Rather than explaining these notions in general, we will describe explicitly what they amount to in the present case.

We first need some notation, in order to describe the representations that occur. We will, however, try to keep the notation at a minimum, which means that some ambiguities may evolve. The group G has an Iwasawa decomposition, $G = KA_0N_0$. The subgroup K is a maximal compact subgroup, by conjugation we may arrange that $\sigma(K) = K$. Since K is the set of fixed points for the associated Cartan involution θ on G, the assumption $\sigma(K) = K$ amounts to assuming that σ and θ commute. Thus we have at our disposal three commuting involutions, σ, θ and their product $\sigma\theta$. We may also arrange that the abelian group A_0 is preserved by these involutions (whereas the nilpotent part N_0 will not be preserved, in general, by any of the involutions) and that $A_0 \cap H$ is of minimal dimension. Let M_0 be the centralizer of A_0 in K. The product $P_0 = M_0A_0N_0$ is a minimal parabolic subgroup of G. The principal series of representations of G is a series of representations induced from P_0. This is the series of representations that contribute to the decomposition of $L^2(G/K)$ as well as $L^2_{\mathrm{mc}}(G)$. However, for the general case of $L^2_{\mathrm{mc}}(G/H)$ we need to induce from a (possibly) different parabolic subgroup of G, in order to ensure the existence of H-fixed (generalized) vectors.

Let $\mathfrak{g}$ be the Lie algebra of G. By differentiation, an involution of G induces an involution of $\mathfrak{g}$, which we will denote by the same symbol. The involutions θ and σ then give rise to the decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q},$$

where $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{h}$ and $\mathfrak{q}$ are the $+1$ and -1 eigenspaces. In particular, $\mathfrak{k}$ and $\mathfrak{h}$ are the Lie algebras of K and H. The Lie algebra $\mathfrak{a}_0$ of A_0 is a maximal abelian subspace of $\mathfrak{p}$. The intersection $\mathfrak{a}_{\mathrm{q}} := \mathfrak{a}_0 \cap \mathfrak{q}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. It determines a parabolic subgroup $P \supset P_0$ whose Levy part is the centralizer of $\mathfrak{a}_{\mathrm{q}}$ in G. As all parabolic subgroups P admits a so-called Langlands decomposition, which is written as $P = MAN$. The Levy part of P is the product of M and A, and $N \subset N_0$ is the nilpotent part. The subgroup A is central in MA, its Lie algebra $\mathfrak{a}$ is contained in $\mathfrak{a}_0$ and satisfies $\mathfrak{a} \cap \mathfrak{q} = \mathfrak{a}_{\mathrm{q}}$. The parabolic subgroup P is characterized among the parabolic subgroups containing P_0 by being minimal among those which are stable for the involution $\sigma\theta$.

It is by induction from P that we construct the principal series of representations for G/H. More precisely, let $(\xi, \mathcal{H}_\xi)$ be a finite dimensional unitary representation of M and $\lambda \in i\mathfrak{a}^*$ an imaginary linear functional on $\mathfrak{a}$. The induced representation

$$\pi_{\xi,\lambda} = \mathrm{Ind}_P^G(\xi \otimes e^\lambda \otimes 1)$$

(where 1 stands for the trivial representation of N) is defined as follows. The representation space consists of functions $f\colon G \to \mathcal{H}_\xi$ satisfying

$$f(manx) = a^{\lambda+\rho}\xi(m)f(x), \quad (m \in M, a \in A, n \in N, x \in G) \tag{1}$$

and the action of G is the right action. Here $\rho \in \mathfrak{a}^*$ denotes half the trace of the adjoint action of $\mathfrak{a}$ on $\mathfrak{n} = \mathrm{Lie}(N)$ (the appearance of this is standard), and $a^{\lambda+\rho} = e^{(\lambda+\rho)(\log a)}$ by definition. It follows from (1) that f is uniquely determined by its restriction to K, since $G = PK$. The Hilbert space $\mathcal{H}_{\xi,\lambda}$ is the space of functions satisfying (1), for which $k \mapsto \|f(k)\|_\xi$ is in $L^2(K)$, and its inner product is given by

$$\langle f, g\rangle = \int_K \langle f(k), g(k)\rangle_\xi \, dk,$$

where dk is Haar measure on K. The space of smooth functions satisfying (1) is denoted $C^\infty_{\xi,\lambda}$, it is dense in $\mathcal{H}_{\xi,\lambda}$. The space of generalized functions (in the sense of distribution theory) satisfying the identity is denoted $C^{-\infty}_{\xi,\lambda}$. Then

$$C^\infty_{\xi,\lambda} \subset \mathcal{H}_{\xi,\lambda} \subset C^{-\infty}_{\xi,\lambda},$$

and $\langle\,\cdot\,,\,\cdot\,\rangle$ makes sense as a sesquilinear pairing between the spaces $C^{-\infty}_{\xi,\lambda}$ and $C^\infty_{\xi,\lambda}$. In particular, it makes sense to form the matrix element $\langle \pi_{\xi,\lambda}(x)u, v\rangle$ with

$u \in C^{-\infty}_{\xi,\lambda}$ and $v \in C^{\infty}_{\xi,\lambda}$. The elements of $C^{\infty}_{\xi,\lambda}$ and $C^{-\infty}_{\xi,\lambda}$ are exactly the smooth and the generalized vectors for the representation $\pi_{\xi,\lambda}$.

We have thus described the representations that occur in the most continuous part of $L^2(G/H)$. The next issue is the H-fixed (generalized) vector u that will be used to form the matrix element by which the appropriate spherical functions are defined. It is an element of $C^{-\infty}_{\xi,\lambda}$. We will not give the details of the construction, but only mention that it requires the extension of the definition of $C^{-\infty}_{\xi,\lambda}$ to $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$, the space of complex linear functionals on $\mathfrak{a}_{\mathrm{q}}$. The construction of the H-fixed vector then involves analytic continuation with respect to λ. See [6] and [8] for details. We note that in contrast to the cases of G/K and G, the space of H-fixed (generalized) vectors for a given irreducible representation can have dimension > 1. This means that in the decomposition of $L^2(G/H)$ there can (and sometimes will) occur multiplicities. However, the multiplicities are finite (see [4], [12]).

In order to proceed with the definition of the generalized spherical functions that will be used in the harmonic analysis, one further observation is needed. A vector v in a representation space $(\pi, \mathcal{H})$ for G is called K-finite if the vectors $\pi(k)v$, where $k \in K$, span a finite dimensional subspace of $\mathcal{H}$. The space of K-finite vectors in $L^2(G/H)$ is dense, and hence, in order to describe the Plancherel decomposition it suffices to consider K-finite functions on G/H. In fact, we shall fix a finite dimensional representation of K, and consider only such functions on G/H, whose left translates span a space on which the action of K is equivalent with the fixed representation. Let (τ, V_τ) be a fixed finite dimensional representation of K. Then for convenience, we shall actually move our focus from scalar valued functions on G/H to V_τ-valued functions f that satisfy $f(kx) = \tau(k)f(x)$ for $k \in K$, $x \in G/H$. Such functions are called τ-spherical. It is not difficult to see that there are natural maps allowing this change (see [13], Lemma 5). Thus, our generalized spherical functions will be τ-spherical functions, and the Fourier transform will be defined for τ-spherical functions. The space of τ-spherical square integrable functions on G/H is denoted $L^2(G/H : \tau)$.

Let (τ, V_τ) be as in the previous paragraph. The generalized spherical function to be defined will depend linearly on an element $\psi \in V_\tau^{K\cap H\cap M}$, the space of $K\cap H\cap M$-fixed vectors in V_τ, and it will depend meromorphically on a parameter $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. Given ψ and λ we define the following V_τ-valued function $\tilde{\psi}_\lambda$ on G. The value of $\tilde{\psi}_\lambda$ at x is to be $a^{\lambda+\rho}\tau(m)\psi$ if $x = manh$ belongs to the product $PH = MANH$, and 0 otherwise. It can be shown that PH is an open subset of G. We then define

$$E(\psi : \lambda)(x) = \int_K \tau(k)\tilde{\psi}_\lambda(k^{-1}x)\, dk \in V_\tau. \tag{2}$$

Then $E(\psi : \lambda)$ is a τ-spherical function on G/H (by the invariance of the Haar measure on K) which is called an *Eisenstein integral.* In fact, the integral in (2) only converges for λ in a certain region of $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$, and to give a proper definition one must allow analytic continuation to $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. The Eisenstein integrals are our basic generalized spherical functions, and they are the corner stones of the harmonic analysis on G/H. For details, see [7] and [13]. In particular, the components of the vector valued Eisenstein integrals are indeed (linear combinations of) matrix coefficients of the $\pi_{\xi,\lambda}$, with vectors $u \in (C^{-\infty}_{\xi,\lambda})^H$ and $v \in C^{\infty}_{\xi,\lambda}$ cf. [13], eq. (25). Thus, the integral over K in (2) is essentially the same as that in the definition of the pairing between $C^{-\infty}_{\xi,\lambda}$ and $C^{\infty}_{\xi,\lambda}$. In fact, a somewhat more general construction is possible, in which one considers all the open subsets of G of the form PxH, where $x \in G$. These open subsets are finite in number, and can be parametrized by taking x in a suitable subset $\mathcal{W}$ of K. Then the parameter ψ will be taken in a finite dimensional Hilbert space ${}^\circ\mathcal{C}$ which contains $V_\tau^{K\cap H\cap M}$ as a subspace. We will not give its definition here, but refer to [13], Sect. 2. The notation ${}^\circ\mathcal{C}$ will nevertheless be used freely in the sequel. The reader may think of it as the space $V_\tau^{K\cap H\cap M}$, to which it is equal under the simplifying assumption on G/H that PH is dense. The Eisenstein integral $E(\psi : \lambda)(x)$ depends linearly on $\psi \in {}^\circ\mathcal{C}$. We write $E(\lambda)$ for the corresponding $\mathrm{Hom}({}^\circ\mathcal{C}, V_\tau)$-valued function on G/H, and $E(\lambda : x)$ for its value at x.

When G/H is a Riemannian symmetric space, the Eisenstein integrals are equal to the generalized spherical functions of [40], Ch. III, 2, and when G/H is a semisimple Lie group viewed as a symmetric space, they are multiples of the Eisenstein integrals introduced by Harish-Chandra.

There is, however, one serious problem associated with the application of these Eisenstein integrals to the harmonic analysis. This problem does not exist seriously in any of the cases G/K and G. The Eisenstein integrals are constructed as meromorphic functions in the parameter $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$, but they will be used eventually for purely imaginary functionals λ. These are the values of λ for which $\pi_{\xi,\lambda}$ is unitary, and it is only these representations that contribute in the decomposition of $L^2_{\mathrm{mc}}(G/H)$. However, the meromorphic function $\lambda \mapsto E(\lambda : x)$ may have (and in general, has) singularities on $i\mathfrak{a}^*_{\mathrm{q}}$. This problem is handled in [13] where a suitable *normalized Eisenstein integral* $E^\circ(\lambda : x) = E(\lambda : x) \circ C^{-1}(\lambda)$ is introduced. The normalization factor $C(\lambda)$, which is a linear operator on ${}^\circ\mathcal{C}$ depending meromorphically on λ, is determined from the asymptotic behaviour of $E(\lambda)$ in a particular direction. The normalized Eisenstein integrals are regular on $i\mathfrak{a}^*_{\mathrm{q}}$, see [13], Thm. 2. See also [9] for a more general result, with a different proof and valid for the intermediate series as well.

The Fourier transform $\mathcal{F}f$ of a τ-spherical function f on G/H, say continuous with compact support, is now defined as follows. For an element $A \in \mathrm{Hom}({}^\circ\mathcal{C}, V_\tau)$ we denote by A^* the adjoint operator in $\mathrm{Hom}(V_\tau, {}^\circ\mathcal{C})$, defined with respect to the inner products of these finite dimensional spaces. Then

$$\mathcal{F}f(\lambda) = \int_{G/H} E^\circ(-\bar{\lambda} : x)^* f(x)\, dx \in {}^\circ\mathcal{C}, \tag{3}$$

where dx is the invariant measure on G/H. Notice that $\mathcal{F}f(\lambda)$ is a meromorphic function of $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$.

As in the ordinary Fourier theory on $\mathbb{R}$, the transpose of the Fourier transform plays an important role in harmonic analysis. The transpose is the map that forms *wave packets* out of functions in λ. More precisely, let $\mathcal{S}(i\mathfrak{a}_{\mathrm{q}}^*)$ denote the space of Schwartz functions on the Euclidean space $i\mathfrak{a}_{\mathrm{q}}^*$. For $\varphi \in \mathcal{S}(i\mathfrak{a}_{\mathrm{q}}^*) \otimes {}^\circ\mathcal{C}$ we define

$$\mathcal{J}\varphi(x) = \int_{i\mathfrak{a}_{\mathrm{q}}^*} E^\circ(\lambda : x)\varphi(\lambda)\, d\lambda \in V_\tau \tag{4}$$

where $d\lambda$ is Lebesgue measure on $i\mathfrak{a}_{\mathrm{q}}^*$, suitably normalized. The function $\mathcal{J}\varphi$ on G/H is obviously τ-spherical. That $\mathcal{J}$ is the transpose of $\mathcal{F}$, in the sense that $\langle \mathcal{F}f, \varphi\rangle = \langle f, \mathcal{J}\varphi\rangle$ for f and φ as above, where the inner products are those of L^2 on $i\mathfrak{a}_{\mathrm{q}}^*$ and G/H, respectively, is essentially an application of Fubini's theorem. However, it requires estimates to see that the integral in (4) converges and produces an element of $L^2(G/H)$. See [15], p. 301.

We can now state what is essentially the Plancherel theorem for $L^2_{\mathrm{mc}}(G/H)$. We define the space $L^2_{\mathrm{mc}}(G/H : \tau)$ as the closure in $L^2(G/H : \tau)$ of the space of all wave packets $\mathcal{J}\varphi$, with φ as above. In order to avoid a constant in the last statement of the following theorem, we assume all measures to be suitably normalized. (The precise normalizations are given in [15], p. 278.)

Theorem 2.1. ([15], Cor. 17.5) *The Fourier transform $f \mapsto \mathcal{F}f$ allows a unique extension to a continuous linear map $\mathcal{F}\colon L^2(G/H : \tau) \to L^2(i\mathfrak{a}_{\mathrm{q}}^*) \otimes {}^\circ\mathcal{C}$, whose kernel is the orthocomplement of $L^2_{\mathrm{mc}}(G/H : \tau)$, and whose restriction to the latter space is an isometry. Likewise, the wave packet transform $\varphi \mapsto \mathcal{J}\varphi$ allows a unique extension to a continuous linear map $\mathcal{J}\colon L^2(i\mathfrak{a}_{\mathrm{q}}^*) \otimes {}^\circ\mathcal{C} \to L^2_{\mathrm{mc}}(G/H : \tau)$. The map $\mathcal{J}\mathcal{F}$ is the orthogonal projection of $L^2(G/H : \tau)$ onto $L^2_{\mathrm{mc}}(G/H : \tau)$.*

The proof of Theorem 2.1 involves a detailed analysis of the algebra $\mathbb{D}(G/H)$ of invariant differential operators. Results of [3] and [11] are used. A central step consists of showing the existence of an element $D \in \mathbb{D}(G/H)$, depending on τ, which acts injectively on $C_c^\infty(G/H : \tau)$ and which satisfies

$$D\mathcal{J}\mathcal{F}f = Df \tag{5}$$

for all $f \in C_c^\infty(G/H : \tau)$. The orthocomplement of $L^2_{\mathrm{mc}}(G/H : \tau)$ in $L^2(G/H : \tau)$ is then identified as the kernel of D (acting on the L^2-space in the distribution sense).

3. Fourier inversion and Paley-Wiener theorem

In addition to Theorem 2.1 it is also proved in [15] (see Thm. 15.1) that the Fourier transform $\mathcal{F}$ is injective, when restricted to the space $C_c^\infty(G/H : \tau)$ of smooth, compactly supported τ-spherical functions. It is therefore an obvious problem to retrieve f from $\mathcal{F}f$. Moreover, it is natural to seek a description of the image $\mathcal{F}(C_c^\infty(G/H : \tau))$. We will now discuss the solution to these problems, the details are to be found in [17] and [19].

The inversion formula for the Fourier transform on $\mathbb{R}$ is the formula $f(x) = \mathcal{J}\mathcal{F}f(x)$. It is valid for all x when $f \in C_c^\infty(\mathbb{R})$ (of course, weaker hypotheses are sufficient). The same holds true for the non-compact Riemannian symmetric spaces G/K, and also for certain other semisimple symmetric spaces, for which $L^2_{\mathrm{mc}}(G/H) = L^2(G/H)$ (see [17], Sect. 11). However, it is false when $L^2_{\mathrm{mc}}(G/H)$ is a proper subspace of $L^2(G/H)$, since a contradiction would arise from the fact that $\mathcal{J}\mathcal{F}f \in L^2_{\mathrm{mc}}(G/H)$ combined with the density of $C_c^\infty(G/H)$ in $L^2(G/H)$. The function $\mathcal{J}\mathcal{F}f \in L^2_{\mathrm{mc}}(G/H)$ is smooth, but it need not have compact support.

In order to explain our inversion formula, let us consider again a non-compact Riemannian symmetric space G/K, with the trivial K-type $\tau = 1$. In this case $C_c^\infty(G/K : 1) = C_c^\infty(K\backslash G/K)$ and $E^\circ(\lambda : x) = c(\lambda)^{-1}\varphi_\lambda(x)$, where $\varphi_\lambda(x)$ is the spherical function and $c(\lambda)$ is Harish-Chandra's c-function. The inversion formula $f(x) = \mathcal{J}\mathcal{F}f(x)$ was established by Harish-Chandra. It takes the following form

$$f(x) = \int_{i\mathfrak{a}_0^*} \varphi_\lambda(x)\mathcal{F}f(\lambda)c(\lambda)^{-1}\, d\lambda. \tag{6}$$

Note that the usual Plancherel measure $|c(\lambda)|^{-2}\, d\lambda$ is hidden by the fact that $\mathcal{F}f(\lambda)$ equals $1/c(-\bar\lambda)^*$ times the usual spherical Fourier transform $\tilde f(\lambda)$; the star denotes complex conjugation. A simpler proof of this formula was later given by J. Rosenberg, [46], based on a part of Helgason's proof of the Paley-Wiener theorem (see [39], Ch. IV, 7). We will now discuss a part of this proof, since its generalization to G/H is used in the statement of the inversion formula.

In Helgason's Paley-Wiener argument, one exploits the Harish-Chandra expansion formula

$$\varphi_\lambda(a) = \sum_{w\in W} c(w\lambda)\Phi_{w\lambda}(a), \tag{7}$$

which is valid for a in the positive chamber A^+ associated with P_0. The function Φ_λ on A^+ is given by a series expansion so that $\Phi_\lambda(a) \sim a^{\lambda-\rho}$ as $a \to \infty$ in A^+. Furthermore, one uses the relations

$$c(\lambda)c(-\bar\lambda)^* = c(w\lambda)c(-w\bar\lambda)^*. \tag{8}$$

Insertion of (7) in (6), and use of (8) together with the W-invariance of $\tilde f$ yields the following expression

$$f(a) = |W| \int_{i\mathfrak{a}_0^*} \Phi_\lambda(a)\mathcal{F}f(\lambda)\, d\lambda. \tag{9}$$

(It requires some estimates to see that the latter integral converges). By Cauchy's theorem one can shift the domain of integration in (9) from $i\mathfrak{a}_0^*$ to $\eta + i\mathfrak{a}_0^*$, where $\eta \in \mathfrak{a}_0^*$ is antidominant (i.e. $\eta(Y) \le 0$ for all $Y \in \mathfrak{a}_0^+$). The idea is to let η pass to infinity, which will allow one to obtain an estimate showing that $\mathcal{J}\mathcal{F}f$ is compactly supported, as the first step towards the equality with f. Here we will, however, leave Rosenberg's proof and return to the general case of G/H.

In the general case of G/H, there exists for the Eisenstein integral an expansion formula as (7), see [14] Thm. 11.1. The c-functions are generalized, and there are identities generalizing (8). In fact, these are the Maass-Selberg relations, mentioned in the introduction. For the most continuous series, these identities are established in [7], Thm. 16.3 (see also [8]). By arguments similar to the above, one can thus rewrite the wave packet (4) in a form similar to (9), provided x belongs to the open dense subset $(G/H)_+ = \cup_{w\in\mathcal{W}} KA_q^+ wH$ of G/H (for details, see [10], p. 211, or [17], Sect. 4). However, if one wants to perform the shift by Cauchy's theorem as before, one must take into account that the integrand is no longer holomorphic in λ. This is due to the fact that the Eisenstein integrals are defined by an analytic continuation that only produces a meromorphic function. Thus, the shifted integral differs from the wave packet by some residual terms. When η is moved sufficiently far in the antidominant direction, the integrand becomes holomorphic, and no more residual terms are produced. From then on, the integral is independent of η. We call the sufficiently far shifted integral a *pseudo wave packet*, and denote it by $\mathcal{TF}f$. The inversion formula for the Fourier transform on $C_c^\infty(G/H:\tau)$ can now be stated as follows.

Theorem 3.1. ([17], Thm. 4.7) *Let $f \in C_c^\infty(G/H:\tau)$ and let $x \in (G/H)_+$. Then*

$$\mathcal{TF}f(x) = f(x).$$

Since by Theorem 2.1 the wave packet $\mathcal{JF}f$ is the projection of f onto the space $L^2_{\mathrm{mc}}(G/H:\tau)$, it follows from Theorem 3.1 that the difference between $\mathcal{TF}f$ and $\mathcal{JF}f$ is the projection of f onto $L^2_{\mathrm{mc}}(G/H:\tau)^\perp$. In the process described above, this projection is thus exhibited as the sum of the residual terms.

The main result of [16] describes a suitable grouping of the residual terms leading to an expression of the form

$$\mathcal{T}\varphi = \sum_{F\subset\Delta} \mathcal{T}_F\varphi, \tag{10}$$

with $\varphi = \mathcal{F}f$ and with $\mathcal{T}_\emptyset\varphi = \mathcal{J}\varphi$. Here $\Delta \subset \mathfrak{a}_q^*$ is the set of simple roots for the root system of $\mathfrak{a}_q$ in $\mathfrak{g}$, determined by the parabolic subgroup P, and $\mathcal{T}_F\varphi \in C^\infty((G/H)_+ : \tau)$ for each $F \subset \Delta$. A crucial part of the proof of Theorem 3.1 consists of showing that the functions $\mathcal{T}_F\mathcal{F}f$ extend smoothly to G/H for $F \subsetneq \Delta$. This involves the analogue of the theorem for a lower dimensional symmetric space determined by F. It is thus possible to proceed by induction. The proof of the theorem is then completed by using (5). Notice that as a consequence of the theorem $\mathcal{TF}f$ extends smoothly to G/H. Hence by (10) also $\mathcal{T}_\Delta\mathcal{F}f$ extends.

Closely related to Theorem 3.1 is the Paley-Wiener theorem, which was conjectured in [15], Rem. 21.8, and proved there for the case that $\dim \mathfrak{a}_q = 1$. Its formulation requires the introduction of a Paley-Wiener space $\mathrm{PW}(G/H:\tau)$, whose precise definition is given in [15], Def. 21.6, see also the forthcoming paper [19]. Here we restrict ourselves to mentioning that $\mathrm{PW}(G/H:\tau)$ consists of meromorphic ${}^\circ\mathcal{C}$-valued functions on $\mathfrak{a}^*_{q\mathbb{C}}$, satisfying (a) suitable estimates of Paley-Wiener-Schwartz type and (b) the so-called Arthur-Campoli relations. The latter are defined as all the relations of a particular type which hold for all the functions $\lambda \mapsto E^\circ(-\bar\lambda : x)^* v$, for $x \in G/H$, $v \in V_\tau$. They generalize the relations used in [2].

Theorem 3.2. (Paley-Wiener theorem, [19]) *The Fourier transform $\mathcal{F}$ is a linear isomorphism from $C_c^\infty(G/H : \tau)$ onto* $\mathrm{PW}(G/H : \tau)$.

The Fourier transform maps injectively into the Paley-Wiener space, so the problem is to establish surjectivity. The proof of this is based on the inversion formula in Theorem 3.1. If $\varphi \in \mathrm{PW}(G/H : \tau)$ then the pseudo-wave packet $\mathcal{T}\varphi$ is a well-defined smooth function on $(G/H)_+$. According to Theorem 3.1, this function is the natural candidate for the inverse image of φ. By the Paley-Wiener shift argument it is seen that the support of $\mathcal{T}\varphi$ is contained in a compact subset of G/H. The main difficulty of the proof is to show that $\mathcal{T}\varphi$ admits a smooth extension to G/H. This is established by observing first that (10) is valid by residue calculus. It is then shown that $\mathcal{T}_F\varphi$ extends smoothly, for each $F \subset \Delta$. For $F = \Delta$ this is a relatively simple consequence of the Arthur-Campoli relations, since the smooth extension is known to hold when φ is already of the form $\mathcal{F}f$. For $F \subsetneq \Delta$ it again involves induction from a lower dimensional symmetric space determined by F. An important ingredient of the proof is that the Arthur-Campoli relations for that space can be induced up to relations for G/H. This principle, which we call induction of relations, is based on a detailed study of asymptotic expansions of meromorphic families of eigenfunctions in [18]. Once it has been established that $\mathcal{T}\varphi \in C_c(G/H : \tau)$, the identity $\mathcal{F}\mathcal{T}\varphi = \varphi$ follows by means of (5).

Our Paley-Wiener theorem generalizes that of Arthur [2] for the group case. However, our approach differs from Arthur's in a number of ways. Firstly, Arthur's result involves unnormalized Eisenstein integrals whereas ours involves normalized ones. Nevertheless, the associated Paley-Wiener theorems are equivalent. Next, Arthur uses residue calculus in the spirit of Langlands' work [43], but at the same time invokes Harish-Chandra's Plancherel theorem for the group. We use the residue calculus to also establish the Plancherel theorem, which is analogous to Langlands' approach. Finally, [2] relies on a lifting principle of Casselman, the proof of which has not been published. In [18] we establish a normalized version of Casselman's principle in close connection with the above mentioned induction of relations.

4. Plancherel decomposition

Theorem 3.1 is also the starting point for our derivation of the Plancherel formula. In order to simplify the presentation, we will from now on assume that PH is the only open P-orbit on G/H so that the set $\mathcal{W}$ discussed in Section 2 has only one element. We refer to [19] for the general case.

In the residue calculus leading to (10) an important role is played by the concept of a residue weight, as introduced in [16], Sect. 3.2. It is a certain function on the set of all $\sigma\theta$-stable parabolic subgroups containing A_{q}, and its purpose is to assign weights according to which residual contributions are counted.

Let W denote the Weyl group of the root system on $\mathfrak{a}_{\mathrm{q}}$ generated by Δ. If $F \subset \Delta$, let $\mathfrak{a}_{F\mathrm{q}}$ be the intersection in $\mathfrak{a}_{\mathrm{q}}$ of the root hyperplanes for the roots from F, and let W_F denote the centralizer of $\mathfrak{a}_{F\mathrm{q}}$ in W. The set F determines a unique $\sigma\theta$-stable parabolic subgroup P_F containing P.

If t is any choice of a residue weight, then from the mentioned procedure of taking residues it follows that, for $f \in C_c^\infty(G/H : \tau)$, $F \subset \Delta$,

$$\mathcal{T}_F \mathcal{F} f(x) = t(P_F)\,|W| \int_{\epsilon_F + i\mathfrak{a}_{F\mathrm{q}}^*} \int_{G/H} K_F(\nu : x : y) f(y)\, dy\, d\nu, \tag{11}$$

with $K_F(\nu : \cdot : \cdot)$ a smooth $\mathrm{End}(V_\tau)$-valued function on $G/H \times G/H$, depending meromorphically on $\nu \in \mathfrak{a}_{F\mathrm{q}\mathbb{C}}^*$. This function turns out to be independent of the particular choice of the residue weight. In the above formula, $d\nu$ is a translate of suitably normalized Lebesgue measure. Finally, ϵ_F is a sufficiently close approximation of the origin in the $\Delta \setminus F$-positive chamber of $\mathfrak{a}_{F\mathrm{q}}^*$; it enters here in order to avoid singularities of the function $\nu \mapsto K_F(\nu : \cdot : \cdot)$ that may possibly lie in the space $i\mathfrak{a}_{F\mathrm{q}}^*$.

From its asymptotic behaviour it is deduced in [19] that the *kernel function* K_F can be expressed as

$$K_F(\nu : x : y) = \frac{1}{|W_F|}\, E_F^\circ(\nu : x) E_F^\circ(-\bar{\nu} : y)^*, \tag{12}$$

where $E_F^\circ(\nu : x) \in \mathrm{Hom}(\mathcal{A}_{2,F}, V_\tau)$ depends smoothly on $x \in G/H$ and meromorphically on ν. Here $\mathcal{A}_{2,F}$ is a finite dimensional Hilbert space, built from the discrete series representations of the lower dimensional symmetric space determined by F. More precisely, let $P_F = M_F A_F N_F$ be the Langlands decomposition of the parabolic subgroup determined by F. Then $\mathcal{A}_{2,F} = L_d^2(M_F/M_F \cap H : \tau)$, where L_d^2 denotes the discrete part of L^2. It follows in the course of the proof that the space $L_d^2(M_F/M_F \cap H : \tau)$ is finite dimensional, i.e., there are only finitely many discrete series representations that admit the existence of τ-spherical functions (for the given K-type τ). For $F = \emptyset$ we have, in particular, that $\mathcal{A}_{2,\emptyset} \simeq {}^\circ\mathcal{C}$. On the other hand, for $F = \Delta$ and if $\mathfrak{a}_{\Delta\mathrm{q}} = \{0\}$, we have $\mathcal{A}_{2,\Delta} = L_d^2(G/H : \tau)$, and $|W| K_\Delta(0)$ turns out to be the integral kernel of the orthogonal projection from $L^2(G/H : \tau)$ onto $L_d^2(G/H : \tau)$.

The functions $E_F^\circ(\nu : x)$ are called normalized Eisenstein integrals determined by the (standard) parabolic subgroup P_F. At a later stage they will appear to be (linear combinations of) matrix coefficients for the generalized principal series determined by P_F. The Eisenstein integrals allow converging series expansions which describe their asymptotic behaviour towards infinity. The highest order asymptotic behaviour is determined by the following asymptotic expression along sets of the form $mA_{F\mathrm{q}}^+$, as $m \in M_F/M_F \cap H$. Let $\psi \in \mathcal{A}_{2,F}$, then

$$a^{\rho_F} E_F^\circ(\nu : ma)\psi \sim \sum_{s \in W(\mathfrak{a}_{F\mathrm{q}})} a^{s\nu} C_{P_F|P_F}^\circ(s : \nu)\psi(m). \tag{13}$$

Here $W(\mathfrak{a}_{F\mathrm{q}}) = W_F^*/W_F$, with W_F^* is the normalizer of $\mathfrak{a}_{F\mathrm{q}}$ in the Weyl group W. Moreover, $C^\circ_{P_F|P_F}(s:\nu) \in \mathrm{End}(\mathcal{A}_{2,F})$ depends meromorphically on $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$. We require that $C^\circ_{P_F|P_F}(1:\nu)$ is the identity operator for all ν; this requirement of normalization, which can be accomplished, together with (12) determines the Eisenstein integral completely. The expression on the right-hand side of (13) is called the *constant term* of the Eisenstein integral (along P_F).

As in Harish-Chandra's theory for the group, a key role is played by the following Maass-Selberg relations for the (matrix-valued) C-functions

$$C^\circ_{P_F|P_F}(s:\nu)C^\circ_{P_F|P_F}(s:-\bar{\nu})^* = I_{\mathcal{A}_{2,F}}.$$

We derive these relations from $W(\mathfrak{a}_{F\mathrm{q}})$-invariance of the function $\nu \mapsto K_F(\nu)$, which in turn is inherited from a similar invariance of the kernel $K_\emptyset$, in view of W-equivariance of our residue calculus. In this fashion, Maass-Selberg relations for the generalized principal series are deduced from the Maass-Selberg relations in the most continuous case (see Section 3) by means of the residue calculus. The Maass-Selberg relations imply that the constant term on the right-hand side of (13) is regular on the set $i\mathfrak{a}^*_{F\mathrm{q}}$, as a meromorphic function of ν. From the theory of the constant term in [22] and from the results of [9], both of which papers generalize similar results of Harish-Chandra for the group case, it then follows that also the Eisenstein integral on the left-hand side of (13) is regular on $i\mathfrak{a}^*_{F\mathrm{q}}$. It should be mentioned that in the application of the theory of the constant term, it is of crucial importance that all the discrete series representations have an infinitesimal $\mathbb{D}(G/H)$-character which is real and regular. This result has been established in [45]. Beyond this, no further information on the discrete series is needed.

The regularity of the Eisenstein integral, together with suitable estimates, allow us to replace ϵ_F in (11) by the limit value zero. Thus we arrive at wave packets over the sets $i\mathfrak{a}^*_{F\mathrm{q}}$, where the Plancherel measure will ultimately turn out to be supported. The estimates mentioned above are of a uniform nature in both ν and x. They are obtained via a detailed analysis of the differential equations satisfied by the Eisenstein integral in [7] and [19]. To conclude that the Eisenstein integrals have tempered behaviour in the variable x we use information on the location of the residues involved in the definition of K_F, see [16] and [17].

The estimates for the Eisenstein integrals also allow us to define a Fourier transform $\mathcal{F}_F\colon C_c^\infty(G/H:\tau) \to \mathcal{S}(i\mathfrak{a}^*_{F\mathrm{q}}) \otimes \mathcal{A}_{2,F}$ and an adjoint wave packet transform $\mathcal{J}_F\colon \mathcal{S}(i\mathfrak{a}^*_{F\mathrm{q}}) \otimes \mathcal{A}_{2,F} \to L^2(G/H:\tau)$ by the formulas (3) and (4) with $\mathfrak{a}_{F\mathrm{q}}$ in place of $\mathfrak{a}_\mathrm{q}$ and with E_F° in place of E°.

Formulas (10), (11) with $\epsilon_F = 0$ for all $F \subset \Delta$, and (12) now lead to the formula

$$I = \sum_{F\subset\Delta} t(P_F)\,[W:W_F]\,\mathcal{J}_F \circ \mathcal{F}_F \qquad \text{on} \quad C_c^\infty(G/H:\tau). \tag{14}$$

Define the equivalence relation $\sim$ on the collection $\mathcal{P}(\Delta)$ of subsets of Δ by $F \sim F'$ if and only if $\mathfrak{a}_{F\mathrm{q}}$ and $\mathfrak{a}_{F'\mathrm{q}}$ are conjugate under the Weyl group. The following result is essentially the Plancherel formula for $L^2(G/H)$.

Theorem 4.1. *If $F \subset \Delta$, the Fourier transform $\mathcal{F}_F$ has a unique extension to a continuous linear operator $L^2(G/H : \tau) \to L^2(i\mathfrak{a}_{F\mathrm{q}}^*) \otimes \mathcal{A}_{2,F}$. Moreover, the wave packet transform $\mathcal{J}_F$ has a unique extension to a continuous linear operator $L^2(i\mathfrak{a}_{F\mathrm{q}}^*) \otimes \mathcal{A}_{2,F} \to L^2(G/H : \tau)$. The operator $\mathcal{J}_F \circ \mathcal{F}_F$ only depends on the class of F in $\mathcal{P}(\Delta)/\sim$; its extension to $L^2(G/H : \tau)$ is $[W : W_F^*]^{-1}$ times the orthogonal projection onto a closed subspace $L^2_{[F]}(G/H : \tau)$. Finally,*

$$L^2(G/H : \tau) = \bigoplus_{[F] \in \mathcal{P}(\Delta)/\sim} L^2_{[F]}(G/H : \tau), \tag{15}$$

with summands that are mutually orthogonal.

We sketch a few important ingredients of the proof. From the Maass-Selberg relations combined with a spectral analysis it is deduced that $\mathcal{J}_F \circ \mathcal{F}_F$ only depends on F through its class in $\mathcal{P}(\Delta)/\sim$; moreover, if $F, F' \subset \Delta$ are not equivalent, then the images of $\mathcal{J}_F \circ \mathcal{F}_F$ and $\mathcal{J}_{F'} \circ \mathcal{F}_{F'}$ are orthogonal. Thus, the identity (14) gives rise to a decomposition of the form (15), where the projection onto $L^2_{[F]}(G/H : \tau)$ is given by the part of the sum in (14) ranging over the elements of the class $[F]$. This part equals $\mathcal{J}_F \circ \mathcal{F}_F$ multiplied by the constant

$$[W : W_F] \sum_{F' \in [F]} t(P_F) = [W : W_F^*],$$

where the equality follows from a straightforward counting argument.

This final argument clarifies the role of the residue weight in the harmonic analysis. It determines the distribution of the projection onto $L^2_{[F]}(G/H : \tau)$ over the elements in the class $[F]$.

In the above discussion, the details of which will appear in [19], Theorem 4.1 has been derived from Theorem 2.1 by a residual calculus, without reference to representation theory. In particular, the Eisenstein integrals $E_F^\circ(\nu)$, for $F \neq \emptyset$, are introduced as residues and not as matrix coefficients of generalized principal series representations. In [19] it is shown that the Eisenstein integral $E_F^\circ(\nu : \cdot)$ indeed is a (generalized) matrix coefficient for the generalized principal series obtained by induction from P_F. This is achieved by an asymptotic analysis in the spirit of the proof of the subrepresentation theorem, see [26]. This then allows us to identify the decomposition described in Theorem 4.1 as the τ-spherical part of the Plancherel decomposition of $L^2(G/H)$ in the sense of representation theory. The fact that the summation in (15) ranges over classes then corresponds to the fact that standard intertwining operators connect the generalized principal series obtained by induction from $P_{F'}$, for $F' \in [F]$.

Delorme's proof in [29] follows a completely different road towards the Plancherel formula. In [27] the normalized Eisenstein integrals are *introduced* as matrix coefficients of generalized principal series representations. Moreover, in the same paper their temperedness is established by an argument involving the use of translation functors. The Maass-Selberg relations for the Eisenstein integrals are established in [24] by the so-called truncation method, see [28], which is inspired by a method used in [1]. In [24] it is then shown that the sum on the right-hand side of (15) is orthogonal. Finally, in [29] it is proved that the orthocomplement of the mentioned sum in $L^2(G/H : \tau)$ is trivial by combining the method of truncation with ideas of [21] that lead to an a priori proof that the Plancherel measure is supported on the representations that are tempered relative to G/H.

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