

Equivariant Cohomology and Stationary Phase

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Preface

This is the text of a survey lecture given at the conference on “Symplectic Geometry and its Applications”, Keio University, Yokohama, July 21, 1993. I have been stimulated by many people, but I would like to thank especially L. Jeffrey for her helpful explanations to me of [17].

1. Equivariant Cohomology

Equivariant cohomology is a structure which is attached to a smooth action of a Lie group G on a smooth manifold M . It can be defined as the cohomology of $EG \times_G M$, in which $EG \rightarrow BG$ is the universal principal G -bundle; BG is the classifying space of the group G .

Although this explains several aspects of equivariant cohomology, cf. Atiyah and Bott [1], for our purposes it is more convenient to use the model of H. Cartan, introduced in [5], [6]. It is a variation of de Rham cohomology, in which the algebra $\Omega(M)$ of smooth differential forms on M is replaced by the algebra

$$(1.1) \quad A := (S(\mathfrak{g}^*) \otimes \Omega(M))^G$$

of G -equivariant polynomial mappings

$$(1.2) \quad \omega : \mathfrak{g} \ni X \mapsto \omega(X) \in \Omega(M),$$

from the Lie algebra \mathfrak{g} of G to $\Omega(M)$. (It will be convenient to allow complex valued differential forms, so all algebras are over \mathbf{C} .) The equivariance of ω means that

$$(1.3) \quad \omega(\text{Ad } g(X)) = (g_M^*)^{-1}(\omega(X)), \quad g \in G, \quad X \in \mathfrak{g}.$$

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Here Ad stands for the *adjoint* action of G on its Lie algebra \mathfrak{g} and g_M^* denotes *pullback* of differential forms by means of the action $g_M : M \rightarrow M$ on M of the element $g \in G$.

In $\Omega(M)$ one has the derivations d and $i(v)$, of exterior differentiation and contraction with a vectorfield v in M , respectively. These are related to the *Lie derivative* by means of the *homotopy formula*

$$(1.4) \quad \mathcal{L}(v) := \frac{d}{dt}_{t=0} (e^{tv})^* = d \circ i(v) + i(v) \circ d.$$

Here e^{tv} denotes the flow in M after time t with velocity field equal to v .

If, for each $X \in \mathfrak{g}$, the vectorfield X_M denotes the infinitesimal action of X in M , then the *equivariant exterior differentiation* D is defined by:

$$(1.5) \quad (D\omega)(X) := d(\omega(X)) - i(X_M)(\omega(X)), \quad X \in \mathfrak{g}, \omega : \mathfrak{g} \rightarrow \Omega(M).$$

Clearly $D : A \rightarrow A$, and one also gets that $D \circ D = 0$. For the latter one uses that A consists of equivariant mappings $\mathfrak{g} \rightarrow \Omega(M)$, which implies, substituting $g = \exp(tX)$ in (1.3) and differentiating with respect to t at $t = 0$, that

$$(1.6) \quad 0 = \mathcal{L}(X_M)(\omega(X)) = (d \circ i(X_M) + i(X_M) \circ d)(\omega(X)),$$

cf. (1.4). The quotient

$$(1.7) \quad \mathbb{H}_G^*(M) := \ker D / \text{im } D$$

is called the *equivariant cohomology* of the G -action on M . It can be shown that if G is *compact*, which we assume from now on, then $\mathbb{H}_G^*(M)$ is canonically isomorphic to the topological equivariant cohomology, cf. [1].

In order to explain the *grading* in $\mathbb{H}_G^*(M)$, let $A^{k,l}$ denote the space of elements of A which are homogeneous polynomial mappings of degree k , from \mathfrak{g} to $\Omega^k(M)$. If $\omega \in A^{k,l}$, then

$$(1.8) \quad X \mapsto d(\omega(X)) \in A^{k,l+1}$$

and

$$(1.9) \quad X \mapsto i(X_M)(\omega(X)) \in A^{k+1,l-1}.$$

So we get $D_p : A^p \rightarrow A^{p+1}$, if we define

$$(1.10) \quad A^p := \bigoplus_{k,l \mid 2k+l=p} A^{k,l}$$

as the space of *equivariant forms of degree p* . We get

$$(1.11) \quad \mathbb{H}_G^*(M) = \bigoplus_{p \geq 0} \mathbb{H}_G^p(M),$$

in which

$$(1.12) \quad \mathbb{H}_G^p(M) := \ker D_p / \text{im } D_{p-1}$$

is the cohomology in degree p . In Section 3 we will see another reason why it is natural to give the indeterminate X degree two.

If $G = \mathbf{R}/\mathbf{Z}$ is the circle, then $\mathfrak{g} = \mathbf{R}$ and we can write, for $\omega \in A$:

$$(1.13) \quad \omega(X) = \sum_{j \geq 0} X^j \omega_j,$$

in which the $\omega_j \in \Omega(M)^G$ form a sequence of G -invariant differential forms on M . The sum is finite: if $\omega \in A^p$, then $\omega_j \in \Omega^{p-2j}(M)^G$, which is equal to zero if $p-2j < 0$ or $p-2j > \dim M$. The equivariant exterior derivative is given by

$$(1.14) \quad (D\omega)_j = d\omega_j - i(v)\omega_{j-1},$$

in which the vectorfield $v = 1_M$ is the infinitesimal action of $1 \in \mathbf{R} = \mathfrak{g}$ on M . So the computation of the equivariant cohomology involves sequences of equations in $\Omega(M)^G$.

A similar remark holds true for torus actions, using a multi-index notation in (1.13). For nonabelian Lie algebras \mathfrak{g} , the choice of the basis is not so obvious. One also has that the monomials $X^j \omega_j$ need not be equivariant, so do not always belong to A .

2. Localization in the Orbit Space

Replacing M by G -invariant open subsets U , we get a sheaf of algebras $A(U)$. The G -invariant open subsets of M correspond to the open subsets of the orbit space M/G , so the $A(U)$ can be viewed as a sheaf over M/G . It is a fine sheaf, because of the existence of partitions of unity by means of G -invariant functions, obtained from arbitrary partitions of unity by averaging these over G . Using Mayer-Vietoris sequences as in Bott and Tu [4, Ch. II], one can think of the equivariant cohomology of M as being built up out of the local equivariant cohomology groups $\mathbb{H}_G^*(U)$.

Each $x \in M$ has a G -invariant open neighborhood U_x and a G -equivariant retraction of U_x to the orbit $G \cdot x \simeq G/G_x$ through x . This leads to

$$(2.1) \quad \mathbb{H}_G^*(U_x) \simeq \mathbb{H}_G^*(G/G_x) \simeq \mathbb{S}(\mathfrak{g}_x^*)^{G_x},$$

the ring of $\text{Ad } G_x$ -invariant polynomials on \mathfrak{g}_x . Here

$$(2.2) \quad G_x := \{g \in G \mid g_M(x) = x\}$$

is the stabilizer of x in G and

$$(2.3) \quad \mathfrak{g}_x = \{X \in \mathfrak{g} \mid X_M(x) = 0\}$$

is its Lie algebra.

Formula (2.1) shows that the local cohomology is not trivial (as for the de Rham cohomology) if $\mathfrak{g}_x \neq 0$. It is even infinite-dimensional over \mathbf{C} ; it is a polynomial algebra of rank equal to the rank of \mathfrak{g}_x . This rank is equal to the

dimension of a maximal abelian subalgebra of \mathfrak{g}_x , or of the orbit space of the adjoint action of G_x in \mathfrak{g}_x .

This is most spectacular if x is a fixed point for the group action, in which case (2.1) is obvious and we get that the equivariant cohomology is equal to the ring

$$(2.4) \quad I := S(\mathfrak{g}^*)^G$$

of Ad G -invariant polynomials on \mathfrak{g} . Note that A and $H_G^*(M)$ are algebras over I , because multiplication with $f \in I$ is a linear mapping $: A \rightarrow A$, which commutes with the algebra structure in A and also with d and $i(X_M)$, hence with D .

3. Locally Free Actions

The other extreme occurs if the action is *locally free*, which means that $\mathfrak{g}_x = 0$ for all $x \in M$. In this case the quotient space is a manifold of dimension equal to $\dim(M) - \dim(G)$ with mild singularities, which locally are those of quotients of a manifold by a finite group action. The concept of such a manifold was introduced by Satake [22] under the name of *V-manifold*, but nowadays the name *orbifold* also has become popular. The point of [22] is that on such a manifold the de Rham theory goes through, practically without any change. One has for instance Poincaré duality defined by integration over the manifold, if the V -manifold is oriented. Note that if the action is free, that is $G_x = \{1\}$ for all $x \in M$, then M/G is a smooth manifold and $\pi : M \rightarrow M/G$ is a smooth fibration, known in the literature as a *principal fiber bundle*. Because of the many interesting examples, it is worthwhile however to allow locally free actions which are not free.

Now $\mathfrak{g}_x = 0$ yields in view of (2.1) that the local cohomology is trivial, and we get that

$$(3.1) \quad H_G^*(M) \xleftarrow[\pi^*]{\simeq} H^*(M/G).$$

In other words: *If the action is locally free, then the equivariant cohomology of M is canonically isomorphic to the de Rham cohomology of the quotient space M/G .*

More precisely, if $\pi : M \rightarrow M/G$ denotes the projection $\pi : x \mapsto G \cdot x$, which assigns to each $x \in M$ the G -orbit through x , then the pullback π^* by π is an isomorphism from $\Omega(M/G)$ onto the subspace $\Omega(M)_{\text{basic}}$ of the so-called *basic* differential forms in M . These are defined as the $\beta \in \Omega(M)$ which are G -invariant and satisfy $i(X_M)\beta = 0$ for all $X \in \mathfrak{g}$. As a constant map from \mathfrak{g} to $\Omega(M)$, such a β belongs to A , and $D\beta = 0$ if and only if $d\beta = 0$. The isomorphism (3.1) now means that if $\omega \in A$ and $D\omega = 0$, then there exists $\nu \in A$ and $\beta \in \Omega(M)_{\text{basic}}$, such that

$$(3.2) \quad \omega(X) = \beta + (D\nu)(X), \quad X \in \mathfrak{g}.$$

We have already observed before that A and $H_G^*(M)$ are modules over the ring I of Ad-invariant polynomials : $\mathfrak{g} \rightarrow \mathbf{C}$. If the action is locally free, then each $f \in I$ corresponds via (3.1) to a cohomology class c_f in $H^{\text{even}}(M/G)$, these cohomology classes of M/G are called the *characteristic classes* of the fibration $M \rightarrow M/G$. In this way the cohomology of M/G , which is finite-dimensional over \mathbf{C} if M is compact, can be viewed as a module over the ring of characteristic classes.

In [6, p. 63] an explicit construction of β and ν is indicated, using a *connection form* θ . That is, a \mathfrak{g} -valued one form in M , which is G -equivariant and which reproduces X when applied to X_M . In formula:

$$(3.3) \quad \theta \in (\mathfrak{g} \otimes \Omega^1(M))^G, \quad i(X_M)\theta \equiv X, \quad X \in \mathfrak{g}.$$

Connection forms exist if (and only if) the action is locally free. They can be constructed first in tubular neighborhoods of orbits and then pieced together by means of G -invariant partitions of unity.

If θ is a connection form in M , then the corresponding *curvature form in M* is defined by

$$(3.4) \quad \Omega := d\theta - [\theta, \theta] \in (\mathfrak{g} \otimes \Omega^2(M))^G.$$

Here $[\theta, \theta] \in (\mathfrak{g} \otimes \Omega^2(M))^G$ is defined by

$$(3.5) \quad [\theta, \theta]_x(v, w) = [\theta_x(v), \theta_x(w)], \quad v, w \in T_x M.$$

The curvature form in M has the property that, for each $f \in I$, $f(\Omega)$ is a closed basic form, of even degree. So $f(\Omega) = \pi^*\gamma$ for a uniquely determined closed form γ in M/G . The corresponding class $[\gamma] \in H^{\text{even}}(M/G)$ is equal to the characteristic class c_f , so in particular it does not depend on the choice of θ . The form γ is called the *characteristic form in M/G* , defined by θ and f .

The relation $X \leftrightarrow \Omega$ explains why the indeterminate X has been given degree two; this is the choice which makes (3.1) into an *isomorphism of graded rings*.

For torus actions, the situation is considerably simpler. We then have

$$(3.6) \quad \Omega := d\theta = \pi^* R$$

for a closed \mathfrak{g} -valued two-form R in M/G , called the *curvature form in M/G* . It defines the *Chern class* $c := [R] \in \mathfrak{g} \otimes H^2(M)$, and we have $c_f = f(c)$.

In the case of the circle $G = \mathbf{R}/\mathbf{Z}$, $\mathfrak{g} = \mathbf{R}$, $\omega(X) = \sum X^j \omega_j$, the form β in (3.3) is given explicitly by:

$$(3.7) \quad \beta = \sum_{j \geq 0} (d\theta)^j \wedge \omega_j - \sum_{j \geq 0} \theta \wedge (d\theta)^j \wedge i(v)\omega_j.$$

If $\omega = f \in I$, then $\beta = f(\Omega)$, confirming the description of the characteristic classes, which we gave above.

Combining (3.1) with the observation that the equivariant cohomology of a point is isomorphic to the ring of Ad-invariant polynomials on the Lie algebra,

one can now also explain the second identity in (2.1). Indeed, if H is a closed Lie subgroup of G , then we can use the left-right action of $G \times H$ on G and write

$$(3.8) \quad \mathbb{H}_G^*(G/H) \xrightarrow{\sim} \mathbb{H}_{G \times H}^*(G) \xleftarrow{\sim} \mathbb{H}_H^*(\text{point}).$$

4. Integration

From now on, we assume that M is compact and oriented and that the G -action preserves the orientation. If $\omega \in A$, then the integral

$$(4.1) \quad \left(\int \omega\right)(X) := \int_M \omega(X)^{[\dim M]}, \quad X \in \mathfrak{g}$$

defines an $\text{Ad } G$ -invariant function on \mathfrak{g} , so $\int \omega \in \mathbb{S}(\mathfrak{g}^*)^G$. Here we have written

$$(4.2) \quad \omega(X) = \sum_{k=0}^{\dim M} \omega(X)^{[k]}, \quad \omega(X)^{[k]} \in \Omega^k(M).$$

Note that $\omega = D\nu$ implies that that

$$(4.3) \quad \omega(X)^{[\dim M]} = d(\nu(X))^{[\dim M]},$$

because

$$(4.4) \quad (i(X_M)\nu(X))^{[\dim M]} = i(X_M)(\nu(X))^{[\dim M+1]} = 0.$$

So Stokes' theorem yields that $\int \omega = 0$ if $\omega \in \text{im } D$, which means that integration yields a map

$$(4.5) \quad \int : \mathbb{H}_G^*(M) \rightarrow I = \mathbb{S}(\mathfrak{g}^*)^G.$$

Because the ring I has no zero divisors, the map \int can only be nonzero if the rank of $\mathbb{H}_G^*(M)$ is equal to the rank of \mathfrak{g} . That is, it is necessary for having $(\int \omega)(X) \neq 0$ for some $\omega \in A$ satisfying $D\omega = 0$, that there exist $x \in M$ at which

$$(4.6) \quad \text{rank } \mathfrak{g}_x = \text{rank } \mathfrak{g}.$$

See [1, §3] for more about the rank of the module $\mathbb{H}_G^*(M)$. The localization of $\int \omega$ at the points where (4.6) holds is expressed in a more explicit way in the *localization formula* (4.13) of *Berline-Vergne* [3] and *Atiyah-Bott* [1]. For its formulation, we need some information about the action of a torus $T \subset G$ near its fixed points in M .

If $X \in \mathfrak{g}$, then the zeroset

$$(4.7) \quad Z = Z_X := \{x \in M \mid X_M(x) = 0\}$$

of X_M in M is equal to the fixed point set

$$(4.8) \quad M^T := \{x \in M \mid t_M(x) = x \text{ for all } t \in T\}$$

of the torus

$$(4.9) \quad T = T_X := \text{closure in } G \text{ of } \{\exp(\tau X) \mid \tau \in \mathbf{R}\}.$$

We write \mathfrak{t} for the Lie algebra of T . For generic X , T is a maximal torus in G and \mathfrak{t} is a maximal abelian subalgebra of \mathfrak{g} .

Using Bochner's local linearization theorem of actions of compact Lie groups near fixed points, one obtains that each connected component F is a smooth compact submanifold of M^T , and there are only finitely many F 's. For each $x \in F$, the normal space $T_x M / T_x F$ splits into two-dimensional T -invariant planes P_j , on which the infinitesimal action of $Y \in \mathfrak{t}$ is equal to $\lambda_j(Y)$ times the standard infinitesimal rotation of a quarter turn. Here

$$(4.10) \quad \lambda_j \in \mathfrak{t}^*, \quad \lambda_j(\ker \exp \cap \mathfrak{t}) \subset 2\pi\mathbf{Z}$$

are (the real versions of) the *weights* of the torus action.

Because of the rigidity in (4.10), the weights do not depend on the choice of the point x in the connected manifold F . Also, writing the quarter turn in the plane P_j as multiplication with i , P_j can be viewed as a complex line bundle over F , with a curvature form in $\Omega^2(F)$ attached to a connection in P_j . If λ_j occurs with multiplicity, then we get a complex vector bundle over F and the Chern form has to be replaced by a curvature matrix. (One may also use the "splitting principle" as in [4, §21], in order to reduce the computations to the case of complex line bundles.) In this way the normal bundle $N(F)$ of F in M may be provided with the structure of a Hermitian complex vector bundle, the infinitesimal action of X on the frame bundle $\text{FN}(F)$ of $N(F)$ will be denoted by LX .

The *equivariant Euler form* of the normal bundle of F is now defined as

$$(4.11) \quad \varepsilon(X) := \det_{\mathbf{C}} \left[\frac{i}{2\pi} (LX - \Omega) \right] \in \Omega^{\text{even}}(F).$$

Here Ω denotes the curvature form in $\text{FN}(F)$, defined by a connection form in the bundle $\text{FN}(F) \rightarrow F$. Because the complex determinant is a conjugacy-invariant polynomial, the characteristic form (4.11) is well-defined.

If $\lambda_j(X) \neq 0$ for all j , then this is an invertible element in the commutative algebra $\Omega^{\text{even}}(F)$, with inverse given by

$$(4.12) \quad \frac{1}{\varepsilon(X)} = \prod_j \frac{2\pi}{i\lambda_j(X)} \cdot \det_{\mathbf{C}} \left[\sum_{l \geq 0} ((LX)^{-1} \Omega)^l \right].$$

Note that the terms in the right hand side can only be nonzero if $2l \leq \dim F$. In particular the sum is finite. If $F = \{x\}$ is an isolated point, or more generally if the normal bundle of F is trivial, then $1/\varepsilon(X)$ is just equal to the scalar $\prod_j (2\pi/i\lambda_j(X))$.

With these notations, the localization formula now reads:

$$(4.13) \quad \left(\int \omega \right)(X) = \sum_F \int_F (i_F^* \omega(X) / \varepsilon(X))^{\dim F}.$$

On F the orientation is chosen such that it is compatible with the orientations of M and $N(F)$. Note that the condition (4.6) just means that \mathfrak{g}_x contains a maximal abelian subalgebra \mathfrak{t} of \mathfrak{g} , so $x \in M^T$. It is also remarkable that the *polynomial* $(\int \omega)(X)$ is equal to a sum of *rational* functions of X , which in general may have quite high order poles. The vanishing of the sum over F of the coefficients of these poles is just one example of the many magic identities which follow from the localization formula.

The proof indicated in [3] uses Stokes' formula in the complement of a small tubular neighborhood of M^T . For the curvature computations, see [10, Sec. 2], which can be turned into a proof of (4.13), if the factor $(-1)^k e^{Jx} \sigma^{n-k} / (n-k)!$ is replaced by ω_k , if $\omega(X) = \sum_k X^k \omega_k$. Note that it is sufficient to prove (4.13) for circle actions, because the rays through the integral lattice $\ker \exp \cap \mathfrak{t}$ form a dense subset of \mathfrak{t} .

The proof of Berline, Getzler and Vergne in [2, Ch. 7] is based on an idea, which potentially has much wider applications. In a general form, due to Witten [23], it is the observation that

$$(4.14) \quad \int_M e^{s \operatorname{D} \lambda(X)} \omega(X) = \int_M \omega(X)$$

for all $s \in \mathbf{C}$, if $\omega, \lambda \in A$, $\operatorname{D} \omega = 0$ and λ is of odd degree. (This makes $\operatorname{D} \lambda(X)$ of even degree, so that its exponential, as a power series, is unambiguously defined.) Indeed, because

$$(4.15) \quad \frac{d}{ds} e^{s \operatorname{D} \lambda} \omega = \operatorname{D} \lambda e^{s \operatorname{D} \lambda} \omega = \operatorname{D}(\lambda e^{s \operatorname{D} \lambda} \omega),$$

its integral over M is equal to zero, which shows that the left hand side in (4.14) is constant as a function of s . Note that the non-polynomial part of $s \mapsto e^{s \operatorname{D} \lambda(X)}$ is given by the exponential function $s \mapsto e^{-s \varphi}$, in which

$$(4.16) \quad \varphi = i(X_M) \lambda(X)^{[1]}.$$

Now we use a G -invariant Riemannian structure β on M and choose

$$(4.17) \quad \lambda(X) := \beta(X_M, \cdot) \in (\mathfrak{g}^* \otimes \Omega^1(M))^G.$$

Then $\varphi = \beta(X_M, X_M)$, and $e^{-s \varphi}$ gets a Gaussian concentration at the zeroset $Z_X = M^T$ of X_M . The right hand side in (4.13) now is equal to the constant term in the asymptotic expansion of (4.14) as $s \in \mathbf{R}$, $s \rightarrow +\infty$. This is easy to prove in the case of isolated fixed points. For the details of the proof in the general case, see [2, pp. 219-223].

5. Hamiltonian Actions

Now assume that M carries a *symplectic form* σ . That is, $\sigma \in \Omega^2(M)$, $d\sigma = 0$, and, for each $x \in M$, σ_x is a nondegenerate antisymmetric bilinear form on $T_x M$. This implies that $\dim M = 2m$ for some integer m . We assume that the action of G on M is *Hamiltonian*, which means that there exists

$$(5.1) \quad \mu \in (\mathfrak{g}^* \otimes \Omega^0(M))^G,$$

such that, for each $X \in \mathfrak{g}$, the vector field X_M is equal to the Hamiltonian vectorfield in M defined by the function $\mu(X)$:

$$(5.2) \quad i(X_M)\sigma = -d(\mu(X)), \quad X \in \mathfrak{g}.$$

This can be summarized in the statement that $D\hat{\sigma} = 0$, if $\hat{\sigma} \in A$ is defined by

$$(5.3) \quad \hat{\sigma}(X) := \sigma - \mu(X), \quad X \in \mathfrak{g}.$$

An immediate consequence is that, for each equivariantly closed form ω , the form

$$(5.4) \quad \alpha(X) := e^{-i\hat{\sigma}(X)}\omega(X)$$

is also equivariantly closed. So the localization formula (4.13) can be applied to write its integral over M as a sum of contributions from the connected components F of Z_X , the zeroset of X_M :

$$(5.5) \quad \begin{aligned} I(X) &:= \int (e^{-i\hat{\sigma}}\omega)(X) = \int_M e^{i\mu(X)} \sum_k \frac{(-i\sigma)^k}{k!} \omega(X)^{[2(m-k)]} \\ &= \sum_F e^{i\langle X, \mu(F) \rangle} r_F(X), \end{aligned}$$

in which

$$(5.6) \quad r_F(X) := \int_F i_F^*(e^{-i\sigma}\omega(X))/\varepsilon(X).$$

Note that Z_X is equal to the set of critical points of the function $\mu(X)$. This also implies that $\mu(X)$ is constant on each connected component F of Z_X , its value on F has been denoted by $\langle X, \mu(F) \rangle$ in (5.5).

The integral on the left hand side of (5.5) is an oscillatory integral with phase function equal to $\mu(X)$. The terms (5.6) coincide with the leading terms of the asymptotic expansion of (5.5) for $X \rightarrow \infty$, given by the method of stationary phase. One says that in this case the method of stationary phase is *exact*. This was observed for $\omega(X) \equiv 1$ in [9]. However, in the next sections we will discuss how the generalization to arbitrary equivariantly closed forms ω can be used in the study of the ring structure of the cohomology of the reduced phase space.

Another observation is that Z_X , being equal to the set of critical points of $\mu(X)$, is always nonvoid. Actually, using the $\mu(X)$ as Morse functions, Ginzburg

[11] proved the very strong statement that integration over M defines a *Poincaré duality* for $\mathbb{H}_G^*(M)$, in the sense that

$$(5.7) \quad [\omega] \mapsto ([\nu] \mapsto \int \omega \nu) : \mathbb{H}_G^*(M) \rightarrow \mathrm{Hom}_I(\mathbb{H}_G^*(M), I)$$

is an *isomorphism of I -modules*. Recall that I stands for the ring of Ad G -invariant polynomials on \mathfrak{g} . This is in extreme contrast with the case that the G -action is locally free, because then $\int_M \omega(X) \equiv 0$ for every equivariantly closed form ω .

6. The Reduced Phase Space

Writing

$$(6.1) \quad \mu(x) : X \mapsto \mu(X)(x) \in \mathfrak{g}^*, \quad x \in M,$$

μ can also be seen as an equivariant mapping from M to \mathfrak{g}^* , this is called the *momentum mapping* of the Hamiltonian action of G on M . We now assume that $0 \in \mathfrak{g}^*$ is a *regular value* of the momentum mapping $\mu : M \rightarrow \mathfrak{g}^*$. This implies that the level set $\mu^{-1}(0)$ is a smooth compact submanifold of M , of codimension equal to $\dim \mathfrak{g}$. It is G -invariant and G acts locally freely on $\mu^{-1}(0)$, so the orbit space

$$(6.2) \quad M_0 := \mu^{-1}(0)/G$$

is an orbifold.

We will write π_0 for the projection $x \mapsto G \cdot x$ from $\mu^{-1}(0)$ to M_0 , and i_0 for the identity from $\mu^{-1}(0)$ to M . Then

$$(6.3) \quad \ker(\mathrm{T}_x \pi_0) = \mathrm{T}_x(G \cdot x) = \ker(i_0^* \sigma_x),$$

and it follows that the unique two-form σ_0 in M_0 , determined by

$$(6.4) \quad i_0^* \sigma = \pi_0^* \sigma_0,$$

is a symplectic form on M_0 . The symplectic orbifold (M_0, σ_0) is called the *Marsden-Weinstein reduced phase space*, at the level 0. This name is inspired by classical mechanics. However, a wealth of examples occur in complex algebraic geometry, where M is a complex projective variety and $M_0 \simeq M//G^{\mathbb{C}}$ is Mumford's geometric quotient by action of the complexification $G^{\mathbb{C}}$ of G , which is a reductive complex algebraic group. See Ness [21, §2]. Also moduli spaces can sometimes be identified with reduced phase spaces.

Using the gradient flow of the function $x \mapsto \|\mu(x)\|^2$ on M , Kirwan [19] proved the fundamental theorem that the first arrow in

$$(6.5) \quad \mathbb{H}_G^*(M) \xrightarrow{i_0^*} \mathbb{H}_G^*(\mu^{-1}(0)) \xleftarrow{\pi_0^*} \mathbb{H}^*(M_0)$$

is *surjective*.

The surjectivity of *Kirwan's homomorphism*

$$(6.6) \quad \kappa_0 := (\pi_0^*)^{-1} \circ i_0^* : \mathbb{H}_G^*(M) \rightarrow \mathbb{H}^*(M_0)$$

raises the hope that the cohomology $\mathbb{H}^*(M_0)$ of the reduced phase space M_0 may be computed from the equivariant cohomology $\mathbb{H}_G^*(M)$ of M . (Not from the ordinary cohomology $\mathbb{H}^*(M)$ of M , which in examples can be much simpler than $\mathbb{H}^*(M_0)$.) In special cases, Kirwan [19] computed the Betti numbers of M_0 in this way.

7. Integration over the Reduced Phase Space

However, also the ring structure of $\mathbb{H}^*(M_0)$ often is very interesting, because the product corresponds to *intersection of cycles*. For any equivariantly closed form ω in M , write

$$(7.1) \quad I_0(\omega) := \int_{M_0} \kappa_0(\omega),$$

for the integral over the reduced phase space of $\kappa_0(\omega)$. Combining the facts that κ_0 is a ring homomorphism and surjective with Poincaré duality in M_0 , we get

$$(7.2) \quad \ker \kappa_0 = \{\omega \in \mathbb{H}_G^*(M) \mid I_0(\omega \nu) = 0 \text{ for all } \nu \in \mathbb{H}_G^*(M)\}.$$

So the *ring*

$$(7.3) \quad \mathbb{H}^*(M_0) \simeq \mathbb{H}_G^*(M) / \ker \kappa_0$$

can be described if the relation

$$(7.4) \quad I_0(\omega \nu) = 0, \quad \omega, \nu \in \mathbb{H}_G^*(M),$$

is known.

In order to get hold of this, Witten [23] showed that (4.14), this time with

$$(7.5) \quad \lambda(X) = \mu(X) \beta(X_M, \cdot),$$

leads to a localization of $I_0(\omega)$ at the critical points of $x \mapsto \|\mu(x)\|^2$. This has been worked out by Wu [24] in the case of a circle action and for $\omega = e^{\hat{\sigma}}$. The result is a formula for the symplectic volume of the reduced phase space, in terms of the fixed points of the circle action.

With a somewhat different proof, Kalkman [18] obtained, also for circle actions but for any $\omega \in \mathbb{H}_G^{\dim M - 2}(M)$, the formula

$$(7.6) \quad \int_{M_0} \kappa_0(\omega) = \sum_{F \mid \mu(F) > 0} \int_F X \iota_F^* \omega(X) / \varepsilon(X).$$

As an application, he computed the ring structure of $\mathbb{H}^*(M_0)$, for a circle action on $M = \mathbf{CP}^n$. (In the sum on the right hand side of (7.6), the condition $\mu(F) > 0$ for the fixed point components may also be replaced by $\mu(F) < 0$, adding a minus sign in front of the sum sign.)

In Kalkman's Ph. D. thesis, (7.6) is proved by observing that $\mu^{-1}(0)$ is the boundary of the domain where $\mu > 0$. Then Stokes' theorem is applied in the complement in this domain of a small tubular neighborhood of the fixed point set. A remarkable feature of this proof is that it works, with $\mu^{-1}(0)$ replaced by ∂M , for an arbitrary (not necessarily Hamiltonian) circle action on any compact oriented manifold with boundary ∂M .

The remainder of this section is an attempt to explain the results of Jeffrey and Kirwan [17]. It contains a generalization of (7.6) to Hamiltonian actions of arbitrary compact Lie groups G . See formula (7.18) below.

The starting point of Jeffrey and Kirwan is the \mathfrak{g} -Fourier transform

$$(7.7) \quad f(\xi) = (\mathcal{F}_{\mathfrak{g}} I)(\xi) = \int_{\mathfrak{g}} \left[\int_M e^{-i(X, \xi - \mu)} e^{-i\sigma} \omega(X) \right] dX$$

of the temperate function $I(X)$ on \mathfrak{g} , which was introduced in (5.5). That is, f is a temperate distribution in \mathfrak{g}^* . Here dX is the Euclidean measure with respect to an $\text{Ad } G$ -invariant inner product in \mathfrak{g} , which in the sequel will also be used in order to identify \mathfrak{g}^* with \mathfrak{g} . Its restriction to the maximal abelian subalgebra \mathfrak{t} defines a Euclidean measure on \mathfrak{t} and an identification of \mathfrak{t}^* with \mathfrak{t} .

Let φ be a test function (smooth and with compact support) on \mathfrak{g}^* . Using the dual measure in \mathfrak{g}^* , interchanging the order of integration and writing

$$(7.8) \quad \omega(X) = \sum_j X^j \omega_j$$

with a multi-index j , we get

$$(7.9) \quad \int_{\mathfrak{g}^*} \varphi(\xi) f(\xi) d\xi = (2\pi)^n \sum_j \int_M (D^j \varphi \circ \mu) e^{-i\sigma} \omega_j.$$

Here $n = \dim \mathfrak{g}$. In other words,

$$(7.10) \quad f = (2\pi)^n \sum_{j,k} (-D)^j \mu_* \left(\frac{(-i\sigma)^k}{k!} \omega_j^{[2(m-k)]} \right).$$

Here μ_* , the transposed of μ^* , denotes the pushforward of measures in M to measures in \mathfrak{g}^* by means of the momentum mapping $\mu : M \rightarrow \mathfrak{g}^*$. It follows that the distribution f is supported by the image of the momentum mapping, a set which is known to intersect \mathfrak{t}^* in a convex polytope, if M is connected. If $\omega = 1$, then f is equal to $(2\pi)^n (-i)^m$ times the pushforward under μ of the canonical (Liouville) measure $\sigma^m/m!$ of M . In particular, it is a measure. For general ω it can be a distribution of arbitrarily high order.

If V is a sufficiently small open neighborhood of 0 in \mathfrak{g}^* , then there exists a G -equivariant retraction ρ from $\mu^{-1}(V)$ onto $\mu^{-1}(0)$ such that $\rho \times \mu$ is a diffeomorphism from $\mu^{-1}(V)$ onto $\mu^{-1}(0) \times V$, and moreover the symplectic form is given by

$$(7.11) \quad \sigma = \rho^* \pi_0^* \sigma_0 + d(\rho^* \theta, \mu).$$

Here θ is a connection form for the locally free G -action on $\mu^{-1}(0)$. This result follows from the normal form of Hamiltonian group actions as obtained by Gotay [12], Marle [20], and Guillemin and Sternberg [15, §41].

Now assume that $\text{supp}(\varphi) \subset V$. Using the normal form and the fact that in $\mu^{-1}(V)$ we may replace $\omega(X)$ by $\rho^* \pi_0^* \kappa_0(\omega)$, one obtains that $\langle \varphi, f \rangle$ is equal to a nonzero universal constant (which involves the volume of the π_0 -fiber) times

$$(7.12) \quad \int_{M_0} \left(\int_{\mathfrak{g}^*} \varphi(\xi) e^{-i(\xi, \Omega)} d\xi \right) e^{-i\sigma_0} \kappa_0(\omega).$$

Here Ω is the curvature form in $\mu^{-1}(0)$ of θ , and we take φ to be $\text{Ad } G$ -invariant in order to obtain that the integral over ξ is a well-defined characteristic form in $M_0 = \mu^{-1}(0)/G$.

It follows that f is equal to an $\text{Ad } G$ -invariant polynomial near the origin in \mathfrak{g}^* . For torus actions and $\omega = 1$, this was actually the way in which it was proved in [9], that the pushforward of the canonical density under the momentum mapping is a piecewise polynomial density in \mathfrak{g}^* . By letting the support of φ shrink to 0, one obtains that the integral of $e^{-i\sigma_0} \kappa_0(\omega)$ over M_0 is equal to a nonzero universal constant times $f(0)$.

The next step is that one would like to use the localization formula (5.5), in order to write $f(0)$ as the sum of contributions from the connected components F of the fixed point set M^T . Now (5.5) is an equation between functions on \mathfrak{t} , so we begin by expressing $f(0)$ in terms of the restriction of I to \mathfrak{t} . Let φ be an $\text{Ad } G^*$ -invariant smooth and compactly supported function in \mathfrak{g}^* with integral equal to one. (Later we shall see that we also could take a Gaussian.) Let

$$(7.13) \quad \psi(X) = \int_{\mathfrak{g}^*} e^{-i\langle X, \xi \rangle} \varphi(\xi) d\xi$$

denote its \mathfrak{g}^* -Fourier transform. ψ is an $\text{Ad } G$ -invariant entire function on the complexification of \mathfrak{g} , satisfying the Paley-Wiener estimates. Note also that $\psi(0) = 1$. Then

$$(7.14) \quad \begin{aligned} f(0) &= \lim_{\epsilon \downarrow 0} \epsilon^{-n} \int_{\mathfrak{g}^*} \varphi(\epsilon^{-1} \xi) (\mathcal{F}_{\mathfrak{g}} I)(\xi) d\xi \\ &= \lim_{\epsilon \downarrow 0} \int_{\mathfrak{g}} \psi(\epsilon X) I(X) dX = c \lim_{\epsilon \downarrow 0} \int_{\mathfrak{t}} \psi(\epsilon X) I(X) \pi(X) dX. \end{aligned}$$

Here c is a universal positive constant and the polynomial $\pi(X) = \pi(-X)$ is equal to the product of all the roots of the Lie algebra \mathfrak{g} with respect to the maximal abelian subalgebra \mathfrak{t} ; these roots are regarded as linear forms on \mathfrak{t} .

The problem which arises now, is that the poles of the rational functions $r_F(X)$ which appear in (5.5) are not locally integrable, so we cannot substitute (5.5) in (7.14) right away. However, using that the integrand in (7.14) is a rapidly decreasing analytic function of X , we can apply Cauchy's integral theorem and replace X by $X + iY$ in the integrand, for any $Y \in \mathfrak{t}$. If Y lies in the complement

$\tilde{\mathfrak{t}}$ of the zeroset of all the weights λ_j , for all j and all F , then we get that $f(0)$ is equal to a nonzero universal constant times the sum over all F of

$$(7.15) \quad \int_{\mathfrak{t}} \psi(\epsilon(X + iY)) e^{i\langle X + iY, \mu(F) \rangle} r_F(X + iY) \pi(X + iY) dX.$$

Because of the Cauchy integral theorem, (7.15) does not change if Y is replaced by any Z in the connected component $C_{F,Y}$ of Y in the complement $\tilde{\mathfrak{t}}_F$ of the weight hyperplanes for the action on the normal bundle of F . Note that $C_{F,Y}$ is an open polyhedral cone, determined by a choice of signs (the same as for Y) of the weights at F . Also, $C_{F,Y}$ does not depend on the choice of Y in the connected component Λ of $\tilde{\mathfrak{t}}$. For this reason, we write $C_{F,\Lambda}$ instead of $C_{F,Y}$, this is just the connected component of $\tilde{\mathfrak{t}}_F$ which contains Λ . Conversely, Λ is equal to the intersection of the chambers $C_{F,\Lambda}$, where F ranges over the connected components of M^T . One might call $C_F = C_{F,\Lambda}$ an *action chamber at F* . The choice of Λ corresponds to a choice $F \mapsto C_F$ of action chambers, such that the intersection of the C_F 's is nonvoid.

If $\langle Z, \mu(F) \rangle > 0$, then the exponential decrease as $t \rightarrow \infty$, which occurs if Z is replaced by tZ , shows that the integral is equal to zero, unless F belongs to

$$(7.16) \quad \mathcal{F}_\Lambda := \{F \mid \langle Z, \mu(F) \rangle \leq 0 \text{ for all } Z \in C_{F,\Lambda}\}.$$

It will be argued below that (7.15) has an asymptotic expansion in integral powers of ϵ as $\epsilon \downarrow 0$; the *constant term* in this expansion will be called the *residue* $\text{Res}_{\varphi,\Lambda}$ of the meromorphic function

$$(7.17) \quad e^{i\langle X, \mu(F) \rangle} \pi(X) r_F(X)$$

of $X \in \mathfrak{t} \otimes \mathbf{C}$. With this notation, we arrive at the following version of the formula of Jeffrey and Kirwan [17, Th. 8.1]:

$$(7.18) \quad \int_{M_0} e^{-i\sigma_0} \kappa_0(\omega) = c \sum_{F \in \mathcal{F}_\Lambda} \text{Res}_{\varphi,\Lambda} [e^{i\langle X, \mu(F) \rangle} \pi(X) r_F(X)] .$$

In order to further investigate the residues, we note that

$$(7.19) \quad X \mapsto r_F(X + itY)$$

converges for $t \downarrow 0$ in the space of temperate distributions on \mathfrak{t} , the limit will be denoted by $r_{F,\Lambda}$. Its \mathfrak{t} -Fourier transform $\mathcal{F}_{\mathfrak{t}} r_{F,\Lambda}$ is a temperate distribution in \mathfrak{t}^* .

In order to express (7.15) in terms of the distribution $\mathcal{F}_{\mathfrak{t}} r_{F,\Lambda}$, it is convenient to write

$$(7.20) \quad \pi(X) = \varpi(X) \varpi(-X) = \pm \varpi(X)^2,$$

in which $\varpi(X)$ denotes the product of a choice of positive roots. We then have, modulo nonzero universal factors:

$$(7.21) \quad \begin{aligned} \varpi(X) \int_{\mathfrak{g}^*} \varphi(\xi) e^{-i\langle X, \xi \rangle} d\xi &= \varpi(X) \int_{\mathfrak{g}^*} \varphi(\xi) \int_{G^0/T} e^{-i\langle X, \text{Adg}^* \xi \rangle} dg d\xi \\ &= \varpi(X) \int_{\mathfrak{t}^*} \varphi(\xi) \int_{G^0/T} e^{-i\langle X, \text{Adg}^* \xi \rangle} dg \pi(\xi) d\xi = \int_{\mathfrak{t}^*} \varphi(\xi) e^{-i\langle X, \xi \rangle} \varpi(\xi) d\xi. \end{aligned}$$

Here we have used the formula

$$(7.22) \quad \int_{G^0/T} e^{-i\langle X, \text{Adg}^* \xi \rangle} dg = \text{const} \sum_{s \in W} \frac{e^{-i\langle X, s^* \xi \rangle}}{\varpi(X) \varpi(s^* \xi)}$$

of Harish-Chandra [16, Corollary]. This can also be viewed as an application of the method of exact stationary phase, cf. Guillemin and Prato [14, Lemma 2.4].

Substituting (7.22) in (7.15), we get that (7.15) is equal to a nonzero universal constant times

$$(7.23) \quad \epsilon^{-n} \int_{\mathfrak{t}^*} \varphi(\epsilon^{-1} \xi) \varpi(\xi) \varpi\left(\frac{\partial}{\partial \xi}\right) (\mathcal{F}_t r_{F, \Lambda})(\xi - \mu_t(F)) d\xi.$$

Here μ_t denotes the momentum mapping for the action of T , so $\mu_t(F) \in \mathfrak{t}^*$ is equal to the restriction of $\mu(F) \in \mathfrak{g}^*$ to \mathfrak{t} . Note that (7.23), for arbitrary φ and $\epsilon = 1$, yields the F -contribution to the whole distribution f , not only to its value at the origin in \mathfrak{g}^* .

The distribution $\mathcal{F}_t r_{F, \Lambda}$ can be described in terms of the convolutions $m_{F, \Lambda}$ of the *halfline measures* m_j , defined by

$$(7.24) \quad \langle \varphi, m_j \rangle = \int_0^\infty \varphi(t \lambda_{j, \Lambda}) dt,$$

where

$$(7.25) \quad \lambda_{j, \Lambda} = \text{sign}\langle Y, \lambda_j \rangle \cdot \lambda_j, \quad Y \in C_{F, \Lambda}.$$

and the λ_j range over the weights of the T -action on the normal bundle of F . In the convolution product, the factors m_j may appear with higher multiplicities, but each has to appear at least once. Such convolutions of halfline measures were introduced by Dufflo, Heckman and Vergne [8]. The support of each such $m_{F, \Lambda}$ is equal to the cone spanned by the $\lambda_{j, \Lambda}$, which in turn is equal to the dual cone $\mathfrak{t}_{F, \Lambda}^*$ of $C_{F, \Lambda}$. It follows from the fact that μ has regular values, that the $\lambda_{j, \Lambda}$ span \mathfrak{t}^* . This implies that $\mathfrak{t}_{F, \Lambda}^*$ has a nonvoid interior and that the measure $m_{F, \Lambda}$ is determined by a locally integrable density, cf. Guillemin, Lerman and Sternberg [13, Prop. 2.4]. Moreover, this density is piecewise polynomial, in the following sense. Let $\mathfrak{t}_{F, \Lambda}^{\text{reg}}$ denote the set of $\eta \in \mathfrak{t}_{F, \Lambda}^*$ which do not belong to a cone spanned by less than $\dim \mathfrak{t}$ of the $\lambda_{j, \Lambda}$. The statement then is that $m_{F, \Lambda}$ is equal to a polynomial in each connected component of $\mathfrak{t}_{F, \Lambda}^{\text{reg}}$, cf. [13, Th. 2.7].

The distribution $\mathcal{F}_t r_{F, \Lambda}$ now can be written as a finite linear combination of derivatives of the $m_{F, \Lambda}$. It follows that the support of $\mathcal{F}_t r_{F, \Lambda}$ is contained in

, F, Λ , and that $\mathcal{F}_t r_{F, \Lambda}$ is equal to a polynomial in each connected component of $\mathfrak{F}_{F, \Lambda}^{\text{reg}}$.

If $-\mu_t(F) \in \mathfrak{F}_{F, \Lambda}^{\text{reg}}$, then we can write

$$(7.26) \quad \text{Res}_{\varphi, \Lambda} [e^{i(X, \mu(F))} \pi(X) r_F(X)] = (\pi(D) \mathcal{F}_t r_{F, \Lambda})(-\mu_t(F)),$$

which is independent of the choice of the test function φ . However, in general the condition that $-\mu_t(F) \in \mathfrak{F}_{F, \Lambda}^{\text{reg}}$ need not hold, one can already find counterexamples for two-dimensional torus actions on \mathbf{CP}^3 .

In general, near $-\mu_t(F)$ the distribution $\mathcal{F}_t r_{F, \Lambda}$ is a linear combination of derivatives of piecewise polynomial densities. Substituting this in (7.23) and transposing all derivatives to $\varphi(\epsilon^{-1}\xi) \varpi(\xi)$ by means of partial integrations, we see that (7.23) has an asymptotic expansion in integral powers of ϵ as $\epsilon \downarrow 0$. The coefficients are equal to sums of integrals over cones $\tilde{\mathfrak{C}}$ of products of polynomials with derivatives of φ . Here the $\tilde{\mathfrak{C}}$ are the cones which near 0 are equal to $\mu_t(F) + \mathfrak{C}$, in which \mathfrak{C} is a connected component of $\mathfrak{F}_{F, \Lambda}^{\text{reg}}$. If $-\mu_t(F) \in \mathfrak{F}_{F, \Lambda}^{\text{reg}}$, then $\tilde{\mathfrak{C}} = \mathfrak{t}^*$ and the derivatives of φ can be transposed to the polynomials by means of partial integrations, but if $-\mu_t(F)$ belongs to the boundary of \mathfrak{C} , then this procedure would lead to additional boundary terms.

In any case, this shows that the residue is always well-defined. It may depend on the choice of φ , although the sum over all F of the residues neither depends on φ , nor on Λ . The description of $\mathcal{F}_t r_{F, \Lambda}$ also shows that, instead of the compactly supported smooth function φ , we could have taken a Gaussian.

The formula (7.18) may be compared with the formula which Guillemin and Prato [14] obtained for f , in the case that $\omega = 1$, the T -fixed points are isolated and their μ_t -images are not in the walls of the Weyl chambers in \mathfrak{t}^* .

Finally, if σ is replaced by $\delta \sigma$, $\delta > 0$, then μ gets replaced by $\delta \mu$ and σ_0 by $\delta \sigma_0$. The local contributions at each F in (7.18) is a polynomial in δ , cf. (5.6) and (7.23). This leads to a formula for $\int_{M_0} \kappa_0(\omega)$ as the sum over F of the constant terms of the local contributions, viewed as polynomials in δ .

One may also note that the topological equivariant cohomology can be defined over \mathbf{Z} . If G acts (locally) freely on $\mu^{-1}(0)$, then κ_0 maps to the integral (rational) cohomology of M_0 , so the explicit computation of the universal factor should confirm that $\int_{M_0} \kappa_0(\omega)$ is integral (rational) for integral equivariant cohomology classes ω .

Further explorations might tell how efficient the formula really is for the computation of the ring structure of the cohomology the reduced phase space. For instance, a natural question is whether this can be used for the computation of the cohomology ring of an arbitrary toric variety, which is a reduced phase spaces for a torus action on a (noncompact) complex vector space. The result may then be compared with the formula of Danilov [7, §10]. In [17], examples have been worked out for the non-Abelian group $G = \text{SU}(2)$.

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