Building Bridges Between Convex Regions*

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Abstract

In the *Euclidean traveling salesman and buyers problem* (TSBP), we are given a set of convex regions in d-dimensional space, and we wish to find a minimum-cost tour that visits all the regions. The cost of a tour depends on the length of the tour itself and on the distance that buyers within each region need to travel to meet the salesman. We show that constant-factor approximations to the TSBP and several similar problems can be obtained by visiting the centers of the smallest enclosing spheres of the regions.

1 Introduction

The Euclidean *traveling salesman and buyers problem* (TSBP) is a generalization of the classical Euclidean traveling salesman problem (TSP). A salesman wants to meet potential buyers, who are scattered in k disjoint convex regions $R_1, R_2, \dots, R_k$ of d-dimensional space. The salesman chooses a *market-place* p_i in each region R_i, where the buyers living in that region will meet him to do business, and a tour visiting all k market-places in turn. We call the maximum distance from all possible buyers in region R_i to the market-place $p_i \in R_i$ the *maximum travel distance* $td(p_i, R_i)$ of region R_i. The cost of a tour is then defined as

$$\ell + \gamma \sum_{i=1}^{k} td(p_i, R_i)$$

where ℓ is the Euclidean length of the tour itself, and $\gamma \geqslant 0$ is a parameter that determines the cost of the buyers' travel relative to the salesman's.

The salesman wants to find a set $\{p_1, \dots, p_k\}$ of market-places and a tour visiting them that minimizes this cost. The usual Euclidean TSP is the special case where each region is a single point, and so the TSBP is NP-hard. The Euclidean TSP with neighborhoods (TSPN) [1, 6, 3] is the special case where $\gamma = 0$: the cost of a tour is simply the length of the tour itself, regardless of the maximum travel distances.

The TSPN in the plane has been studied recently by Dumitrescu and Mitchell [3], who presented a PTAS for the case of disjoint unit disk neighborhoods, and a constant-factor approximation algorithm for arbitrary (possibly overlapping) connected (not necessarily convex) regions with the same diameter. No approximation results appear to be known in more than two dimensions, except for the case of disjoint unit spheres.

*Part of this research was done when the three authors were at Hong Kong University of Science & Technology. It was partially supported by the Hong Kong Research Grants Council.

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We show that a constant factor approximation for the TSBP for any fixed $\gamma > 0$ and any dimension $d \geqslant 2$ can be obtained by choosing the market places at the *center* of each region, which we define as the center of its smallest enclosing sphere.

As the first step in proving this result, we consider the case $k = 2$. We are given two disjoint convex regions that we need to connect using a bridge, such that the diameter of the (now connected) union of the two regions is minimized. The planar case of this *minimum diameter bridge problem* (MDBP) has been first considered in the literature for two convex polygons [2]. Kim and Shin [5] gave a linear time algorithm for this case. Given two convex polyhedra in 3-dimension, Tan [9] gave a quadratic-time algorithm. Recently Tokuyama [10] adapted the parametric search technique [7] to solve min-max optimization problems, and applied this to obtain a linear-time algorithm for the MDBP for convex polytopes in any fixed dimension $d \geqslant 2$. Due to the complexity of the method, this algorithm has presumably only theoretical value. It is also unclear how it could be applied to non-polyhedral convex regions.

We show that the bridge that connects the centers of the two regions has cost at most $\sqrt{2}$ times the optimal cost, for any fixed dimension $d \geqslant 2$.

We then consider two variants of the TSBP studied by Tokuyama [10]. Tokuyama gave linear time algorithms for these variants based on parametric search as well.

In the *geometric network-base location problem* (GNLP), the cost of a set of market places $p_1, \ldots, p_k$ (here called "network-bases") is

$$|\mathrm{MST}(p_1, \ldots, p_k)| + \sum_{i=1}^{k} td(p_i, R_i),$$

where $|\mathrm{MST}(\cdot)|$ is the length of a minimum spanning tree for the points p_i. We prove that choosing the centers of the regions as network-bases results in a cost at most $3\sqrt{2}$ the optimal.

The *minimum diameter spanning tree problem* (MDSTP) for k disjoint convex regions is a generalization of the *minimum diameter spanning tree problem* [4] for points. The task is to construct a *spanning tree of regions*: a node of the tree corresponds to a region, and each edge in the tree connects two regions. The addition of these "bridges" turns the union of regions into a simply-connected set. We wish to choose the bridges such that the diameter of this resulting set is as small as possible. We prove that a solution with cost at most $2\sqrt{2}$ times the optimal cost can be obtained as follows: first construct a minimum spanning tree on the centers of regions, and then build bridges along the edges of this tree.

Our proofs use only the convexity of the regions. If the center of the smallest enclosing sphere for each region is known, no further computation involving the region is necessary to compute an approximate solution to each problem. Note that the smallest enclosing sphere for a convex polytope can be computed in time linear in the number of vertices, in any dimension [11]. Consequently, for a set of k disjoint convex polytopes, all our approximations can be computed in time linear in their total complexity using standard techniques, and we will not discuss algorithms in this paper at all.

2 Preliminaries

For a region A in d-dimensional space, we denote by $\mathrm{int}(A)$, $\mathrm{cl}(A)$, and ∂A the interior, closure, and the boundary of A, respectively. We use $|\cdot|$ to denote the length of a line segment or path, and the total length of all edges of a tour or tree.

Throughout the paper, R denotes a compact convex region in d-dimensional space. The center of the smallest enclosing sphere of R is called its *center*, and denoted $c(R)$. Likewise, we define $r(R)$

to be the radius of R's smallest enclosing sphere. Given a point $p \in R$, we define the *farthest point* $f(p) = f_R(p)$ as the lexicographically smallest point $q \in R$ that maximizes $|pq|$. Note that since R is uniquely determined by the point p, we will usually suppress the subscript. The *maximum travel distance* for p in R is defined as $td(p, R) := |pf_R(p)|$.

3 The minimum diameter bridge problem

The *minimum diameter bridge problem* is formally defined as follows:

> **Problem MDBP:** Given two disjoint convex regions R_1 and R_2. find points $p_i \in R_i$, $i = 1, 2$, such that
>
> $$\Pi(p_1, p_2) := td(p_1, R_1) + |p_1p_2| + td(p_2, R_2)$$
>
> is minimized.

A 2-approximation. Cai et al. [2] showed that the shortest bridge between two convex polygons in the plane has cost at most two times the cost of the optimal bridge. This is in fact true for convex regions in any dimension, as we quickly prove now.

Lemma 1 *Given two disjoint compact convex regions R_1 and R_2 in d-dimensional space, the shortest bridge between them is a 2-approximation to the MDBP.*

Proof. Let $\overline{p}_1\overline{p}_2$ be the shortest bridge for the two regions, and let p_1p_2 be any bridge. Let C_i be the sphere with center p_i and radius $|p_if(p_i)|$. By definition, R_i is contained in C_i, and so $|\overline{p}_if(\overline{p}_i)| \leqslant 2|p_if(p_i)|$. Since $|\overline{p}_1\overline{p}_2| \leqslant |p_1p_2|$, we have $\Pi(\overline{p}_1, \overline{p}_2) \leqslant 2\Pi(p_1, p_2)$. ◻

A $\sqrt{2}$–approximation. We now prove that the bridge connecting the centers of R_1 and R_2 has cost at most $\sqrt{2}$ the optimal cost. We start with a simple lemma.

Lemma 2 *The center of a compact convex region R lies in R.*

Proof. Let S be the minimum enclosing sphere for R. Assume that the center c of S is not in R. Then there is a plane h containing c but not intersecting R. Since R lies completely in the interior of one half-space bounded by h, we can translate S into this half-space such that R is completely contained in its interior. This contradicts the assumption that S is a minimum enclosing sphere for R. ◻

The following lemma is the core of all our results.

Lemma 3 *Let p_1p_2 be a bridge for R_1, R_2, let $c_i := c(R_i)$, $r_i := r(R_i)$, and let p_i' be the point on c_1c_2 closest to p_i, for $i = 1, 2$. Then*

$$|c_ip_i'| \leqslant \frac{1}{2}\sqrt{2} \cdot |p_if(p_i)|, \quad (1)$$

$$|c_ip_i'| + r_i \leqslant \sqrt{2} \cdot |p_if(p_i)|. \quad (2)$$

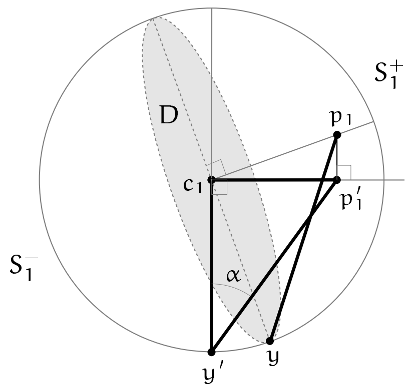

Figure 1: The cross section containing c_1, c_2, and p_1.

Proof. Let S_i be the smallest enclosing sphere for R_i. Figure 1 (b) shows a cross section of the situation containing c_1, c_2 and p_1. By Lemma 2 we have $c_i \in R_i$. Without loss of generality, we prove the inequalities for $i = 1$ only.

Let D be a $(d-1)$-dimensional disk of radius r_1 centered at c_1 and orthogonal to p_1c_1. The disk D divides S_1 into two hemi-spheres S_1^+ (containing p_1) and S_1^- as in Figure 1 (b). Let y be any point on ∂D. Since S is a smallest enclosing sphere of R_1, there must be a point $z \in R_1$ on $\mathrm{cl}(S_1^-)$. We have

$$|p_1y| \leqslant |p_1z| \leqslant |p_1f(p_1)|.$$

Let $y' \in S_1$ be such that $\angle c_2c_1y' = 90°$. Consider the right triangles $\triangle p_1c_1y$ and $\triangle p_1'c_1y'$. Since $|c_1y'| = |c_1y|$ and $|c_1p_1'| \leqslant |c_1p_1|$, we have $|p_1'y'| \leqslant |p_1y| \leqslant |p_1f(p_1)|$.

Let $\alpha := \angle c_1y'p_1'$. Since p_1' lies inside or on the sphere S_1, we have $0 \leqslant \alpha \leqslant 45°$, and so $\sin\alpha \leqslant \frac{1}{2}\sqrt{2}$. Therefore

$$|c_1p_1'| = (\sin\alpha)\cdot|p_1'y'| \leqslant \frac{1}{2}\sqrt{2}\cdot|p_1'y'| \leqslant \frac{1}{2}\sqrt{2}\cdot|p_1f(p_1)|,$$

proving the first inequality. Furthermore,

$$|p_1'c_1| + r_1 = |p_1'c_1| + |c_1y'| = (\sin\alpha + \cos\alpha)\cdot|p_1'y'| \leqslant \sqrt{2}\cdot|p_1'y'| \leqslant \sqrt{2}\cdot|p_1f(p_1)|,$$

which completes the proof. ∎

The main result of this section is the following theorem.

Theorem 1 *Let p_1p_2 be a bridge for R_1, R_2, and let $c_i := c(R_i)$. Then $\Pi(c_1,c_2) \leqslant \sqrt{2}\cdot\Pi(p_1,p_2)$. In other words, the bridge connecting the centers of the two regions is a $\sqrt{2}$-approximation to the MDBP. The bound is tight.*

Proof. Let p_i' be the point on c_1c_2 closest to p_i, for $i = 1, 2$. We have $|p_1'p_2'| \leqslant |p_1p_2|$. We can now apply equation (2) as follows:

$$\begin{aligned}
\Pi(c_1,c_2) &= r_1 + |c_1c_2| + r_2 \\
&= r_1 + (|c_1p_1'| + |p_1'p_2'| + |p_2'c_2|) + r_2 \\
&\leqslant \sqrt{2}\cdot|p_1f(p_1)| + |p_1p_2| + \sqrt{2}\cdot|p_2f(p_2)| \\
&\leqslant \sqrt{2}\cdot\Pi(p_1,p_2).
\end{aligned}$$

Figure 2 shows a lower bound example that proves that this bound is tight. Here R_1 and R_2 are tetrahedra in 3-space such that their centers lie on the midpoint of the longest edge of R_1 and R_2, respectively. We assume the radii of two spheres to be unit, and their distance to be an arbitrarily small $\epsilon > 0$. The optimal bridge is p^*q^*, its cost is at most $2\sqrt{2} + \epsilon$. The bridge connecting the two centers has cost $4 + \epsilon$. ◻

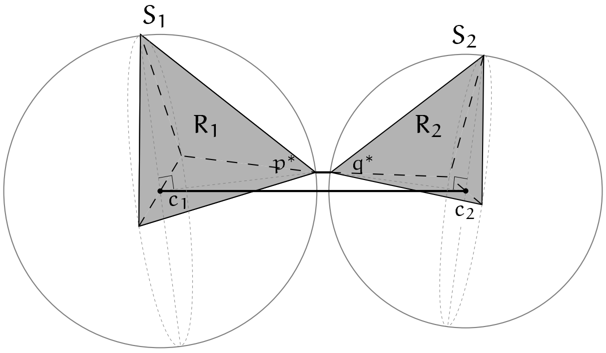

Figure 2: A tight lower bound example.

Obviously, the endpoints of the optimal bridge for two convex regions must lie on the boundaries of the two regions. This leads us to a heuristic improvement to our approximation method: Instead of connecting the two centers directly, we use the bridge p_1p_2, where p_i is the intersection of the segment c_1c_2 with the boundary of R_i. Clearly the cost of this bridge is no worse than that of c_1c_2, and so it is again a $\sqrt{2}$-approximation to the MDBP.

4 The traveling salesman and buyers problem

We now generalize our approximation result to problems involving more than two regions. We start with the *Euclidean traveling salesman and buyers problem* (TSBP):

> **Problem TSBP:** Given a set of disjoint compact convex regions R_i for $i = 1, 2, \ldots, k$, find points $p_i \in R_i$, $i = 1, \ldots, k$ such that
>
> $$\Pi_T(p_1, p_2, ..., p_k) = |TSP(p_1, p_2, \ldots, p_k)| + \gamma \sum_{i=1}^{k} td(p_i, R_i)$$
>
> is minimized. Here $\gamma \geqslant 0$ is a parameter defining the relative weight of the salesman's and the buyers' travel, and $|TSP(\cdot)|$ is the cost of an optimal TSP for the points.

Since the TSBP is a generalization of the Euclidean TSP for points, it is NP-hard. The traveling salesman problem with neighborhoods (TSPN) is the special case where $\gamma = 0$.

Theorem 2 *Let* $c_i := c(R_i)$, *for* $i = 1, \ldots, k$, *and let* $p_i \in R_i$, $i = 1, \ldots, k$. *Then*

$$\Pi_T(c_1, c_2, \ldots, c_k) \leqslant C_\gamma \cdot \Pi_T(p_1, p_2, \ldots, p_k),$$

where $C_\gamma = \gamma\sqrt{2}/2$ *for* $\gamma \geqslant 2$ *and* $C_\gamma = 3\sqrt{2}/\min(2, 2\gamma)$ *for* $\gamma < 2$. *In other words, the shortest TSP tour of the region centers is a constant factor approximation to the TSBP for any fixed* $\gamma > 0$.

Proof. Let T be the optimal TSP tour of $p_1, \ldots, p_k$. We assume without loss of generality that T visits the points in the order $p_1, p_2, \ldots, p_k, p_1$. Let now T' be the tour visiting $c_1, c_2, \ldots, c_k, c_1$ in this order. To simplify the notation, we let $R_{k+1} := R_1$, $p_{k+1} := p_1$, $c_{k+1} := c_1$. We distinguish two cases.

If $\gamma \geqslant 2$, we employ Theorem 1.

$$\begin{aligned}
\Pi_T(c_1, c_2, \ldots, c_k) &= |\mathrm{TSP}(c_1, c_2, \ldots, c_k)| + \gamma \sum_{i=1}^{k} td(c_i, R_i) \\
&\leqslant |T'| + \gamma \sum_{i=1}^{k} td(c_i, R_i) \\
&= \sum_{i=1}^{k} \left(\frac{\gamma}{2} td(c_i, R_i) + |c_i c_{i+1}| + \frac{\gamma}{2} td(c_{i+1}, R_{i+1}) \right) \\
&\leqslant \frac{\gamma}{2} \sum_{i=1}^{k} \left(td(c_i, R_i) + |c_i c_{i+1}| + td(c_{i+1}, R_{i+1}) \right) \\
&\leqslant \frac{\gamma}{2} \sum_{i=1}^{k} \sqrt{2} \left(td(p_i, R_i) + |p_i p_{i+1}| + td(p_{i+1}, R_{i+1}) \right) \\
&\leqslant \frac{\gamma}{2} \sqrt{2} \cdot \sum_{i=1}^{k} \left(\frac{\gamma}{2} td(p_i, R_i) + |p_i p_{i+1}| + \frac{\gamma}{2} td(p_{i+1}, R_{i+1}) \right) \\
&= \frac{\gamma}{2} \sqrt{2} \cdot \Pi_T(p_1, p_2, \ldots, p_k)
\end{aligned}$$

If, on the other hand, $\gamma < 2$, we employ equations (1) and (2) directly. Let p_i' and p_i'' be the points on $c_i c_{i+1}$ closest to p_i and p_{i+1}, respectively, and recall that $|p_i' p_i''| \leqslant |p_i p_{i+1}|$.

$$\begin{aligned}
\Pi_T(c_1, c_2, \ldots, c_k) &\leqslant |T'| + \gamma \sum_{i=1}^{k} td(c_i, R_i) \\
&= \sum_{i=1}^{k} \left(\gamma td(c_i, R_i) + |c_i c_{i+1}| \right) \\
&\leqslant \max(1, \gamma) \sum_{i=1}^{k} \left(r_i + |c_i p_i'| + |p_i' p_i''| + |p_i'' c_{i+1}| \right) \\
&\leqslant \max(1, \gamma) \sum_{i=1}^{k} \left(\sqrt{2} |p_i f(p_i)| + |p_i p_{i+1}| + \frac{1}{\sqrt{2}} |p_{i+1} f(p_{i+1})| \right) \\
&= \max(1, \gamma) \sum_{i=1}^{k} \left((\sqrt{2} + \frac{1}{\sqrt{2}}) td(p_i, R_i) + |p_i p_{i+1}| \right) \\
&\leqslant \max(1, \gamma) \cdot \frac{3\sqrt{2}}{2\gamma} \cdot \sum_{i=1}^{k} \left(\gamma td(p_i, R_i) + |p_i p_{i+1}| \right) \\
&= \frac{3\sqrt{2}}{2 \min(1, \gamma)} \cdot \Pi_T(p_1, \ldots, p_k)
\end{aligned}$$

Here we've used that $1 < 3\sqrt{2}/2\gamma$ for $\gamma < 2$. ◻

The most interesting cases are probably $\gamma = 2$ (taking into account that buyers need to make a roundtrip) and $\gamma = 1$. The approximation factors for these cases are $C_2 = \sqrt{2} \approx 1.41$ and $C_1 = 1.5\sqrt{2} \approx 2.12$. Note that we do not obtain a constant approximation factor for the case $\gamma = 0$ (the traveling salesman problem with neighborhoods).

Figure 3 shows a lower bound example for the TSBP problem: There are three right-angled, isosceles triangles in the plane. The three right-angled vertices x_1, x_2 and x_3 are very close to each other. The minimum enclosing sphere for each triangle is a unit circle. The centers c_1, c_2, c_3 are the midpoints of the long edges of the triangles. We have

$$\begin{aligned} \Pi_T(x_1, x_2, x_3) &= 3\gamma\sqrt{2} + \epsilon, \\ \Pi_T(c_1, c_2, c_3) &= 3\sqrt{3} + 3\gamma, \end{aligned}$$

and so the approximation factor is at least 1.93 for $\gamma = 1$ and 1.31 for $\gamma = 2$.

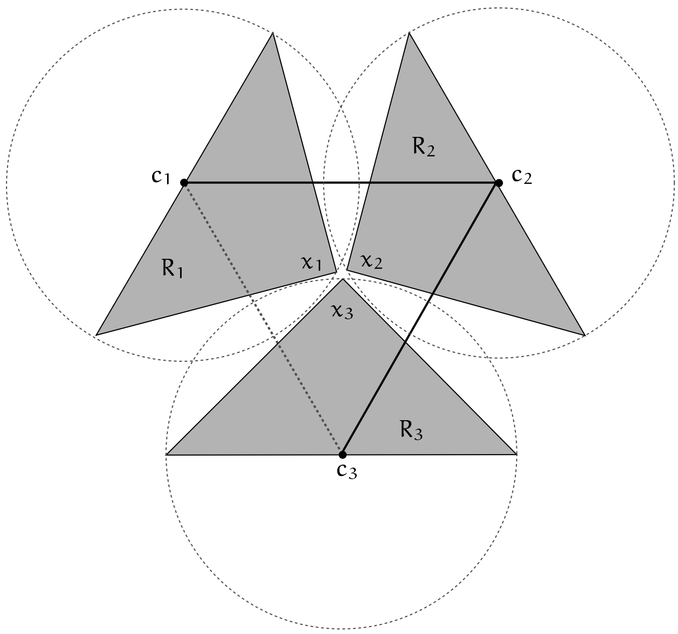

Figure 3: An example giving lower bounds for the TSBP and the GNLP.

5 The geometric network-base location problem

The geometric network-base location problem is defined as follows:

Problem GNLP: Given a set of disjoint compact convex regions R_i for $i = 1, 2, \ldots, k$, find points $p_i \in R_i$, $i = 1, \ldots, k$ such that

$$\Pi_N(p_1, p_2, ..., p_k) = |\text{MST}(p_1, p_2, \ldots, p_k)| + \sum_{i=1}^{k} td(p_i, R_i)$$

is minimized. Here $|\text{MST}(\cdot)|$ is the cost of a minimum spanning tree of the points.

Theorem 3 *Let* $c_i := c(R_i)$, *for* $i = 1, \ldots, k$, *and let* $p_i \in R_i$, $i = 1, \ldots, k$. *Then*

$$\Pi_N(c_1, c_2, \ldots, c_k) \leqslant 3\sqrt{2} \cdot \Pi_N(p_1, p_2, \ldots, p_k).$$

In other words, a MST of the region centers is a $3\sqrt{2}$*-approximation to the GNLP.*

Proof. Let T be a tour of $p_1, \ldots, p_k$ such that $|T| \leqslant 2|MST(p_1, \ldots, p_k)|$. Such a tour can be obtained from an Euler tour of the MST [8]. Without loss of generality, we assume that T visits the points in the order $p_1, p_2, \ldots, p_k, p_1$, and we define T' to be the tour visiting $c_1, c_2, \ldots, c_k, c_1$ in this order. We apply the argument from the proof of Theorem 2 for $\gamma = 1$:

$$\begin{aligned}
\Pi_N(c_1, \ldots, c_k) &\leqslant |T'| + \sum_{i=1}^{k} td(c_i, R_i) \\
&\leqslant \frac{3\sqrt{2}}{2}\left(|T| + \sum_{i=1}^{k} td(p_i, R_i)\right) \\
&\leqslant \frac{3\sqrt{2}}{2}\left(2|MST(p_1, \ldots, p_k)| + \sum_{i=1}^{k} td(p_i, R_i)\right) \\
&\leqslant 3\sqrt{2} \cdot \Pi_N(p_1, \ldots, p_k).
\end{aligned}$$

□

The example of Figure 3 also serves as a lower bound for our GNLP approximation. We have

$$\begin{aligned}
\Pi_N(x_1, x_2, x_3) &= 3\sqrt{2} + \epsilon, \\
\Pi_N(c_1, c_2, c_3) &= 2\sqrt{3} + 3 + \epsilon,
\end{aligned}$$

and so the approximation factor is at least 1.52.

6 The minimum diameter spanning tree problem

The *minimum diameter spanning tree problem* (MDSTP) is a generalization of the problem for points considered by Ho et al. [4]. It is defined as follows.

Problem MDSTP: Given disjoint compact convex regions R_i, $i = 1, \ldots, k$, find a set S of $k-1$ bridges connecting pairs of regions such that $U(S) := S \cup \bigcup_{i=1}^{k} R_i$ is simply connected and such that

$$\Pi_S(S) := \max_{p,q \in U(S)} |\pi_S(p, q)|,$$

is minimized, where $\pi_S(p, q)$ is the shortest path in $U(S)$ connecting p and q.

In other words, the nodes of the tree to be build are the regions, and an edge in the tree connects two regions. Different bridges incident to a region can have different end points.

Theorem 4 *Let* $c_i := c(R_i)$, *for* $i = 1, \ldots, k$. *For* $j = 2, \ldots, k$, *build a bridge connecting* R_1 *and* R_j *along the line segment* c_1c_j. *The resulting tree is a* $2\sqrt{2}$*-approximation to the MDSTP.*

Proof. Let S be the set of bridges constructed, and let $p, q \in \mathcal{U}(S)$ be such that $\Pi_S(S) = |\pi_S(p, q)|$. Assume $p \in R_i$, $q \in R_j$, and let $p_1 p_i$, $q_1 q_j$ be the minimum diameter bridges for the pairs (R_1, R_i) and (R_1, R_j), respectively. By Theorem 1 we have

$$
\begin{aligned}
|\pi_S(pc_1)| &\leqslant \sqrt{2}(|p_1 f(p_1)| + |p_1 p_i| + |p_i f(p_i)|) \leqslant \sqrt{2} \cdot \Pi_S(S^*) \\
|\pi_S(qc_1)| &\leqslant \sqrt{2}(|q_1 f(q_1)| + |q_1 q_j| + |q_j f(q_j)|) \leqslant \sqrt{2} \cdot \Pi_S(S^*),
\end{aligned}
$$

where S^* is an optimal solution to the MDSTP. It follows that $|\pi_S(pq)| \leqslant 2\sqrt{2} \cdot \Pi_S(S^*)$. ◻

The reader may be surprised that our approximation simply connects all regions to the one region R_1, creating a tree of link diameter 2 (where the link diameter is the maximum number of edges of a path in the tree). The construction is less surprising if one knows that the optimal MDST for a set of points has link diameter 2 or 3 [4].

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