

# Phase space quantization and Loop Quantum Cosmology: A Wigner function for the Bohr-compactified real line

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## Abstract

We give a definition for the Wigner function for quantum mechanics on the Bohr compactification of the real line and prove a number of simple consequences of this definition. We then discuss how this formalism can be applied to loop quantum cosmology. As an example, we use the Wigner function to give a new quantization of an important building block of the Hamiltonian constraint.

## 1 Introduction

The Wigner function [1] has long been recognized as a tool in quantum mechanics. For a wave function  $\Psi(x)$  on the real line it is defined as

$$W(\Psi)(x, p) \doteq \int \overline{\Psi}\left(x + \frac{1}{2}x'\right) \Psi\left(x - \frac{1}{2}x'\right) e^{ipx'} dx'.$$

It is a function on phase space that comes, in a certain sense, as close to being a classical probability distribution corresponding to  $\Psi$  on phase space as possible. It can therefore be used to analyze the extent to which a given quantum state can be described in classical terms. Furthermore the Wigner function figures prominently in Weyl quantization, a map that assigns symmetric operators to real functions (subject to smoothness and fall-off criteria) on phase space in a systematic fashion. In physics parlance,

Weyl quantization is referred to as *totally symmetric ordering*. A comprehensive mathematical treatment of the Wigner function and its properties in quantum mechanics can be found in [2].

Loop quantum cosmology (LQC for short, see [3] for a review) is a theory of quantum cosmology developed in close connection with loop quantum gravity [4, 5, 6], and can be viewed as a symmetry reduced version of the latter. It has been used as a testbed for techniques used in loop quantum gravity, but it can be argued that it also makes physical predictions in its own right. One large set of results shows that the classical singularities of cosmology are resolved in the quantum theory. As an example, we refer to [7] for a beautiful result in this direction.

Technically LQC started out as a quantum theory on a circle, but it was later realized that it is actually more appropriately formulated as a quantum theory on the Bohr compactification  $\mathbb{R}_B$  of the real line. While we will review some of the mathematics of  $\mathbb{R}_B$  and of the functions on this space in section 3 below, we refer to [8] for a good overview over both mathematical and physical aspects of these developments.

Given that quantum mechanics on  $\mathbb{R}_B$  is the foundation for LQC, it is an interesting question whether the Wigner function can be generalized to this setting. A look at the literature shows that the Wigner function can be, and has been, generalized in a number of ways, for example to quantum mechanics on  $U(1)$ , and more generally, certain non-Abelian groups (see for example [9, 10]). In fact, the Wigner function for  $U(1)$  has made a brief appearance in LQG [11], where it was used in the study of the semiclassical limit. In these generalizations, the role of Fourier analysis is played by its natural generalizations for harmonic analysis on groups (Pontryagin duality in the Abelian case, Peter-Weyl theory for compact non-Abelian groups). It must however be said that such generalizations are generically neither unique, nor do they share all the properties of the Wigner function on  $\mathbb{R}$ .

To the best of our knowledge however, a generalization of the Wigner function to  $\mathbb{R}_B$  has not yet been considered. The present paper intends to fill this gap. As it turns out, the generalization of the Wigner function to  $\mathbb{R}_B$  is quite straightforward. What is more, its properties mirror that of its cousin on  $\mathbb{R}$  extremely closely. This is due on the one hand to the Abelian nature of  $\mathbb{R}_B$ , on the other hand to a useful property of the Pontryagin dual of  $\mathbb{R}_B$ , namely that the operation of “taking a square root” with respect to its group product is well defined. Such square roots (or divisions by two, in additive notation) will naturally show up when proving properties of the Wigner function.

We will also demonstrate the applicability of the Wigner function to issues in LQC. In particular we will use it to obtain the Weyl quantization of the

modified holonomy from [12], which in turn could be used to define a modified quantum dynamics. We will compare the properties of this quantization to the standard one, but we will not yet use it to complete the quantization of the Hamiltonian constraint and attempt an analysis of the physical differences that would result.

We should say that there are other conceivable applications of the Wigner functions besides the one we demonstrate in this paper. To give an example we recall that recently a method has been established in LQC that allows to calculate effective equations of motion within a systematic approximation scheme [13, 14]. Weyl ordering figures prominently in this method and thus we expect that the Wigner function techniques from the present paper may also be useful in that context.

The paper is organized as follows: We start by giving a brief review of the properties of the Wigner function in ordinary quantum mechanics in section 2. In section 3 we generalize its definition to quantum mechanics on the Bohr compactification  $\mathbb{R}_B$ , and list analogous properties. In section 4 we sketch an application to LQC. We finish with a discussion of our results and the possibility of a generalization to loop quantum gravity in section 5.

## 2 The Wigner function on $\mathbb{R}$

In the present section we will recall the definition of the Wigner function for quantum mechanics on the real line. We will follow closely the exposition in [2] (and refer to it for proofs and details) although with slightly different conventions. Our Fourier transform convention will be

$$\tilde{f}(k) \doteq \int f(x) e^{-ikx} dx, \quad f(x) = \int \tilde{f}(k) e^{ikx} \frac{dk}{2\pi};$$

we write the scalar product on  $L^2(\mathbb{R}, dx)$  as  $\langle \cdot, \cdot \rangle$ , and denote the Schwartz test functions on  $\mathbb{R}$  by  $\mathcal{S}(\mathbb{R})$ . We introduce the usual position and momentum operators  $\mathbf{X}$  and  $\mathbf{P}$

$$\mathbf{X}\Psi(x) = x\Psi(x) \quad \text{and} \quad \mathbf{P}\Psi(x) = \frac{1}{i} \frac{d}{dx} \Psi(x), \quad (1)$$

for  $\Psi \in \mathcal{S}(\mathbb{R})$ .

The Wigner function  $W(\Psi)$  of a wave function is conveniently defined as a special case  $W(\Psi) = W(\Psi, \Psi)$  of the Wigner transform  $W(\Psi, \Psi')$  of a *pair*

of wave functions, in turn defined by the following equivalent expressions:

$$W(\Psi, \Psi')(x, p) \doteq \int \overline{\Psi}(x + x'/2) \Psi'(x - x'/2) e^{ipx'} dx' \quad (2)$$

$$= 2 \int \overline{\tilde{\Psi}}(k) \tilde{\Psi}'(2p - k) e^{2ix(p-k)} \frac{dk}{2\pi} \quad (3)$$

$$= \int \overline{\tilde{\Psi}}(p - q/2) \tilde{\Psi}'(p + q/2) e^{ixq} \frac{dq}{2\pi} \quad (4)$$

Some of its basic properties are as follows (see Props. 1.92 and 1.96 in [2]):

**Proposition 2.1.** *Eq.(2) defines the Wigner transform as a sesquilinear map between the following spaces:*

$$\begin{aligned} \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) &\longrightarrow \mathcal{S}(\mathbb{R}^2) \\ W : L^2(\mathbb{R}) \times L^2(\mathbb{R}) &\longrightarrow L^2(\mathbb{R}^2) \cap C_\infty(\mathbb{R}^2) \\ \mathcal{S}'(\mathbb{R}) \times \mathcal{S}'(\mathbb{R}) &\longrightarrow \mathcal{S}'(\mathbb{R}^2), \end{aligned}$$

with the last of these a continuous extension of the others, and  $C_\infty(\mathbb{R}^2)$  denoting continuous functions vanishing at infinity. On  $L^2(\mathbb{R}) \times L^2(\mathbb{R})$  the Wigner transform has the overlap property

$$\iint \overline{W(\Psi_1, \Psi_2)} W(\Phi_1, \Phi_2) \frac{dx dp}{2\pi} = \overline{\langle \Psi_1, \Phi_1 \rangle} \langle \Psi_2, \Phi_2 \rangle \quad (5)$$

and is hermitian:

$$\overline{W(\Psi, \Psi')} = W(\Psi', \Psi). \quad (6)$$

Furthermore, for  $\Psi \in \mathcal{S}(\mathbb{R})$ ,  $W(\Psi, \Psi)$  has marginal distributions

$$\int W(\Psi, \Psi)(x, k) dx = |\tilde{\Psi}|^2(k), \quad \int W(\Psi, \Psi)(x, k) \frac{dk}{2\pi} = |\Psi|^2(x) \quad (7)$$

and the same is true (modulo technical details) for  $\Psi \in L^2(\mathbb{R})$ .

We note in particular that according to (6),  $W(\Psi, \Psi)$  is real. Moreover, its “marginals” are the quantum mechanical probability distributions for measurements of position and momentum according to (7). However,  $W(\Psi, \Psi)$  fails to be a joint probability distribution because it is not positive in general. It is positive precisely for Gaussian states, arguably the most classical quantum states:

**Proposition 2.2** (Hudson’s theorem [15]). *For  $\Psi \in L^2(\mathbb{R})$ ,  $\Psi \neq 0$  it holds that*

$$W(\Psi, \Psi) \geq 0 \quad \Longleftrightarrow \quad \Psi \text{ is a Gaussian.}$$

In this sense the Wigner function is as close to a classical joint probability distribution in position and momentum as quantum mechanics allows.

The Wigner function can also be used in quantization: As per (7),

$$\begin{aligned}\iint x W(\Psi, \Psi')(x, p) \frac{dx dp}{2\pi} &= \langle \Psi, \mathbf{X} \Psi' \rangle, \\ \iint p W(\Psi, \Psi')(x, p) \frac{dx dp}{2\pi} &= \langle \Psi, \mathbf{P} \Psi' \rangle\end{aligned}$$

whenever  $\Psi'$  is in the domain of the respective operators. Also it is easy to calculate

$$\iint xp W(\Psi, \Psi')(x, p) \frac{dx dp}{2\pi} = \left\langle \Psi, \frac{1}{2}(\mathbf{P}\mathbf{X} + \mathbf{X}\mathbf{P})\Psi' \right\rangle$$

which shows a relation to quantization by symmetric ordering. In fact this relation carries on much further: For phase space functions  $\sigma(x, p)$  in  $L^1(\mathbb{R}^2)$ , we may define a quantization  $\sigma$  of  $\sigma$  by the Bochner integral

$$\sigma \doteq \int \tilde{\sigma}(p, x) \exp[i(p\mathbf{X} + x\mathbf{P})] \frac{dx dp}{2\pi}$$

where

$$\tilde{\sigma}(p, x) \doteq \int \sigma(x', p') e^{-i(px' + xp')} \frac{dx' dp'}{2\pi}.$$

One can even extend this definition to  $\sigma(x, p)$  which are in  $\mathcal{S}'(\mathbb{R}^2)$ . The matrix elements of this operator are determined in a very simple manner from the Wigner function and the symbol  $\sigma$ :

**Proposition 2.3** (Prop. (2.5) in [2]). *For  $\sigma \in \mathcal{S}'(\mathbb{R}^2)$  and  $\Psi, \Psi' \in \mathcal{S}(\mathbb{R})$ ,*

$$\langle \Psi, \sigma \Psi' \rangle = \iint \sigma(x, p) W(\Psi, \Psi')(x, p) \frac{dx dp}{2\pi}.$$

After this brief exposition of the properties of the Wigner function on  $\mathbb{R}$ , we can turn now to the actual topic of this paper.

### 3 The Wigner function on $\mathbb{R}_B$

#### 3.1 The Bohr compactification $\mathbb{R}_B$

In the present section we define the Wigner function for wave functions on the Bohr compactification of the real line, and derive some of its properties. We will start by recalling some basic facts about harmonic analysis on Abelian groups and the definition of  $\mathbb{R}_B$ . A good reference for these matters is [16].

Given any locally compact Abelian group  $G$ , one can form the dual group  $\widehat{G}$  as the Abelian group of (continuous) characters of  $G$ . Multiplication in

$\widehat{G}$  is given by pointwise multiplication of characters, the inverse by complex conjugation, and the topology by uniform convergence on compact sets. With this topology  $\widehat{G}$  itself becomes a locally compact group. There is a natural isomorphism between  $G$  and its double-dual.

As locally compact Abelian groups,  $G$  and  $\widehat{G}$  have unique (up to scaling) Haar measures  $d\mu$ ,  $d\widehat{\mu}$ . Fourier transform can be defined as

$$\widehat{f}(\chi) = \int_G d\mu f(x) \overline{\chi}(x)$$

for a character  $\chi$  of  $G$ . The normalization of the Haar measures can be chosen such that Fourier transform becomes an isomorphism

$$L^2(G, d\mu) \longrightarrow L^2(\widehat{G}, d\widehat{\mu}).$$

A locally compact Abelian group  $G$  is compact iff  $\widehat{G}$  is discrete. This is used to define the Bohr compactification  $B(G)$  of a locally compact group  $G$ :  $B(G)$  is defined as the dual group of  $\widehat{G}_{\text{discr}}$ , with the latter being  $\widehat{G}$  as far as group structure is concerned, but equipped with the *discrete* topology. For the reals this works out as follows.

Let  $G$  be the additive group of real numbers  $G = (\mathbb{R}, +)$  with its usual topology, which we think of as the configuration space of ordinary quantum mechanics in one dimension. The characters of  $G$  are precisely the functions  $h_\mu : G \rightarrow \mathbb{C}$  given by

$$h_\mu(c) = \exp[i\mu c] \tag{8}$$

labelled by  $\mu \in \mathbb{R}$ , and form a group  $\widehat{G}$  isomorphic to  $(\mathbb{R}, +)$  with the usual topology, i.e., the usual momentum space. Thus  $\widehat{G}_{\text{discr}}$  is the additive group of real numbers with the discrete topology. Since this group is discrete and Abelian, its characters form a compact Abelian group  $B(\mathbb{R})$ , which we will also denote  $\mathbb{R}_B$ . Now each real number  $c$  defines an obvious character  $\widehat{\mathbb{R}}_B \ni \mu \mapsto h_\mu(c) \in \mathbb{C}$  of  $\widehat{\mathbb{R}}_B$  and this correspondence embeds  $(\mathbb{R}, +)$  as a dense subgroup of  $\mathbb{R}_B$ , justifying the description of  $\mathbb{R}_B$  as a compactification of the real line. The reason that there are more characters of  $\widehat{\mathbb{R}}_B$  than of  $\widehat{G}$  is that there is now no continuity requirement in the  $\mu$  variable (or more precisely, continuity is required with respect to the discrete topology). It will be convenient to denote the character on  $\widehat{\mathbb{R}}_B$  corresponding to any  $c \in \mathbb{R}_B$  by  $\mu \mapsto h_\mu(c)$ ; on the other hand the maps  $c \mapsto h_\mu(c)$  for  $\mu \in \mathbb{R}$  define characters on  $\mathbb{R}_B$ , which continuously extend the formula (8) from  $\mathbb{R}$  to  $\mathbb{R}_B$ .

Both  $\mathbb{R}_B$  and  $\widehat{\mathbb{R}}_B$  carry Haar-measures  $dc$ ,  $d\mu$ :  $d\mu$  is just the counting measure on  $\mathbb{R}$ ,

$$\int_{\widehat{\mathbb{R}}_B} \widehat{f}_\mu d\mu = \sum_{\mu \in \mathbb{R}} \widehat{f}_\mu$$

and  $dc$  is characterized by

$$\int_{\mathbb{R}_B} h_\mu(c) dc = \delta_{\mu,0}.$$

Fourier transformation

$$\widehat{f}_{\mu'} \doteq \int f(c) h_{-\mu'}(c) dc$$

is an isomorphism  $L^2(\mathbb{R}_B, dc) \rightarrow L^2(\widehat{\mathbb{R}}_B, d\mu)$ . The characters  $h_\mu(\cdot)$  form an uncountable orthonormal basis in  $\mathcal{H} = L^2(\mathbb{R}_B, dc)$ , which is therefore inseparable. We will also use the Hilbert spaces  $L^2(\mathbb{R}_B \times \widehat{\mathbb{R}}_B, dc d\mu)$  and  $L^2(\widehat{\mathbb{R}}_B \times \widehat{\mathbb{R}}_B, d\mu d\mu)$  which are isomorphic under the partial Fourier transform in the first variable

$$\widehat{F}(\nu, \mu) = \int_{\mathbb{R}_B} F(c, \mu) h_{-\nu}(c) dc.$$

Let us define some further function spaces (which roughly correspond to the Schwartz spaces occurring in the theory on  $\mathbb{R}$ ).

**Definition 3.1.** *Denote by*

$$\begin{aligned} \text{Cyl}(\mathbb{R}_B) : & \quad \text{the finite span of characters on } \mathbb{R}_B, \\ \text{Cyl}(\widehat{\mathbb{R}}_B) : & \quad \text{the image of } \text{Cyl}(\mathbb{R}_B) \text{ under Fourier transform,} \\ \text{Cyl}(\mathbb{R}_B \times \widehat{\mathbb{R}}_B) : & \quad \text{the algebraic tensor product } \text{Cyl}(\mathbb{R}_B) \otimes \text{Cyl}(\widehat{\mathbb{R}}_B). \end{aligned}$$

Some remarks about this definition: first, any element  $\Psi \in \text{Cyl}(\mathbb{R}_B)$  may be written as a finite sum

$$\Psi = \sum_{\mu \in \mathbb{R}} \widehat{\Psi}_\mu h_\mu$$

where the Fourier coefficients

$$\widehat{\Psi}_\mu = \int \Psi(c) \overline{h_\mu(c)} dc$$

of  $\Psi$  vanish for all but finitely many  $\mu \in \mathbb{R}$ . Accordingly,  $\text{Cyl}(\widehat{\mathbb{R}}_B)$  consists of all complex-valued functions on  $\mathbb{R}$  which are nonzero only at finitely many points. Second,  $\text{Cyl}(\mathbb{R}_B \times \widehat{\mathbb{R}}_B)$ , the finite span of functions  $f_1 \otimes f_2$  on  $\mathbb{R}_B \times \widehat{\mathbb{R}}_B$  with  $f_1 \in \text{Cyl}(\mathbb{R}_B)$ ,  $f_2 \in \text{Cyl}(\widehat{\mathbb{R}}_B)$ , can be described equivalently as the set of functions  $f(c, \lambda)$  on  $\mathbb{R}_B \times \widehat{\mathbb{R}}_B$  that are in  $\text{Cyl}(\mathbb{R}_B)$  for fixed  $\lambda$  and in  $\text{Cyl}(\widehat{\mathbb{R}}_B)$  for fixed  $c$ .  $\text{Cyl}(\mathbb{R}_B \times \widehat{\mathbb{R}}_B)$  is a  $*$ -algebra under pointwise linear combination, products and complex conjugation; its elements are absolutely integrable with respect to the product measure  $dc d\mu$  on  $\mathbb{R}_B \times \widehat{\mathbb{R}}_B$  and hence can be integrated as nested integrals in either order.

We shall also make use of the algebraic duals of these spaces, which can be easily characterised: the dual  $\text{Cyl}(\widehat{\mathbb{R}}_B)^*$  of  $\text{Cyl}(\widehat{\mathbb{R}}_B)$  consists of all functionals  $f : \text{Cyl}(\widehat{\mathbb{R}}_B) \rightarrow \mathbb{C}$  of the form

$$f(\Xi) \doteq \sum_{\mu} f_{\mu} \Xi_{\mu}, \quad (\Xi \in \text{Cyl}(\widehat{\mathbb{R}}_B))$$

where  $\mu \mapsto f_{\mu}$  is any complex-valued function on  $\mathbb{R}$ ; while  $\text{Cyl}(\mathbb{R}_B)^*$  is the image of  $\text{Cyl}(\widehat{\mathbb{R}}_B)^*$  under the dual of the Fourier transform. Thus  $\text{Cyl}(\mathbb{R}_B)^*$  consists of all functionals  $\Gamma : \text{Cyl}(\mathbb{R}_B) \rightarrow \mathbb{C}$  of the form

$$\Gamma(\Phi) \doteq \sum_{\mu} \widehat{\Gamma}_{\mu} \widehat{\Phi}_{-\mu} \quad (\Phi \in \text{Cyl}(\mathbb{R}_B))$$

where the Fourier coefficients  $\widehat{\Gamma}_{\mu} = \Gamma(h_{-\mu})$  form an arbitrary complex-valued function  $\mu \mapsto \widehat{\psi}_{\mu}$  on  $\mathbb{R}$ . The Fourier transform then extends to a map from  $\text{Cyl}(\mathbb{R}_B)^*$  to  $\text{Cyl}(\widehat{\mathbb{R}}_B)^*$  so that

$$\widehat{\Gamma}(\Xi) = \sum_{\mu} \widehat{\Gamma}_{\mu} \Xi_{\mu} \quad (\Xi \in \text{Cyl}(\widehat{\mathbb{R}}_B)),$$

whereupon the Parseval identity holds in the form

$$\Gamma(\overline{\Phi}) = \widehat{\Gamma}(\widehat{\Phi}) \quad (\Gamma \in \text{Cyl}(\mathbb{R}_B)^*, \Phi \in \text{Cyl}(\mathbb{R}_B)).$$

There is a particular class of distributions over  $\text{Cyl}(\mathbb{R}_B)$  whose action can be expressed in terms of the restriction of cylindrical functions to the real line.

**Lemma 3.2.** *Suppose  $\Gamma \in \text{Cyl}(\mathbb{R}_B)^*$  has the property that  $\widehat{\Gamma} : \mu \mapsto \widehat{\Gamma}_{\mu}$  is the Fourier transform of a finite complex measure  $\rho$  on  $\mathbb{R}$ ,*

$$\widehat{\Gamma}_{\mu} = \int_{\mathbb{R}} e^{-i\mu x} d\rho(x)$$

*(in particular, this holds if  $\mu \mapsto \widehat{\Gamma}_{\mu}$  is a Schwartz function on  $\mathbb{R}$ ). Then the action of  $\Gamma$  on any  $\Psi \in \text{Cyl}(\mathbb{R}_B)$  is*

$$\Gamma(\Psi) = \int_{\mathbb{R}} \Psi|_{\mathbb{R}}(x) d\rho(x) \tag{9}$$

*which may be written*

$$\Gamma(\Psi) = \int_{\mathbb{R}} (\mathcal{F}^{-1}\widehat{\Gamma})(x) \Psi|_{\mathbb{R}}(x) dx$$

*if  $\mu \mapsto \widehat{\Gamma}_{\mu}$  is of Schwartz class, where  $\mathcal{F}$  is the usual Fourier transform on  $\mathbb{R}$ ,  $(\mathcal{F}\psi)(p) = \widehat{\psi}(p)$ , and the measure  $dx$  is the usual Lebesgue measure on  $\mathbb{R}$ .*



*Proof.* By linearity it is enough to consider the case  $\Psi(c) = h_{-\mu}(c)$ , for which  $\Psi|_{\mathbb{R}}(x) = e^{-i\mu x}$ . By definition, the left-hand side of (9) is  $\widehat{\Gamma}_\mu$ , and the result follows.  $\square$

In other words, this type of distribution over  $\text{Cyl}(\mathbb{R}_B)$  acts on a cylindrical function as integration of the cylindrical function restricted to  $\mathbb{R}$  with respect to a measure on  $\mathbb{R}$ .

For later use, we note a simple application of Bochner's theorem.

**Lemma 3.3.** *Under the hypotheses of Lemma 3.2, if  $\Gamma$  is positive (i.e.,  $\Gamma(\Psi) \geq 0$  for all pointwise nonnegative  $\Psi \in \text{Cyl}(\mathbb{R}_B)$ ) then  $\rho$  is a finite positive measure.*

*Proof.* Considering any cylindrical function of the form  $\Psi(c) = \left| \sum_{i=1}^N \xi_i h_{\lambda_i}(c) \right|^2$ , and applying Lemma 3.2, we see that

$$0 \leq \Gamma(\Psi) = \sum_{i,j=1}^N \bar{\xi}_i \xi_j \int_{\mathbb{R}} e^{i(\lambda_i - \lambda_j)x} d\rho(x)$$

Thus the Fourier transform of  $\rho$  is a function of positive type, and hence  $\rho$  is a positive measure by Bochner's theorem (Theorem IX.9 in [17]).  $\square$

Finally we define the operators  $\mathbf{h}_\mu$  ( $\mu \in \mathbb{R}$ ) and  $\mathbf{p}$  by their actions

$$(\mathbf{h}_\mu \Psi)(c) = h_\mu(c) \Psi(c), \quad (\mathbf{p} \Psi)(c) = \sum_{\mu} \mu \widehat{\Psi}_\mu h_\mu(c)$$

on wave functions  $\Psi \in \text{Cyl}(\mathbb{R}_B)$ ; each  $\mathbf{h}_\mu$  extends to a bounded operator on  $L^2(\mathbb{R}_B)$ . We note that  $\mathbf{h}_\mu$  and  $\mathbf{p}$  have the same commutation relations as  $\exp[i\mu \mathbf{X}]$  and  $\mathbf{P}$ ,  $\mathbf{X}$  and  $\mathbf{P}$  being the operators of the Schrödinger representation from (1). In the latter case, however, the generator  $\mathbf{X}$  may be recovered by differentiation

$$\mathbf{X} \Psi = -i \frac{d}{d\mu} \exp[i\mu \mathbf{X}] \Psi \Big|_{\mu=0};$$

here, however,  $\mu \mapsto \mathbf{h}_\mu \Psi$  is not differentiable on a dense domain of  $\Psi$  in  $L^2(\mathbb{R}_B)$ . In other words, there is no operator on  $L^2(\mathbb{R}_B)$  corresponding to the position operator  $\mathbf{X}$ .

### 3.2 The Wigner transform on $\mathbb{R}_B$

We now come to the definition of the Wigner transform in this setting. At first glance, it is not clear how to generalize the standard definition (2) on  $\mathbb{R}$ ,

because it is not clear how to divide an element of  $\mathbb{R}_B$  by two. The equivalent expressions (3) and (4) do not suffer from this problem. They contain Fourier transforms, but those can be replaced by the Fourier transform (3.1) on  $\mathbb{R}_B$ .

**Definition 3.4.** For states  $\Psi, \Psi' \in \text{Cyl}(\mathbb{R}_B)$  the Wigner transform is defined as a complex-valued function on  $\mathbb{R}_B \times \mathbb{R}_B$  by

$$W(\Psi, \Psi')(c, \mu) \doteq \int_{\widehat{\mathbb{R}}_B} \overline{\widehat{\Psi}}_{\mu'} \widehat{\Psi}'_{2\mu-\mu'} h_{2(\mu-\mu')}(c) d\mu' \quad (10)$$

and may be written equivalently as

$$W(\Psi, \Psi')(c, \mu) = \int_{\widehat{\mathbb{R}}_B} \overline{\widehat{\Psi}}_{\mu-\nu/2} \widehat{\Psi}'_{\mu+\nu/2} h_\nu(c) d\nu.$$

We remark that the second of these expressions is a direct analogue of (4) (apart from the normalising  $2\pi$  factor), while the first is analogous to (3); the factor of 2 appearing in the latter expression arises from a Jacobian determinant that is not needed in the present setting. While this definition seems reasonable, its merits should ultimately be found in its properties. So let us look at some of those, next.

**Proposition 3.5.** The Wigner transform is a sesquilinear map

$$W : \text{Cyl}(\mathbb{R}_B) \times \text{Cyl}(\mathbb{R}_B) \longrightarrow \text{Cyl}(\mathbb{R}_B \times \widehat{\mathbb{R}}_B) \quad (11)$$

and extends to maps between the following spaces:

$$W : \begin{array}{ll} L^2(\mathbb{R}_B) \times L^2(\mathbb{R}_B) & \longrightarrow L^2(\mathbb{R}_B \times \widehat{\mathbb{R}}_B) \\ \text{Cyl}(\mathbb{R}_B)^* \times \text{Cyl}(\mathbb{R}_B)^* & \longrightarrow \text{Cyl}(\mathbb{R}_B \times \widehat{\mathbb{R}}_B)^* . \end{array}$$

On  $L^2(\mathbb{R}_B) \times L^2(\mathbb{R}_B)$  the Wigner transform has the overlap property

$$\iint \overline{W(\Psi_1, \Psi_2)} W(\Phi_1, \Phi_2) dc d\mu = \overline{\langle \Psi_1, \Phi_1 \rangle} \langle \Psi_2, \Phi_2 \rangle . \quad (12)$$

and is hermitean:

$$\overline{W(\Psi, \Psi')} = W(\Psi', \Psi). \quad (13)$$

Furthermore, for  $\Psi \in \text{Cyl}(\mathbb{R}_B)$ ,

$$\int W(\Psi, \Psi)(c, \mu) dc = |\widetilde{\Psi}_\mu|^2, \quad \int W(\Psi, \Psi)(c, \mu) d\mu = |\Psi|^2(c). \quad (14)$$

*Proof.* First look at the assertion (11). For  $\Psi, \Psi' \in \text{Cyl}(\mathbb{R}_B)$  and any given  $\mu$ ,  $\overline{\widehat{\Psi}}_{\mu'} \widehat{\Psi}'_{2\mu-\mu'}$  is nonzero for only finitely many  $\mu'$ , hence the integral in (10) amounts to a finite sum, and thus for fixed  $\mu$ ,  $W(\Psi, \Psi')(c, \mu)$  is in  $\text{Cyl}(\mathbb{R}_B)$ . On the other hand  $\overline{\widehat{\Psi}}_{\mu'} \widehat{\Psi}'_{2\mu-\mu'}$  is nonzero only for finitely many  $\mu$ , with

$\mu'$  held fixed. So the integral amounts to a finite sum of terms with finite support in  $\mu$ . This proves (11).

The extension to a map from  $\text{Cyl}(\mathbb{R}_B)^* \times \text{Cyl}(\mathbb{R}_B)^*$  into  $\text{Cyl}(\mathbb{R}_B \times \widehat{\mathbb{R}}_B)^*$  is obtained as follows: for  $\Psi, \Psi' \in \text{Cyl}(\mathbb{R}_B)$ , and  $F \in \text{Cyl}(\mathbb{R}_B \times \widehat{\mathbb{R}}_B)$  it is easy to calculate

$$\int_{\mathbb{R}_B \times \widehat{\mathbb{R}}_B} W(\Psi, \Psi')(c, \mu) F(c, \mu) dc d\mu = \int \widehat{\Psi}_{\mu+\nu/2} \widehat{\Psi}'_{\mu-\nu/2} \widehat{F}(\nu, \mu) d\nu d\mu$$

where  $\widehat{F} \in \text{Cyl}(\widehat{\mathbb{R}}_B) \otimes \text{Cyl}(\widehat{\mathbb{R}}_B)$  is the partial Fourier transform

$$\widehat{F}(\nu, \mu) = \int_{\mathbb{R}_B} F(c, \mu) h_{-\nu}(c) dc$$

Noting that above expression converges even if  $\Psi$  and  $\Psi'$  are replaced by elements of  $\text{Cyl}(\mathbb{R}_B)^*$ , we then define  $W(\Gamma, \Gamma')$  for  $\Gamma, \Gamma' \in \text{Cyl}(\mathbb{R}_B)^*$  as the element of  $\text{Cyl}(\mathbb{R}_B \times \widehat{\mathbb{R}}_B)^*$  with action

$$W(\Gamma, \Gamma')[F] = \int \widehat{\Gamma}_{\mu+\nu/2} \widehat{\Gamma}'_{\mu-\nu/2} \widehat{F}(\nu, \mu) d\nu d\mu \quad (F \in \text{Cyl}(\mathbb{R}_B \times \widehat{\mathbb{R}}_B)). \quad (15)$$

Short calculations shows that we have the properties

$$W(\Gamma, \Gamma')[\overline{W(\Psi, \Psi')}] = \overline{\Gamma}(\Psi) \Gamma'(\overline{\Psi'}) \quad (16)$$

and

$$\widehat{W}(\Gamma, \Gamma')(\nu, \mu) \doteq W(\Gamma, \Gamma')[h_{-\nu} \otimes \delta_\mu] = \widehat{\Gamma}_{\mu-\nu/2} \widehat{\Gamma}_{\mu+\nu/2} \quad (17)$$

for  $\Gamma, \Gamma' \in \text{Cyl}(\mathbb{R}_B)^*$ ,  $\Psi, \Psi' \in \text{Cyl}(\mathbb{R}_B)$ .

Restricting  $\Gamma, \Gamma'$  to  $L^2(\mathbb{R}_B)$  (regarded as a subspace of  $\text{Cyl}(\mathbb{R}_B)^*$ ) it is easy to see that  $\widehat{W}(\Gamma, \Gamma')(\nu, \mu)$  is square summable, so  $\widehat{W}(\Gamma, \Gamma') \in L^2(\widehat{\mathbb{R}}_B \times \widehat{\mathbb{R}}_B)$  is the partial Fourier transform of an element  $W(\Gamma, \Gamma')$  of  $L^2(\mathbb{R}_B \times \widehat{\mathbb{R}}_B)$ . Thus the Wigner transform maps  $L^2(\mathbb{R}_B) \times L^2(\mathbb{R}_B)$  to  $L^2(\mathbb{R}_B \times \widehat{\mathbb{R}}_B)$ , and (12) holds.

Properties (13) and (14) are confirmed by a short calculation. The latter also holds modulo technical refinements in the case  $\Psi \in L^2(\mathbb{R}_B)$ , but we will not pursue this here.  $\square$

### 3.3 Positivity properties

In the standard setting, Hudson's theorem shows that the Wigner function is not a probability distribution except for Gaussian states. Our purpose in this subsection is to investigate this issue for the Wigner functions of elements of  $\text{Cyl}(\mathbb{R}_B)$  and  $\text{Cyl}(\mathbb{R}_B)^*$ . Again, Gaussians will play an important role; however, these must now be treated as distributions because they are not elements of  $L^2(\mathbb{R}_B)$ . We define them as follows:

**Definition 3.6.**  $\Gamma \in \text{Cyl}(\mathbb{R}_B)^*$  is called *Gaussian* if its Fourier transform is of the form

$$\widehat{\Gamma}_\mu = \exp[-a\mu^2 + b\mu + c] \quad (18)$$

where  $a, b, c \in \mathbb{C}$  and  $\text{Re}(a) > 0$ .

This notion is justified because, according to Lemma 3.2,

$$\Gamma(\Psi) = \frac{1}{\sqrt{4\pi a}} \int_{\mathbb{R}} \exp[-(x - ib)^2/(4a) + c] \Psi|_{\mathbb{R}}(x) dx \quad (19)$$

for any cylindrical function  $\Psi$ .

Just as in the standard theory, Gaussians have nice positivity properties:

**Proposition 3.7.** *For  $\Gamma$  Gaussian,  $W(\Gamma, \Gamma)$  is positive in the sense that  $W(\Gamma, \Gamma)[F] \geq 0$  for any pointwise nonnegative  $F \in \text{Cyl}(\mathbb{R}_B \times \widehat{\mathbb{R}}_B)$ . Furthermore, equality holds if and only if  $F = 0$ .*

*Proof.* By linearity it is enough to prove this for  $F$  of the form  $F = \Phi \otimes \delta_{\mu_0}$  for pointwise nonnegative  $\Phi \in \text{Cyl}(\mathbb{R}_B)$ . Let  $\Gamma$  be Gaussian as in (18). Then

$$\begin{aligned} \widehat{\Gamma}_{\mu+\nu/2} \widehat{\Gamma}_{\mu-\nu/2} &= \exp \left[ -a\left(\mu - \frac{\nu}{2}\right)^2 - \bar{a}\left(\mu + \frac{\nu}{2}\right)^2 \right. \\ &\quad \left. + b\left(\mu - \frac{\nu}{2}\right) + \bar{b}\left(\mu + \frac{\nu}{2}\right) + 2\text{Re } c \right] \\ &= f(\mu, a, b, c) \exp \left[ -\frac{\text{Re } a}{2} \nu^2 + i(2\text{Im}(a)\mu - \text{Im}(b))\nu \right] \end{aligned}$$

where

$$f(\mu, a, b, c) = \exp[-2\text{Re}(a)\mu^2 + 2\text{Re}(b)\mu + 2\text{Re}(c)]$$

is a *positive* expression that does not depend on  $\nu$ , while the remaining factor is in the Schwartz class. Using (15) and  $\widehat{F}(\nu, \mu) = \widehat{\Phi}_\nu \delta_{\mu\mu_0}$ , we therefore have

$$W(\Gamma, \Gamma)[F] = f(\mu_0, a, b, c) \sum_{\nu \in \mathbb{R}} \exp \left[ -\frac{a}{2} \nu^2 + i(2\text{Im}(a)\mu_0 - \text{Im}(b))\nu \right] \widehat{\Phi}_\nu \quad (20)$$

and applying Lemma 3.2, we have:

$$W(\Gamma, \Gamma)[F] = \frac{f(\mu_0, a, b, c)}{\sqrt{2\pi \text{Re}(a)}} \int_{\mathbb{R}} \exp \left[ -\frac{(x + 2\text{Im}(a)\mu_0 - \text{Im}(b))^2}{2\text{Re}(a)} \right] \Phi|_{\mathbb{R}}(x) dx$$

which is manifestly positive as  $\Phi \geq 0$ , and vanishes if and only if  $\Phi = 0$ . Every pointwise positive element of  $\text{Cyl}(\mathbb{R}_B \times \widehat{\mathbb{R}}_B)$  is a convex combination of functions of the above form, which completes the proof.  $\square$

We now present a converse to this result, which is analogous to Hudson's theorem (Prop. 2.2) and indeed makes use of the classical result. The hypotheses can be weakened further, but we do not pursue this for simplicity.

**Proposition 3.8.** *Suppose  $\gamma \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \setminus \{0\}$  and define  $\Gamma \in \text{Cyl}(\mathbb{R}_B)^*$  by  $\widehat{\Gamma}_\mu = \widetilde{\gamma}(\mu)$ . If  $W(\Gamma, \Gamma)$  is positive then  $\Gamma$  is a Gaussian.*

*Proof.* We begin by noting that the usual Wigner function of  $\gamma$ ,  $W(\gamma, \gamma)(x, p)$ , has the property that  $W(\gamma, \gamma)(\cdot, \mu) \in L^1(\mathbb{R})$  for each  $\mu$  because  $\gamma \in L^1(\mathbb{R})$ , and is also continuous because  $\gamma \in L^2(\mathbb{R})$ . Next, observe that

$$\begin{aligned} \overline{\widehat{\Gamma}}_{\mu+\nu/2} \widehat{\Gamma}_{\mu-\nu/2} &= \int_{\mathbb{R}} e^{i(x+y)\nu/2} e^{i\mu(x-y)} \overline{\gamma}(x) \gamma(y) \, dx \, dy \\ &= \int_{\mathbb{R}} e^{-ix\nu} W(\gamma, \gamma)(-x, \mu) \, dx \end{aligned}$$

which for each fixed  $\mu \in \mathbb{R}$  exhibits the left-hand side as the Fourier transform of a measure obtained from  $W(\gamma, \gamma)$ ; this measure is finite by the  $L^1$  property mentioned above. Accordingly, by Lemma 3.2,

$$W(\Gamma, \Gamma)[\Phi \otimes \delta_\mu] = \int_{\mathbb{R}} \Phi|_{\mathbb{R}}(x) W(\gamma, \gamma)(-x, \mu) \, dx \quad (21)$$

for any cylindrical function  $\Phi$ . As the left-hand side is nonnegative for every  $\mu \in \mathbb{R}$  and each positive  $\Phi \in \text{Cyl}(\mathbb{R}_B)$ , it follows by Lemma 3.3 and continuity that  $W(\gamma, \gamma)$  is pointwise nonnegative. As  $\gamma \in L^2(\mathbb{R}) \setminus \{0\}$ , Hudson's theorem entails that  $\gamma$  and hence  $\widetilde{\gamma}$  are Gaussian. Thus  $\Gamma$  is a Gaussian element of  $\text{Cyl}(\mathbb{R}_B)^*$ .  $\square$

We can also examine the positivity properties of Wigner functions of cylindrical functions. Although  $\text{Cyl}(\mathbb{R}_B)$  may be embedded in  $\text{Cyl}(\mathbb{R}_B)^*$ , the resulting distributions do not satisfy the hypotheses of Prop. 3.8. Indeed, we may immediately observe that the statement of Prop. 3.8 cannot hold for all elements of  $\text{Cyl}(\mathbb{R}_B)$ :

**Lemma 3.9.** *The Wigner function for a pure character  $\Phi(c) \doteq ah_\mu(c)$  is positive,*

$$W(\Phi, \Phi) = |a|^2 \delta_{\mu_0}(\mu).$$

*Proof.* Let  $\Phi = h_{\mu_0}$ . Then

$$W(\Phi, \Phi) = \int_{\widehat{\mathbb{R}}_B} \overline{\widehat{\Phi}}_{\mu'} \widehat{\Phi}_{2\mu-\mu'} h_{2(\mu-\mu')}(c) \, d\mu'. \quad (22)$$

The first factor in the integrand vanishes unless  $\mu' = \mu_0$ , so

$$W(\Phi, \Phi) = \overline{\widehat{\Phi}}_{\mu_0} \widehat{\Phi}_{2\mu-\mu_0} h_{2(\mu-\mu_0)}(c) \quad (23)$$

and we see that the second factor in this expression vanishes unless  $2\mu - \mu_0 = \mu_0$ , i.e.,  $\mu = \mu_0$ . Thus

$$W(\Phi, \Phi)(c, \mu) = |\widehat{\Phi}_{\mu_0}|^2 h_0(c) \delta_{\mu_0}(\mu) = |a|^2 \delta_{\mu_0}(\mu) \geq 0, \quad (24)$$

so  $W(\Phi, \Phi)$  is a positive element of  $\text{Cyl}(\mathbb{R}_B \times \widehat{\mathbb{R}}_B)$ .  $\square$

This is at first very surprising. After all, the characters are rather quantum mechanical states with ‘infinite uncertainty’ for multiplication operators on  $\mathbb{R}_B$ . On the other hand, they are eigenstates for the operator  $\mathbf{p}$ , and in fact one should interpret them as ‘degenerate Gaussian states’ as follows: In the usual setting of analysis on  $\mathbb{R}$ , consider a Gaussian on Fourier space and its Fourier transform on position space. We are interested in the limit of bringing its width in Fourier space to zero, while keeping its integral fixed. The limit is not a square integrable function, neither in Fourier- nor in position space: In Fourier space, it is the delta distribution, in position space it is a function of constant modulus. For analysis on  $\mathbb{R}_B$ , the situation is however drastically different. The limit is well defined, giving a Kronecker delta on  $\widehat{\mathbb{R}}_B$  and a character on  $\mathbb{R}_B$ . Given this, the statement of Lemma 3.9 is perhaps less surprising.

Next we present a converse to Prop. 3.7:

**Proposition 3.10.** *Suppose that  $\Phi \in \text{Cyl}(\mathbb{R}_B)$ . The following are equivalent:*

1.  $W(\Gamma, \Gamma)(\overline{W(\Phi, \Phi)}) > 0$  for all Gaussians  $\Gamma \in \text{Cyl}(\mathbb{R}_B)^*$ ;
2.  $\Phi$  is of the form  $\Phi = ah_\mu$  for some  $\mu$ , i.e., a scalar multiple of a character.

*In particular,  $W(\Phi, \Phi)$  is pointwise nonnegative for  $\Phi \in \text{Cyl}(\mathbb{R}_B)$  if and only if  $\Phi$  is a scalar multiple of a character.*

*Proof.* We adapt the standard proof of Hudson’s theorem [15]. First observe that with the definitions of Wigner function for  $\Gamma \in \text{Cyl}(\mathbb{R}_B)^*$  and Fourier transform we have

$$W(\Gamma, \Gamma)(\overline{W(\Phi, \Phi)}) = |\Gamma(\overline{\Phi})|^2 \quad (25)$$

by (16). So property 1 implies that  $\Gamma(\overline{\Phi})$  is nonvanishing for any Gaussian  $\Gamma$ .

Now consider the family of Gaussians  $\Gamma_z$  with Fourier transform  $(\widehat{\Gamma_z})_\mu = \exp(-\mu^2 - i\mu z)$ . We have

$$G(z) \doteq \Gamma_z(\overline{\Phi}) = \sum_{\mu} \overline{\Phi}_\mu \exp(-\mu^2 - i\mu z) \quad (26)$$

which is clearly an entire function of exponential type (i.e.,  $|G(z)| \leq Ae^{B|\text{Im } z|}$  for constants  $A, B$ ). As it is also nonvanishing by the previous observation we may conclude by a result of Hadamard (Theorem VIII.10 in [18]) that  $G(z)$  is the exponential of a polynomial of at most first degree, i.e.,

$$G(z) = ae^{ibz} \quad (27)$$

for complex constants  $a \neq 0$  and  $b$ . Now the restriction of  $G(z)$  to the real line is bounded, so we may conclude that  $b$  is real.

We therefore have

$$\sum_{\mu} \overline{\Phi}_{\mu} \exp(-\mu^2 - i\mu t) = ae^{ibt} \quad (28)$$

for all real  $t$ , from which it follows that  $\widehat{\Phi}_{\mu}$  is nonzero only for  $\mu = b$ . Accordingly  $\Phi$  is a scalar multiple of the character  $h_b$ . Thus  $1 \implies 2$ .

In the converse direction we set  $\Phi = ah_{\mu_0}$  and refer to Lemma 3.9, which tells us that  $W(\Phi, \Phi)$  is a positive element of  $\text{Cyl}(\mathbb{R}_B \times \widehat{\mathbb{R}}_B)$ . Using Prop. 3.7 we therefore have  $W(\Gamma, \Gamma)(\overline{W(\Phi, \Phi)}) > 0$  for all Gaussians  $\Gamma$ , and the equivalence of 1 and 2 is established. The proof is concluded by remarking that if  $W(\Phi, \Phi)$  is pointwise positive and not identically zero then  $W(\Gamma, \Gamma)[W(\Phi, \Phi)] > 0$  for all Gaussians  $\Gamma$  by Prop. 3.7, and hence  $\Phi$  is a scalar multiple of a character; the converse is given by Lemma 3.9. It is also trivial that  $W(\Phi, \Phi)$  is identically zero if and only if  $\Phi = 0$ .  $\square$

It is worth remarking that the essential difference between this result and the standard line of argument is that the support of  $\widehat{\Phi}$  is supposed to be bounded, which permits us to obtain an exponential bound of first order. In the usual proof of Hudson's theorem one does not have this luxury and the quadratic bound arises by completing a square to bound the Gaussian term.

### 3.4 Quantization

Now we discuss the Wigner transform and quantization. Consider a distribution  $\sigma \in \text{Cyl}(\mathbb{R}_B \times \widehat{\mathbb{R}}_B)^*$ . It defines a sesquilinear form  $B_{\sigma}$  on  $\text{Cyl}(\mathbb{R}_B)$  via

$$B_{\sigma}(\Psi_1, \Psi_2) = \sigma(W(\Psi_1, \Psi_2)).$$

Motivated by Prop. 2.3 we ask the following question: When is the form  $B_{\sigma}$  induced by an operator  $\sigma$  on  $\text{Cyl}(\mathbb{R}_B)$ ? A partial answer to this question can be given as follows.

Let  $\sigma$  be in  $\text{Cyl}(\mathbb{R}_B \times \widehat{\mathbb{R}}_B)$ . Then we can *define* an operator  $\sigma$  on  $\text{Cyl}(\mathbb{R}_B)$  by

$$(\sigma\Psi)(c) \doteq \iint_{\widehat{\mathbb{R}}_B \times \widehat{\mathbb{R}}_B} \widehat{\Psi}_{\nu} \widehat{\sigma}(\mu - \nu, (\nu + \mu)/2) h_{\mu}(c) d\mu d\nu. \quad (29)$$

This definition is justified by the following fact.

**Lemma 3.11.** *The matrix elements of the operator  $\sigma$  as defined by (29) are given by*

$$\langle h_{\mu}, \sigma h_{\nu} \rangle = B_{\sigma}(h_{\mu}, h_{\nu}) \quad (\equiv \sigma(W(h_{\mu}, h_{\nu}))).$$

In particular,  $\sigma$  is symmetric if  $\sigma$  is real.

The proof is straightforward, noting that

$$B_\sigma(h_\mu, h_\nu) = \widehat{\sigma}(\mu - \nu, (\nu + \mu)/2). \quad (30)$$

This answers the question, albeit only for symbols  $\sigma$  in  $\text{Cyl}(\mathbb{R}_B \times \widehat{\mathbb{R}}_B)$ .

What about more general symbols? If  $\sigma$  is only in  $\sigma \in \text{Cyl}(\mathbb{R}_B \times \widehat{\mathbb{R}}_B)^*$  it is not clear whether the integration in (29) converges or not, so this seems to be too general. On the other hand there are functions  $\sigma$  that are not even in  $L^2(\mathbb{R}_B \times \widehat{\mathbb{R}}_B)$ , for which (29) *does* make sense. For our purposes, it will be sufficient to restrict to the symbol classes defined in the following result:

**Proposition 3.12.** (a) Let  $\text{Symb} \subset \text{Cyl}(\mathbb{R}_B \times \widehat{\mathbb{R}}_B)^*$  be the set of distributions  $\sigma$  such that

$$\text{for any fixed } \beta \in \widehat{\mathbb{R}}_B: \quad M_{\alpha\beta} \doteq \widehat{\sigma}(\alpha - \beta, (\alpha + \beta)/2) \text{ is in } L^1(\widehat{\mathbb{R}}_B). \quad (31)$$

Then for any  $\sigma \in \text{Symb}$ , (29) defines a (possibly unbounded) operator  $\sigma$  with domain  $\text{Cyl}(\mathbb{R}_B)$ .

(b) Let  $\text{Symb}_\infty \subset \text{Symb}$  be the set of distributions  $\sigma$  for which there exist constants  $A, B \geq 0$  such that

$$\sum_{\beta} |M_{\alpha\beta}| \leq A \quad \text{for all } \alpha \in \widehat{\mathbb{R}}_B \quad (32)$$

and

$$\sum_{\alpha} |M_{\alpha\beta}| \leq B \quad \text{for all } \alpha \in \widehat{\mathbb{R}}_B. \quad (33)$$

For each such  $\sigma$ , the operator  $\sigma$  extends to a bounded operator (also denoted  $\sigma$ ) on  $L^2(\mathbb{R}_B)$  with

$$\|\sigma\| \leq \sqrt{AB}.$$

Moreover, the adjoint  $\sigma^*$  is the quantization of  $\overline{\sigma}$ .

*Proof.* (a) is trivial; while (b) is immediate from the Schur test (Theorem 5.2 in [19]). The statement about the adjoint follows on noting that

$$B_{\overline{\sigma}}(h_\mu, h_\nu) = \widehat{\overline{\sigma}}(\mu - \nu, (\mu + \nu)/2) = \overline{\widehat{\sigma}(\nu - \mu, (\mu + \nu)/2)} = \overline{B_\sigma(h_\nu, h_\mu)}.$$

□

We remark that for example any function  $\sigma(c, \lambda)$  on  $\mathbb{R}_B \times \widehat{\mathbb{R}}_B$  which is in  $\text{Cyl}(\mathbb{R}_B)$  for fixed  $\lambda$  belongs to  $\text{Symb}$ .

What does the quantization (29) give? Straightforward calculations show the following



**Lemma 3.13.** For  $\sigma_1(c, \lambda) = h_\mu(c)$  and  $\sigma_2(c, \lambda) = \lambda$

$$\sigma_1 = h_\mu, \quad \sigma_2 = p.$$

For  $\sigma_3(c, \lambda) = \lambda h_\mu(c)$

$$\sigma_3 = \frac{1}{2}(h_\mu p + p h_\mu).$$

We note that the latter is the totally symmetric ordering of  $p \exp(i\mu x)$ , thus one should think of the quantization given by (29) as Weyl quantization. Weyl ordering is interesting because of its properties (see [2] for a discussion), and it has become important in loop quantum cosmology because it was used in a novel method for obtaining effective equations of motion from the quantum theory [13, 14].

## 4 Application to the quantization of the Hamiltonian constraint

As we have pointed out above, the Wigner function can be used to quantize phase space functions in a systematic way. In the following we will apply this technique in a case that is of importance to LQC. More generally we expect that this formalism will be useful in the context of obtaining effective equations of motion from the quantum theory [13, 14], and for cases in which complicated phase space functions have to be quantized.

The example we discuss here is important in the context of describing homogenous and isotropic cosmology coupled to a scalar field in the framework of LQC [7]. It was found there that the standard way of quantizing the Hamiltonian constraint (e.g. [8]) led to physically unacceptable results, and a new quantization was introduced in [12]. In very brief terms, it can be described as follows: As customary in LQC, before quantization the curvature is expressed in terms of the connection along a small edge. Formerly this edge was taken to have a length proportional to the smallest quantum of length in the full theory, as measured in a fiducial background metric. The basic idea of Ashtekar, Pawłowski and Singh (APS) [12] is to determine it in a similar way, but with respect to the physical metric which is subject to quantization. On a technical level this requires quantization of the phase space function

$$e(c, \mu) = h_{\bar{\mu}(\mu)}(c) \tag{34}$$

where  $\bar{\mu}$  is a function fulfilling

$$\bar{\mu}(\mu)^2 = \frac{3\sqrt{3}}{2} |\mu|^{-1}. \tag{35}$$

The new symbol  $e(c, \mu)$  replaces the function  $e_0(c) = \exp(i\mu_0 c)$  in the old quantization of the constraint, which is constant in  $\mu$ .<sup>1</sup> According to [12],  $\mu_0$  was chosen as  $3\sqrt{3}/2$ .

Obviously quantization of (34) necessitates a choice of ordering. In [12] a quantization was arrived at in the following fashion: Naively  $c$  would be quantized by a derivative in  $\mu$ . That derivative, and hence an operator corresponding to  $c$ , fail to exist on functions in  $\text{Cyl}(\widehat{\mathbb{R}}_B)$  however. Nevertheless one can study the action of the operator

$$e_{\text{APS}} \doteq \sum_{n=0}^{\infty} \frac{1}{n!} [\overline{\mu}(\mu) \frac{d}{d\mu}]^n$$

on smooth functions on  $\mathbb{R}$ . This action is given by pullback with a certain diffeomorphism of  $\mathbb{R}$ , and it continues to make sense on functions in  $\text{Cyl}(\widehat{\mathbb{R}}_B)$ . The upshot is that APS define an operator  $e_{\text{APS}}$  on  $\text{Cyl}(\mathbb{R}_B)$  by

$$(\widehat{e_{\text{APS}} \Psi})_{\mu} = \widehat{\Psi}_{\text{Sign}(\mu')|\mu'|^{2/3}} \quad \text{with} \quad \mu' = \text{Sign}(\mu)|\mu|^{3/2} + \frac{1}{K} \quad (36)$$

where  $K$  is a specific numerical constant.

This quantization is very plausible for a number of reasons. First,  $e_{\text{APS}}$  is a unitary operator. Second, although (36) looks very complicated, it can be given a simple interpretation:  $e_{\text{APS}}$  is a constant shift on wave functions over the *volume*. Third, maybe most importantly, it ultimately leads to a Hamiltonian constraint that is physically viable.

Still, since (36) is at least partly motivated by reference to a differentiable structure on  $\widehat{\mathbb{R}}_B$ , – something that does not exist – it may be interesting to consider alternatives. Therefore we proceed now to quantize the same classical function using the Wigner transform. The symbol  $e(c, \mu)$  from equation (34) is undefined at  $\mu = 0$ , and we will remedy this by setting  $e(c, 0) = 0$ . Although this appears *ad hoc*, it will be shown below that the same results are obtained by taking limits of quantized operators formed from regularised versions of  $\overline{\mu}$ . With that end in view, let us first consider general symbols of the form

$$e_f(c, \mu) = h_{f(\mu)}(c)$$

where  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Noting that  $\widehat{e}_f(\nu, \mu) = \delta_{\nu, f(\mu)}$ , we have

$$\widehat{e}_f(\alpha - \beta, (\alpha + \beta)/2) = \delta_{\alpha - \beta, f((\alpha + \beta)/2)}$$

---

<sup>1</sup>Eq. (35) determines  $\overline{\mu}$  only up to sign. We will choose  $\overline{\mu}$  positive, in agreement with [12].

Accordingly, the properties of the quantisation  $e_f$  are closely related to the properties of equation

$$\alpha - \beta = f\left(\frac{\alpha + \beta}{2}\right). \quad (37)$$

In particular, if there are constants  $A$  and  $B$  such that (37) has at most  $A$  (resp.,  $B$ ) solutions for  $\beta$  (resp.,  $\alpha$ ) for each fixed  $\alpha$  (resp.,  $\beta$ ) then  $e_f \in \text{Symb}_\infty$  and  $\|e_f\| \leq \sqrt{AB}$ . We will now restrict to functions  $f$  for which this condition holds. One may also note that

$$e_f h_\beta = \sum_{\alpha \in S_f(\beta)} h_\alpha$$

where  $S_f(\beta)$  is the set of  $\alpha$  solving (37) for the given  $\beta$ . It follows that  $e_f$  is unitary if and only if (37) implicitly defines a bijection  $\beta \mapsto \alpha(\beta)$  of  $\mathbb{R}$ . Indeed,  $e_{\text{APS}}$  is precisely of this form for a suitable  $f$ .

For the particular symbol  $e(c, \mu)$  of interest, the above remarks are valid modulo the special treatment of  $\mu = 0$ ; the upshot is that

$$e h_\beta = \sum_{\alpha \in S(\beta)} h_\alpha \quad (38)$$

where  $S(\beta)$  is the set of solutions  $\alpha \in \mathbb{R} \setminus \{-\beta\}$  to  $\alpha - \beta = \bar{\mu}((\alpha + \beta)/2)$ . Equivalently, these are the solutions to

$$|\alpha + \beta|(\alpha - \beta)^2 = 3\sqrt{3}. \quad (39)$$

with  $\alpha > \beta$ ; analysis of this equation reveals that there are one, two, or three solutions for fixed  $\beta$  according to whether  $\beta$  is greater than, equal to, or less than  $-3^{3/2}2^{5/3}$ . (This is illustrated in the first diagram in Figure 1, in which the solutions would be the intersections of an  $\beta = \text{const.}$  line with the graph.) The same is true for solutions in  $\beta$  for fixed  $\alpha$ ; it therefore follows that  $e \in \text{Symb}_\infty$  and that we have  $\|e\| = 3$ . In contrast to  $e_{\text{APS}}$ , then,  $e$  is not unitary.

The relationship between  $e$  and  $e_{\text{APS}}$  will be discussed further below; first, we show how  $e$  may be obtained as a limit of quantizations based on regularised versions of  $\bar{\mu}$ . A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  will be called an  $\epsilon$ -regularisation of  $\bar{\mu}$  if  $f(\mu) = \bar{\mu}(\mu)$  for  $|\mu| > \epsilon$  and  $f$  is concave on  $|\mu| \leq \epsilon$ . (Concavity is adopted here for convenience; much weaker conditions would also suffice.) Thus, for example, taking  $f(\mu) = \bar{\mu}(\epsilon)$  on  $|\mu| \leq \epsilon$  would give an  $\epsilon$ -regularisation, but there are many other possibilities.

**Lemma 4.1.** *Let  $f$  be any  $\epsilon$ -regularisation of  $\bar{\mu}$ . Then  $e_f \in \text{Symb}_\infty$  and  $\|e_f\| \leq 5$ .*

*Proof.* First note that every solution to (37) with  $|\alpha + \beta| > 2\epsilon$  is a solution to (39) with  $\alpha > \beta$ . We have already seen that there are at most 3 solutions to this equation for  $\alpha$  (resp.,  $\beta$ ) at fixed  $\beta$  (resp.,  $\alpha$ ). It remains to consider solutions with  $|\alpha + \beta| \leq 2\epsilon$ . Fixing  $\beta$ , these solutions correspond to intersections of the graph of  $f$  with a straight line, and there can be at most two of these in this region because  $f$  is concave on  $[-\epsilon, \epsilon]$ . The same is true if we fix  $\alpha$ . Accordingly there are at most 5 solutions to (37) on lines of constant  $\beta$  or  $\alpha$ . The result follows by the foregoing discussion.  $\square$

**Proposition 4.2.** *Let  $f_n$  be any sequence of  $\epsilon_n$ -regularisations of  $\bar{\mu}$  with  $\epsilon_n \rightarrow 0^+$ . Then the sequence of operators  $e_{f_n}$  converges strongly to the operator  $e$  defined above.*

*Proof.* As the operators in the sequence are all bounded with norm less than 5, it is enough to prove strong convergence on the dense subspace  $\text{Cyl}(\mathbb{R}_B)$  of  $L^2(\mathbb{R}_B)$ . In turn, it therefore suffices to establish strong convergence of the sequence applied to each character  $h_\beta$ . Fix  $\beta \in \mathbb{R}$  and choose  $N$  large enough that  $S(\beta)$  has no intersection with  $[-\beta - 2\epsilon_n, -\beta + 2\epsilon_n]$  for  $n > N$ . This is possible because  $S(\beta)$  is finite and excludes  $-\beta$ . For such  $n$ ,  $S_{f_n}(\beta)$  is the union of disjoint sets  $S(\beta)$  and

$$T_n(\beta) \doteq S_{f_n}(\beta) \cap [-\beta - 2\epsilon_n, -\beta + 2\epsilon_n]$$

so we may write

$$e_{f_n} h_\beta = e h_\beta + \sum_{\alpha \in T_n(\beta)} h_\alpha.$$

We now claim that  $T_n(\beta)$  is empty for all sufficiently large  $n$ , thus establishing that  $e_{f_n} h_\beta \rightarrow e h_\beta$  and hence (as  $\beta$  is arbitrary) the required result.

The claim is proved as follows: if  $\alpha \in T_n(\beta)$  then  $|\alpha + \beta| \leq 2\epsilon_n$ , and hence  $\alpha - \beta < 2(|\beta| + \epsilon_n)$ . But by concavity  $f_n((\alpha + \beta)/2) \geq f_n(\epsilon_n) = 3^{3/4}(2\epsilon_n)^{-1/2}$  for  $|\alpha + \beta| \leq 2\epsilon_n$ . As  $\epsilon_n \rightarrow 0$  it is therefore clear that for all sufficiently large  $n$  there are no solutions to  $\alpha - \beta = f_n((\alpha + \beta)/2)$ ; hence  $T_n(\beta)$  is empty as required.  $\square$

In view of these results, we are encouraged to regard the choice  $e(c, 0) = 0$  as well-motivated, and  $e$  as an appropriate Weyl quantization of the original symbol defined (for  $\mu \neq 0$ ) in (34). To visualize the results of the above proposition, and to get a better feeling for the action of  $e$ , it is helpful to plot the space of solutions to (37) with  $f = \bar{\mu}$ . Instead of solving that equation directly, we will give a parametrization of its solution space: It is easily checked that

$$\{(\alpha, \beta) | \alpha - \beta = \bar{\mu}\left(\frac{\alpha + \beta}{2}\right)\} = \left\{\left(x + \frac{1}{2}\bar{\mu}(x), x - \frac{1}{2}\bar{\mu}(x)\right) | x \in \mathbb{R}\right\}. \quad (40)$$

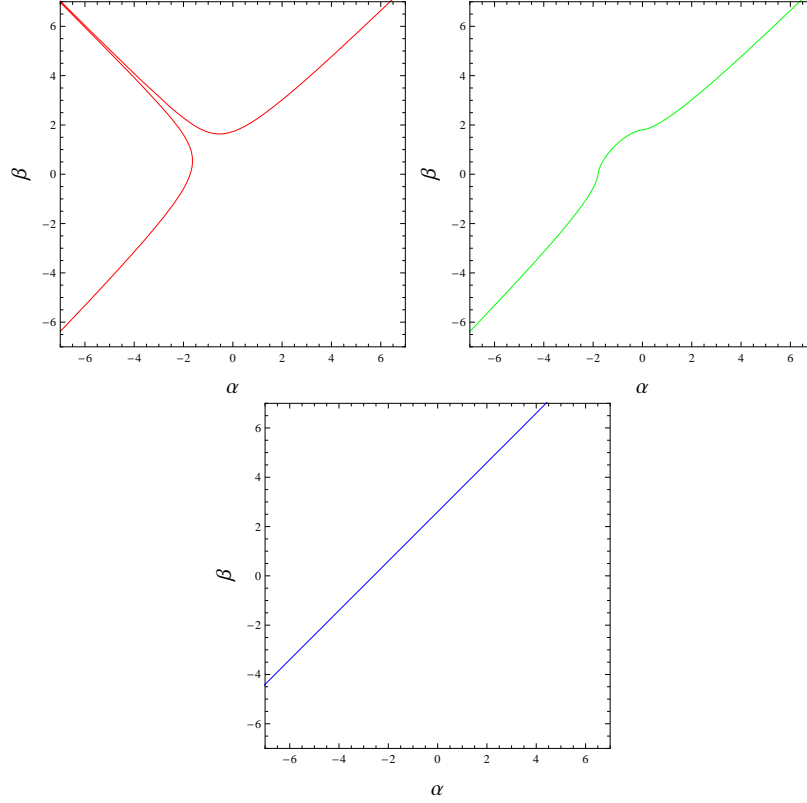


Figure 1: Matrix representations of the operators  $\mathbf{e}$ ,  $\mathbf{e}_{\text{APS}}$ ,  $\mathbf{h}_{\mu_0}$

This set is plotted as a graph in the  $\alpha - \beta$ -plane in the first diagram of Figure 1. Many of the properties of  $\mathbf{e}$  stated in the previous lemma can also be obtained from a visual inspection of what we will call its ‘matrix representation’. It will also be useful in the comparison of  $\mathbf{e}$ ,  $\mathbf{e}_{\text{APS}}$ , and  $\mathbf{h}_{\mu_0}$ . Note that all these operators have only matrix elements equal to 1 or 0 in the basis  $\text{Cyl}(\mathbb{R}_B)$ . So we can visualize them by plotting the set of matrix elements  $\langle h_\beta, \cdot h_\alpha \rangle$  that are equal to 1 as a set in the plane.<sup>2</sup> This is done in Figure 1 for the operators separately, and in Figure 2, for easy comparison, into one diagram. One can see how, for example,  $\mathbf{e}$  is not unitary, because its graph is that of a *multi-valued* function.

Let us compare the operator  $\mathbf{e}$  that we obtained here with  $\mathbf{e}_{\text{APS}}$  of [12]: The first difference is that  $\mathbf{e}_{\text{APS}}$  is unitary in contrast to  $\mathbf{e}$ . Another difference is the “spike” in the graph for  $\mathbf{e}$ : Whereas the graph of  $\mathbf{e}$  has a part (solid lines in Fig. 3, that is very similar to  $\mathbf{e}_{\text{APS}}$ , and can be characterized by  $\alpha \approx \beta$  for large  $\alpha$  and  $\beta$ , the graph also has a part that is very different (drawn as a dashed line in Figure 3), that can be characterized by  $\beta \approx -\alpha$ ,  $\alpha < 0$ .

<sup>2</sup>In other words, we can plot the graphs of the operators as the graphs of functions.

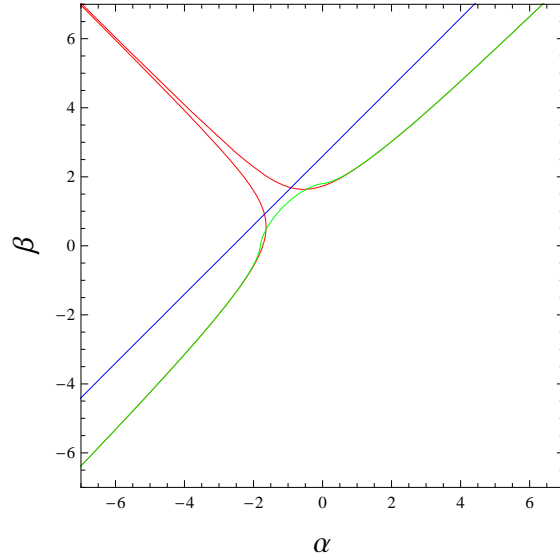


Figure 2: Comparison of the operators  $\mathbf{e}$ ,  $\mathbf{e}_{\text{APS}}$ ,  $\mathbf{h}_{\mu_0}$

This new feature may point to difficulties with the semiclassical limit for this operator because, loosely speaking, states with large momentum eigenvalues correspond to a universe with large spatial extension, and the operator is obviously changed on those large volume states, as compared to  $\mathbf{e}_{\text{APS}}$  and  $\mathbf{h}_{\mu_0}$ . On the other hand, in the present model eigenstates of momentum  $\mu$  and  $-\mu$  are physically identical<sup>3</sup> whence the two parts of the graph of  $\mathbf{e}$  (dashed and solid respectively, in Fig. 3) may act in a very similar way on the physical level.

Ultimately one will have to construct the full Hamiltonian constraint using the operator  $\mathbf{e}$  and compare the physical results to those obtained in [12], and we see no problems of principle for doing this. In particular we note that using  $\mathbf{e}$  will give a Hamiltonian constraint that commutes with the action of the parity operator  $\Pi$ , since one finds that

$$\widehat{\sin(\overline{\mu}c)} \Pi = -\Pi \widehat{\sin(\overline{\mu}c)}, \quad \widehat{\cos(\overline{\mu}c)} \Pi = \Pi \widehat{\cos(\overline{\mu}c)}$$

also with this quantization. The analysis will be substantially more complicated since it does not seem that there is a basis in which the action of  $\mathbf{e}$  drastically simplifies (such as is the case for  $\mathbf{e}_{\text{APS}}$  with respect to the volume eigenvector basis). In particular it does not seem to be likely that the kinematical Hilbert space will be decomposable into different superselection sectors as was the case in other quantizations. That said, we will leave a

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<sup>3</sup>They are related by the parity transform  $\Pi$  which is a symmetry of the system, see [7] for details.

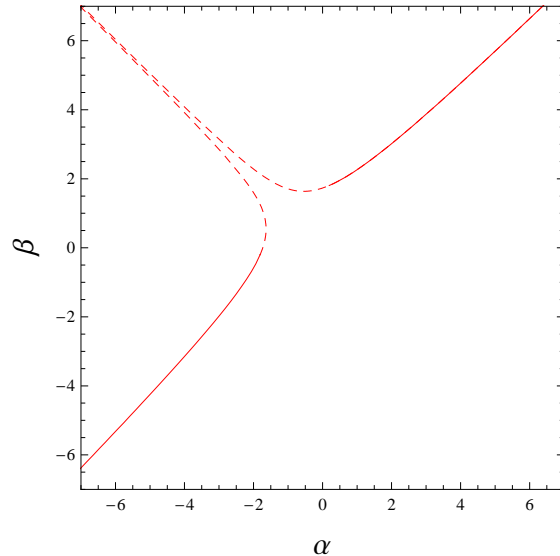


Figure 3: The different parts of the matrix-representation of  $\mathbf{e}$ : The part that differs very little from the operator  $\mathbf{e}_{\text{APS}}$  (solid), and the “spike” that is rather different (dashed)

detailed analysis for the future, and turn now to a discussion of the results of the present paper.

## 5 Closing remarks

In the present paper we have given a definition of the Wigner function for wave functions over the Bohr compactification  $\mathbb{R}_B$  of the real line and shown that our definition possesses many properties analogous to the Wigner function in ordinary quantum mechanics.

Since wave functions over the Bohr compactification figure prominently in loop quantum cosmology, the Wigner function should be of use in that context. To demonstrate this, we used the Wigner function machinery to give an alternative quantization of an important building block of the Hamiltonian constraint for homogenous isotropic cosmology as treated in LQC [7, 12]. We should stress again that the ordering chosen in [12] has many desirable properties and we do not want to claim in any way that it is wrong or inappropriate. Rather, by adding Weyl quantization to the toolbox of those working in LQC, we provide a quantization method that is applicable to a wide variety of situations, without the need to make any *ad hoc* choices. Whether the results are physically viable must still be determined in each instance separately.

The content of the present paper could be further developed in several directions: On the mathematical side, a more detailed investigation of the properties of the quantization map (e.g., with respect to products, or the semiclassical limit) could be undertaken. On the physical side, the quantization of the Hamiltonian constraint using the operator  $\mathfrak{e}$  from (38) should be completed. Then its physical implications need to be analyzed, along the lines of, say [7]. One should also consider application of the Weyl quantization in cases in which the method from [12] cannot be directly applied. An example for this would be homogenous but non-isotropic cosmologies.

But arguably the most interesting extension of the present work would consist in finding a generalization of the Wigner function to the quantum field theoretic context of full loop quantum gravity. This seems to be, at the same time, a very challenging undertaking. In the full theory wave functions live, roughly speaking, on a certain inductive limit of products of the Lie group  $SU(2)$ . So the two main problems we expect are 1) the definition of a “good” Wigner function on  $SU(2)$  that 2) interacts well with taking the inductive limit.

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