

Strongly Stable Real Infinitesimally Symplectic Mappings

R. CUSHMAN

Mathematics Institute, Rijksuniversiteit Utrecht, Utrecht, The Netherlands

AND

AL KELLEY

Department of Mathematics, University of California, Santa Cruz, California 95064

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We prove that a map $A \in sp(\sigma, \mathbb{R})$, the set of infinitesimally symplectic maps, is strongly stable if and only if its centralizer $C(A)$ in $sp(\sigma, \mathbb{R})$ contains only semisimple elements. Using the theorem that every B in $sp(\sigma, \mathbb{R})$ close to A is conjugate by a real symplectic map to an element of $C(A)$, we give a new proof of the openness of the set of strongly stable maps. Then we prove that the set of strongly stable maps is the interior of the set of all infinitesimally symplectic maps with purely imaginary or zero eigenvalues, and the connected components of this set are described. Finally, we give a new proof of the analytic conjugacy theorem for an analytic curve through a given strongly stable map.

0. BASIC TERMINOLOGY

Let V be a vector space over \mathbb{R} and let $\sigma: V \times V \rightarrow \mathbb{R}$ be a skew symmetric bilinear form which is nondegenerate, that is, the map $\sigma^*: V \rightarrow L(V, \mathbb{R}): x \rightarrow \{y \rightarrow \sigma(x, y)\}$ is an isomorphism. The pair (V, σ) is a real symplectic vector space. A basis $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ of (V, σ) is called a standard symplectic basis if and only if $\sigma(x_i, y_j) = \delta_{ij}$ and $\sigma(x_i, x_j) = \sigma(y_i, y_j) = 0$, that is the matrix of σ^* is $J_{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. A real linear map $S: V \rightarrow V$ is *symplectic* if $\sigma(Sx, Sy) = \sigma(x, y)$ for all $x, y \in V$. The set of all real symplectic maps on (V, σ) is denoted $Sp(\sigma, \mathbb{R})$. A real linear mapping $A: V \rightarrow V$ is *infinitesimally symplectic* if $\sigma(Ax, y) + \sigma(x, Ay) = 0$ for all $x, y \in V$. The set of all real infinitesimally symplectic maps on (V, σ) is denoted $sp(\sigma, \mathbb{R})$.

1. NORMAL FORMS FOR STABLE INFINITESIMALLY SYMPLECTIC MAPPINGS

In this section we define the concept of a stable infinitesimally symplectic map, and we find a normal form for these maps. This was done in [2] but is included here for the reader's convenience.

A map $A \in sp(\sigma, \mathbb{R})$ is *stable* if it is semisimple and has purely imaginary eigenvalues. Recall that A is semisimple if its minimal polynomial m_A is a product of irreducible polynomials over \mathbb{R} of multiplicity 1. Thus A is stable if and only if $m_A(s) = \prod_{i=1}^m (s^2 + \alpha_i^2)$ where $\alpha_i > 0$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. A stable map $A \in sp(\sigma, \mathbb{R})$ is *simply stable* if and only if $m_A(s) = s^2 + \alpha^2, \alpha > 0$.

If A is simply stable, then A is invertible because $A^2 + \alpha^2 = 0$. Therefore the bilinear form

$$\tau_A: V \times V \rightarrow \mathbb{R}: (x, y) \rightarrow \sigma(Ax, y)$$

is nondegenerate and symmetric since $A \in sp(\sigma, \mathbb{R})$. τ_A is *sign definite* if it is either positive or negative definite.

The next proposition gives a normal form for simply stable mappings.

PROPOSITION 1.1. *Suppose $A \in sp(\sigma, \mathbb{R})$ is simply stable on (V, σ) , then there is a symplectic basis such that the matrix of τ_A is $\begin{bmatrix} \alpha I_{p,n} & 0 \\ 0 & -I_{p,n} \end{bmatrix}$ where $I_{p,n} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with $2(p+n) = \dim V$ and the matrix of A is $\begin{bmatrix} 0 & -\alpha I_{p,n} \\ \alpha I_{p,n} & 0 \end{bmatrix}$.*

Proof [2]. Because τ_A is nondegenerate and symmetric there is a nonzero vector $x \in V$ such that $\tau_A(x, x) = \epsilon$ where $\epsilon^2 = 1$. Let W be the plane spanned by $\{x, (\epsilon/\alpha)Ax\}$, then both σ and τ_A restricted to W are nondegenerate. W is A -invariant and the matrix of A on W is $\begin{bmatrix} 0 & -\epsilon\alpha \\ \epsilon\alpha & 0 \end{bmatrix}$.

Since σ and τ_A are nondegenerate on W , σ is nondegenerate on $W^\sigma = \{x \in V \mid \sigma(x, w) = 0 \text{ for all } w \in W\}$, and τ_A is nondegenerate on $W^{\tau_A} = \{x \in V \mid \tau_A(x, w) = 0 \text{ for all } w \in W\}$. Note that for $x \in W$ and $y \in V$

$$\tau_A(y, x) = \sigma(Ay, x) = -\sigma(y, Ax),$$

and since $A^2 + \alpha^2 = 0$ on V ,

$$\tau_A(y, Ax) = \sigma(Ay, Ax) = \sigma(y, \alpha^2 x).$$

Thus $y \in W^\tau$ if and only if $y \in W^\sigma$. Put $W^\perp = W^\tau = W^\sigma$. Since W is A -invariant and $A \in sp(\sigma, \mathbb{R})$, W^\perp is A -invariant. Repeating the above argument for A on W^\perp , we eventually obtain $V = W_1 + \dots + W_n$ where W_i are planes invariant under A which are σ and τ_A nondegenerate and σ and τ orthogonal. Moreover each plane W_i has a symplectic basis $\{x_i, y_i\}$ such that the matrix of τ_A on W_i is $\begin{bmatrix} \epsilon_i \alpha_i & 0 \\ 0 & -\epsilon_i \alpha_i \end{bmatrix}$ and the matrix of A is $\begin{bmatrix} 0 & -\epsilon_i \alpha_i \\ \epsilon_i \alpha_i & 0 \end{bmatrix}$ where $\epsilon_i^2 = 1$.

A suitable permutation π on m symbols gives a basis $\{x_{\pi(1)}, \dots, x_{\pi(m)}, y_{\pi(1)}, \dots, y_{\pi(m)}\}$ of V where A and τ_A have the desired form. ■

Suppose that A is stable and has characteristic polynomial $\chi_A(s) = \prod_{i=1}^m (s^2 + \alpha_i^2)^{m_i}$ where $\alpha_i > 0$ and $\alpha_i \neq \alpha_j$ if $i \neq j$. Let $V_i = \prod_{j \neq i} (A^2 + \alpha_j^2)^{m_j} V$ be a *primary subspace* of A on V . Then V_i is invariant under A , $V = \sum_{i=1}^m \oplus V_i$, and the characteristic polynomial of A on V_i is $(s^2 + \alpha_i^2)^{m_i}$. Let $q_i(s) =$

$\prod_{j \neq i} (s^2 + \alpha_j^2)^{m_j}$, then $q_i(-s) = q_i(s)$. If $i \neq j$ then $V_i \subseteq V_j^\sigma$. For if $x_i \in V_i$ and $y_j \in V_j$, there are $x, y \in V$ such that $x_i = q_i(A)x$, $y_j = q_j(A)y$. Therefore, since $A \in sp(\sigma, \mathbb{R})$ and $\chi_A(A) = 0$,

$$\begin{aligned}\sigma(x_i, y_j) &= \sigma(q_i(A)x, q_j(A)y) = \sigma(x, q_i(-A)q_j(A)y) \\ &= \sigma(x, q_i(A)q_j(A)y) = \sigma\left(x, \prod_{k \neq i, j} (A^2 + \alpha_k^2)^{m_k} \chi_A(A)y\right) \\ &= 0.\end{aligned}$$

Suppose that for some i , σ on V_i is degenerate; that is for some nonzero vector $z \in V_i$, $\sigma(x_i, z) = 0$ for all $x_i \in V_i$. Since $V_i \subseteq V_j^\sigma$ for all $j \neq i$, $\sigma(x_j, z) = 0$ for all $x_j \in V_j$ and all j . Since $V = \sum_{j=1}^m \oplus V_j$, $\sigma(x, z) = 0$ for all $x \in V$, which implies that σ is degenerate and this is a contradiction. On the symplectic vector space $(V_i, \sigma|_{V_i})$ the infinitesimally symplectic stable mapping A is simply stable because the minimal polynomial of A on V_i is $s^2 + \alpha_i^2$ since A is stable.

Therefore using Proposition 1.1 we obtain

COROLLARY 1.1. *Let $A \in sp(\sigma, \mathbb{R})$ be stable with minimal polynomial $m_A(s) = \prod_{i=1}^m (s^2 + \alpha_i^2)$ where $\alpha_i > 0$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. Then there is a symplectic basis of (V, σ) such that the matrix of A is*

$$\begin{bmatrix} & & & & -\alpha_1 I_{p_1, n_1} & & & \\ & & & & & \ddots & & \\ & & & & & & \ddots & \\ & & & & & & & -\alpha_m I_{p_m, n_m} \\ & & & & & & & \\ \alpha_1 I_{p_1, n_1} & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & 0 \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \alpha_m I_{p_m, n_m} \end{bmatrix}$$

where $2(p_i + n_i) = \dim V_i$ and on each primary subspace V_i of A the matrix of τ_A is

$$\begin{bmatrix} \alpha_i I_{p_i, n_i} & 0 \\ 0 & \alpha_i I_{p_i, n_i} \end{bmatrix}.$$

A stable mapping $A \in sp(\sigma, \mathbb{R})$ is *strongly stable* if on each primary subspace V_i , $\tau_{A|_{V_i}}$ is sign definite. By Corollary 1.1 a strongly stable mapping has a normal form where either $p_i = 0$ or $n_i = 0$, but not both $p_i = 0$ and $n_i = 0$, for all i .

2. ALGEBRAIC CHARACTERIZATION OF STRONGLY STABLE MAPPINGS

In this section we will show that strongly stable mappings in $sp(\sigma, \mathbb{R})$ are precisely those stable mappings of $sp(\sigma, \mathbb{R})$ whose centralizer in $sp(\sigma, \mathbb{R})$ consists of semisimple mappings with purely imaginary or zero eigenvalues.

We begin by computing the centralizer $C(A)$ in $sp(J_{2m}, \mathbb{R})$ of the simply stable mapping

$$A = \begin{bmatrix} 0 & -\alpha I_{p,n} \\ \alpha I_{p,n} & 0 \end{bmatrix}$$

where $\alpha > 0$ and $p + n = m > 0$. Note that $B \in C(A)$ if and only if

$$B^t J_{2m} + J_{2m} B = 0, \quad (*)$$

$$BA - AB = 0, \quad (**)$$

where B^t is the transpose of B . Write $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c, d \in gl(m, \mathbb{R})$, the space of $m \times m$ real matrices. Then equation (*) gives $d = -a^t$, $b = b^t$, $c = c^t$, while equation (**) gives

$$\begin{aligned} bI_{p,n} &= -I_{p,n}c, & -aI_{p,n} &= I_{p,n}a^t, \\ -a^tI_{p,n} &= I_{p,n}a, & I_{p,n}b &= -cI_{p,n}, \end{aligned}$$

which in turn are equivalent to

$$c = -I_{p,n}bI_{p,n}, \quad -a^t = I_{p,n}aI_{p,n}, \quad (***)$$

because $I_{p,n}^2 = I_m$. Write $a = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ where $\alpha \in gl(p, \mathbb{R})$, $\delta \in gl(n, \mathbb{R})$, β is a $p \times n$ real matrix, γ is an $n \times p$ real matrix. Then equation (***) gives $\alpha = -\alpha^t$, $\beta = \gamma^t$, $\delta = -\delta^t$. Therefore

$$C(A) = \left\{ \begin{bmatrix} a & -b \\ I_{p,n}bI_{p,n} & I_{p,n}aI_{p,n} \end{bmatrix} \in sp(\sigma, \mathbb{R}) \mid a, b \in gl(m, \mathbb{R}), \right. \\ \left. aI_{p,n} = -I_{p,n}a^t, b = b^t \right\}$$

and

$$\begin{aligned} \dim_{\mathbb{R}} C(A) &= \left\{ \frac{1}{2}p(p-1) + \frac{1}{2}n(n-1) + np \right\} \\ &+ \frac{1}{2}m(m+1) = m^2 \end{aligned}$$

where the sum of the terms in the braces is the number of linearly independent solutions of the second equation in (***) and the other term is the number of linearly independent $m \times m$ real symmetric matrices.

Let

$$u(p, n) = \{B \in gl(m, \mathbb{C}) \mid BI_{p,n} + I_{p,n}\bar{B}^t = 0\}$$

be the set of skew hermitian matrices with respect to the hermitian form whose matrix is $I_{p,n}$. Write $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a \in gl(p, \mathbb{C})$, $d \in gl(n, \mathbb{C})$, b is a $p \times n$ complex matrix, c is an $n \times p$ complex matrix. Then $B \in u(p, n)$ if and only if $a = -\bar{a}^t$, $d = -\bar{d}^t$, $b = \bar{c}^t$. Therefore

$$\dim_{\mathbb{R}} u(p, n) = p^2 + n^2 + 2pn = m^2.$$

The following argument shows that the mapping

$$\psi: C(A) \rightarrow u(p, n): B = \begin{bmatrix} a & -b \\ I_{p,n} b I_{p,n} & I_{p,n} a I_{p,n} \end{bmatrix} \rightarrow \psi B = a + ib I_{p,n}$$

is an isomorphism of real vector spaces. First, since $I_{p,n}^2 = I_m$ and $B \in C(A)$, we have

$$\begin{aligned} (a + ib I_{p,n}) I_{p,n} + I_{p,n} \overline{(a + ib I_{p,n})}^t \\ = (a + ib I_{p,n}) I_{p,n} + I_{p,n} (a^t - i I_{p,n} b^t) \\ = (a I_{p,n} + I_{p,n} a^t) + i(b - b^t) \\ = 0 \end{aligned}$$

which shows that $\psi B \in u(p, n)$. Second, ψ is real linear and injective, for $0 = \psi B$ implies $a = b = 0$ and thus $B = 0$. Finally ψ is surjective because $\dim_{\mathbb{R}} C(A) = m^2 = \dim_{\mathbb{R}} u(p, n)$.

Thus we have

PROPOSITION 2.1. *Let $A \in sp(\sigma, \mathbb{R})$ be simply stable and in normal form. Then $C(A)$ is isomorphic to $u(p, n)$ where $m = n + p$ and the signature of τ_A is $2(p - n)$.*

Suppose that either $p = 0$ or $n = 0$ (but not both since $m > 0$). Since every element of $u(0, n) = u(m) = u(p, 0)$ is semisimple, every element of $C(A)$ is semisimple. Since τ_A is positive definite if $n = 0$ and negative definite if $p = 0$, A is strongly stable. Now suppose that $p > 0$ and $n > 0$. Then τ_A is not sign definite, that is, A is not strongly stable. Let $I_{(p,n)}$ be the $p \times n$ matrix given by

$$I_{(p,n)} = \begin{cases} I_p & \text{if } p = n, \\ [I_p \mid 0] & \text{if } p < n, \\ \begin{bmatrix} I_n \\ 0 \end{bmatrix} & \text{if } p > n, \end{cases}$$

and let

$$N = \begin{bmatrix} 0 & I_{(p,n)} & I_p & 0 \\ I_{(n,p)} & 0 & 0 & I_n \\ -I_p & 0 & 0 & -I_{(p,n)} \\ 0 & -I_n & -I_{(n,p)} & 0 \end{bmatrix}.$$

Because of the identities

$$I_{(p,n)}^t = I_{(n,p)}, \quad I_{(p,n)} I_{(n,p)} = I_p, \quad I_{(n,p)} I_{(p,n)} = I_n,$$

we have $N \in \mathfrak{sp}(J_{2m}, \mathbb{R})$ and $N \in C(A)$. But N is not semisimple, because $N^2 = 0$, that is N is nilpotent. Hence we have

COROLLARY 2.1. *Let $A \in \mathfrak{sp}(J_{2m}, \mathbb{R})$ be simply stable and in normal form. Then A is strongly stable if and only if $C(A)$ consists of semisimple elements. Moreover, $C(A)$ is isomorphic to $\mathfrak{u}(m)$.*

To compute the centralizer in $\mathfrak{sp}(\sigma, \mathbb{R})$ of a stable mapping $A \in \mathfrak{sp}(\sigma, \mathbb{R})$ we need the following

LEMMA 2.1. *Let $D = \begin{bmatrix} \beta I_{p,n} & 0 \\ 0 & E_r \end{bmatrix}$ be a real diagonal $(m+r) \times (m+r)$ matrix where $\beta > 0$ and $\beta, -\beta$ are not eigenvalues of E_r . Let B, C be $(m+r) \times (m+r)$ real matrices such that $B = \epsilon B^t, C = \epsilon C^t$ with $\epsilon^2 = 1$. If $DC + BD = 0$, then*

$$C = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix} \begin{matrix} m \\ r \end{matrix}, \quad B = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} \begin{matrix} m \\ r \end{matrix}.$$

Proof. Let

$$C = \begin{bmatrix} C_{11} & C_{12} \\ \epsilon C_{12}^t & C_{22} \end{bmatrix} \begin{matrix} m \\ r \end{matrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ \epsilon B_{12}^t & C_{22} \end{bmatrix} \begin{matrix} m \\ r \end{matrix},$$

$$C_{jj} = \epsilon C_{jj}^t, \quad B_{jj} = \epsilon B_{jj}^t, \quad (j = 1, 2),$$

where $m = p + n$. Then $0 = DC + BD$ implies

$$\beta I_{p,n} C_{12} + B_{12} E_r = 0$$

$$\epsilon E_r C_{12}^t + \epsilon \beta B_{12}^t I_{p,n} = 0$$

which is equivalent to

$$\beta I_{p,n} C_{12} + B_{12} E_r = 0$$

$$C_{12} E_r + \beta I_{p,n} B_{12} = 0$$

since $E_r = E_r^t$ and $I_{p,n} = I_{p,n}^t$. From the second equation we have $\beta B_{12} = -I_{p,n} C_{12} E_r$ since $I_{p,n}^2 = I_m$. Substituting this into the first equation we obtain $0 = I_{p,n} C_{12} (\beta^2 - E_r^2)$, which implies $C_{12} = 0$ since $I_{p,n}$ is invertible and $\beta^2 - E_r^2$ is invertible by hypothesis. Then $B_{12} = 0$ also. ■

Let

$$A = \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix}, \quad D = \begin{bmatrix} \beta I_{p,n} & 0 \\ 0 & E_r \end{bmatrix},$$

where D is as above, and let

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in sp(J_{2(m+r)}, \mathbb{R}),$$

which implies $d = -a^t$, $b = b^t$, $c = c^t$. Then B commutes with A if and only if

$$\begin{aligned} Dc + bD &= 0, & aD + Da^t &= 0, \\ cD + Db &= 0, & a^tD + Da &= 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} Dc + bD &= 0, \\ (a + a^t)D + D(a + a^t) &= 0, \\ (a - a^t)D + D(a - a^t) &= 0. \end{aligned}$$

Applying Lemma 2.1 to the first equation gives

$$b = \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix} \begin{matrix} m \\ r \end{matrix}, \quad c = \begin{bmatrix} c_{11} & 0 \\ 0 & c_{22} \end{bmatrix} \begin{matrix} m \\ r \end{matrix},$$

while applying the same lemma to the last two equations gives $a_{12} + a_{21}^t = 0$ and $a_{12} - a_{21}^t = 0$ so that

$$a = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \begin{matrix} m \\ r \end{matrix}.$$

Note that the results for the last two equations can also be obtained from a well known theorem about commuting matrices [4]. Therefore B commutes with A if and only if $\begin{bmatrix} a_{11} & b_{11} \\ c_{11} & -a_{11}^t \end{bmatrix} \in sp(J_{2m}, \mathbb{R})$ commutes with $\begin{bmatrix} 0 & -\alpha I_{p,n} \\ \alpha I_{p,n} & 0 \end{bmatrix}$ and $\begin{bmatrix} a_{22} & b_{22} \\ c_{22} & -a_{22}^t \end{bmatrix} \in sp(J_{2r}, \mathbb{R})$ commutes with $\begin{bmatrix} 0 & E_r \\ -E_r & 0 \end{bmatrix}$. Repeating this argument gives

PROPOSITION 2.2. *If $A \in sp(\sigma, \mathbb{R})$ is stable with normal form given in Corollary 1.1, then the centralizer of A in $sp(\sigma, \mathbb{R})$ is isomorphic to $u(p_1, n_1) \times \cdots \times u(p_m, n_m)$.*

An immediate consequence is

COROLLARY 2.2. *Let $A \in sp(\sigma, \mathbb{R})$ be stable and in normal form. A is strongly stable if and only if its centralizer in $sp(\sigma, \mathbb{R})$ contains only semisimple elements. Moreover $C(A)$ is isomorphic to $u(p_1 + n_1) \times \cdots \times u(p_m + n_m)$.*

Observe that if $P \in Sp(\sigma, \mathbb{R})$ then $PAP^{-1} \in sp(\sigma, \mathbb{R})$ and $C(PAP^{-1}) = PC(A)P^{-1}$. Therefore if $C(A)$ has only semisimple elements then so does $C(PAP^{-1})$. Consequently A need not be in normal form for the first statement in Corollary 2.2 to hold.

3. OPENNESS OF STRONGLY STABLE MAPPINGS

In this section we show that if $A \in sp(\sigma, \mathbb{R})$ is strongly stable, then any $B \in sp(\sigma, \mathbb{R})$ sufficiently close to A is strongly stable.

We begin by recalling some definitions from Lie algebras. A linear Lie group G is a closed subgroup of $Gl(V, \mathbb{R})$, the set of real invertible linear maps of a real vector space V into itself. The Lie algebra L_G of G is a subalgebra of $gl(V, \mathbb{R})$, the set of real linear maps of V into itself with Lie bracket defined by $[a, b] = a \circ b - b \circ a$ for $a, b \in L_G \subseteq gl(V, \mathbb{R})$, where \circ denotes composition of linear maps. Since $L_G \subseteq gl(V, \mathbb{R})$ it makes sense to say that $A \in L_G$ is semisimple. Recall that A is semisimple if and only if A is diagonalizable on $V^{\mathbb{C}}$, the complexification of V . If $A \in L_G$ is semisimple, then $ad_A: L_G \rightarrow L_G: B \rightarrow [A, B]$ is semisimple [9].

We now prove the basic conjugacy theorem.

THEOREM 3.1. *If $A \in L_G$ is semisimple, then the mapping*

$$\mathcal{O}: \ker ad_A \times \exp(\text{im } ad_A) \subseteq L_G \times G \rightarrow L_G: (C, S) \rightarrow S^{-1}(C + A)S$$

is a local real analytic diffeomorphism near $(0, I)$.

Proof. We compute the tangent to \mathcal{O} at $(0, I)$ in the direction (D, B) as follows. Consider the partial mapping

$$\mathcal{O}(0, \cdot): \exp(\text{im } ad_A) \subseteq G \rightarrow L_G: S \rightarrow S^{-1}AS.$$

If $B \in \text{im } ad_A$ then

$$\gamma: \mathbb{R} \rightarrow \exp(\text{im } ad_A) \subseteq G: t \rightarrow \exp tB$$

is an analytic curve with $\gamma(0) = I$. Hence

$$\begin{aligned} T_I \mathcal{O}(0, \cdot): \text{im } ad_A \rightarrow L_G: B &\rightarrow \left. \frac{d}{dt} \mathcal{O}(0, \gamma(t)) \right|_{t=0} \\ &= \left. \frac{d}{dt} \exp(-tB)A \exp(tB) \right|_{t=0} = A \circ B - B \circ A \\ &= ad_AB. \end{aligned}$$

Next consider the partial mapping

$$\mathcal{O}(\cdot, I): \ker ad_A \subseteq L_G \rightarrow L_G: C \rightarrow A + C.$$

Then

$$T_0\mathcal{O}(\cdot, I): \ker ad_A \subseteq L_G \rightarrow L_G: D \rightarrow D.$$

Therefore

$$T_{(0,I)}\mathcal{O}(D, B) = T_0\mathcal{O}(\cdot, I)D + T_1\mathcal{O}(0, \cdot)B = D + ad_AB,$$

and

$$T_{(0,I)}\mathcal{O}: \ker ad_A \times \operatorname{im} ad_A \rightarrow L_G: (D, B) \rightarrow D + ad_AB.$$

Since A is semisimple, ad_A is semisimple. Thus $L_G = \ker ad_A \oplus \operatorname{im} ad_A$. Consequently $T_{(0,I)}\mathcal{O}$ is an isomorphism. Since \mathcal{O} is a real analytic mapping, by the inverse function theorem \mathcal{O} is a local real analytic diffeomorphism from a neighborhood of $(0, I)$ to a neighborhood of $\mathcal{O}(0, I) = A$. ■

The above theorem can be rephrased as follows. If A is a semisimple element of L_G , then every $B \in L_G$ close to A is conjugate by a unique $S_B \in \exp(\operatorname{im} ad_A) \subset G$ to an element $A + C_B$ in the centralizer of A in L_G .

We now show that strongly stable mappings are in the interior of the set of stable mappings.

COROLLARY 3.1. *Suppose that $A \in sp(\sigma, \mathbb{R})$ is strongly stable. Then every $B \in sp(\sigma, \mathbb{R})$ close to A is stable.*

Proof. Because $A \in sp(\sigma, \mathbb{R})$ is strongly stable, A is invertible; that is $A \in Gl(V, \mathbb{R})$. Since $Gl(V, \mathbb{R})$ is an open subset of the set of all linear maps of V into itself, every $B \in sp(\sigma, \mathbb{R})$ close to A is in $Gl(V, \mathbb{R})$. Because A is semisimple, by Theorem 3.1 B is conjugate to some $C_B \in C(A)$ by some $P_B \in Sp(\sigma, \mathbb{R})$. Since $B \in Gl(V, \mathbb{R}) \cap sp(\sigma, \mathbb{R})$, $C_B \in Gl(V, \mathbb{R}) \cap sp(\sigma, \mathbb{R})$. Because A is strongly stable, there is $P \in Sp(\sigma, \mathbb{R})$ such that $\tilde{A} = PAP^{-1}$ is in the normal form given by Corollary 1.1 with either $p_i = 0$ or $n_i = 0$ for all i . By Corollary 2.2 the centralizer of \tilde{A} in $sp(\sigma, \mathbb{R})$ is $C(\tilde{A}) = u(p_1 + n_1) \times \cdots \times u(p_m + n_m)$ which consists of linear maps which are semisimple with eigenvalues which are zero or purely imaginary. Because the centralizer of A in $sp(\sigma, \mathbb{R})$ is $C(A) = P^{-1}C(\tilde{A})P$, $C_B \in C(A) \cap Gl(V, \mathbb{R})$ is an invertible semisimple linear map with purely imaginary eigenvalues; that is C_B is stable. ■

We now show that the above corollary can be strengthened to

THEOREM 3.2. *Strongly stable mappings are an open subset of the set of stable mappings.*

Proof. Let $A \in sp(\sigma, \mathbb{R})$ be strongly stable on (V, σ) . Then by Corollary 1.1 there are A -invariant subspaces V_i , $1 \leq i \leq n$, of dimension $2m_i$ such that (1) $\sigma|_{V_i}$ is a symplectic form on V_i ; (2) V_i and V_j are σ -orthogonal if $i \neq j$;

(3) $V = V_1 \oplus \cdots \oplus V_n$; (4) $A_i = A|_{V_i}$ has minimal polynomial $s^2 + \alpha_i^2$ where $\alpha_i > 0$ and $\alpha_i \neq \alpha_j$ if $i \neq j$; and (5) the map

$$\tau_{A_i}: V_i \times V_i \rightarrow \mathbb{R}: (v_i, w_i) \rightarrow \sigma(A_i v_i, w_i)$$

is sign definite. Let $\|\cdot\|$ be a norm for $gl(V, \mathbb{R})$. If $B \in sp(\sigma, \mathbb{R})$ and $\|B - A\|$ is sufficiently small, then by Theorem 3.1 and its corollary and by Proposition 2.2 there is a unique $P_B \in Sp(\sigma, \mathbb{R})$ such that

$$\tilde{B} = P_B B P_B^{-1} \in C(A) \cap Gl(V, \mathbb{R}) = (u(m_1) \times \cdots \times u(m_n)) \cap Gl(V, \mathbb{R}).$$

Moreover, the mapping

$$sp(\sigma, \mathbb{R}) \rightarrow Sp(\sigma, \mathbb{R}): B \rightarrow P_B$$

is smooth and $P_A = I$. Actually the proof of Proposition 2.2 shows that V_i is \tilde{B} -invariant and

$$\tilde{B}_i = \tilde{B}|_{V_i} \in u(m_i) \cap Gl(V_i, \mathbb{R}).$$

Let $W_i^B = P_B^{-1}|_{V_i}$. Then (1) W_i^B is B -invariant; (2) $\sigma|_{W_i^B}$ is a symplectic form; (3) W_i^B and W_j^B are σ -orthogonal if $i \neq j$; (4) $V = W_1^B \oplus \cdots \oplus W_n^B$; and (5) $B_i = B|_{W_i^B}$ is stable.

Now define a positive definite inner product γ on V by

$$\gamma(v, w) = \delta_1 \tau_{A_1}(v_1, w_1) + \cdots + \delta_n \tau_{A_n}(v_n, w_n)$$

where

$$\begin{aligned} v &= v_1 + \cdots + v_n \in V_1 \oplus \cdots \oplus V_n, \\ w &= w_1 + \cdots + w_n \in V_1 \oplus \cdots \oplus V_n, \\ \delta_i &= \begin{cases} 1/\alpha_i & \text{if } \tau_{A_i} \text{ is positive definite} \\ -1/\alpha_i & \text{if } \tau_{A_i} \text{ is negative definite.} \end{cases} \end{aligned}$$

Clearly V_i and V_j are γ -orthogonal if $i \neq j$. Consider the inner product

$$\tau_{B_i}: W_i^B \times W_i^B \rightarrow \mathbb{R}: (w_i, w'_i) \rightarrow \sigma(B_i w_i, w'_i).$$

For $w_i \in W_i^B$ there is a unique $v_i \in V_i$ such that $w_i = P_B^{-1} v_i$. Then

$$\sigma(B_i w_i, w_i) = \sigma(B P_B^{-1} v_i, P_B^{-1} v_i) = \sigma(P_B B P_B^{-1} v_i, v_i)$$

where the last equality follows because $P_B \in Sp(\sigma, \mathbb{R})$. Now put

$$\hat{B} = P_B B P_B^{-1} - A,$$

and write

$$\begin{aligned}\tau_{A_i} &> 0 && \text{if } \tau_{A_i} \text{ is positive definite,} \\ \tau_{A_i} &< 0 && \text{if } \tau_{A_i} \text{ is negative definite.}\end{aligned}$$

Then

$$\begin{aligned}\sigma(P_B B P_B^{-1} v_i, v_i) &= \sigma(A_i v_i, v_i) + \sigma(\hat{B} v_i, v_i) \\ \left\{ \begin{aligned} &\geq \sigma(A_i v_i, v_i) - |\sigma(\hat{B} v_i, v_i)| && \text{if } \tau_{A_i} > 0, \\ &\leq \sigma(A_i v_i, v_i) + |\sigma(\hat{B} v_i, v_i)| && \text{if } \tau_{A_i} < 0, \end{aligned} \right. \\ \left\{ \begin{aligned} &\geq \alpha_i \gamma(v_i, v_i) - \|\sigma\| \|\hat{B}\| \gamma(v_i, v_i) && \text{if } \tau_{A_i} > 0, \\ &\leq -\alpha_i \gamma(v_i, v_i) + \|\sigma\| \|\hat{B}\| \gamma(v_i, v_i) && \text{if } \tau_{A_i} < 0. \end{aligned} \right.\end{aligned}$$

Because the map $B \rightarrow P_B$ is continuous and $P_A = I$, it follows that the map $\mathcal{E}: B \rightarrow \|\hat{B}\|$ is continuous and $\mathcal{E}(A) = 0$. Therefore there is a $\delta > 0$ such that $\mathcal{E}(B) < \frac{1}{2} \min \alpha_i / \|\sigma\|$ for all $\|B - A\| < \delta$. Thus

$$\tau_{\hat{B}_i}: V_i \times V_i \rightarrow \mathbb{R}: (v_i, y_i) \rightarrow \sigma(\hat{B}_i v_i, y_i) = \sigma(P_B B P_B^{-1} v_i, y_i)$$

is positive definite (negative definite) if τ_{A_i} is positive definite (negative definite). Since τ_{B_i} is equivalent to $\tau_{\hat{B}_i}$ by $P_B^{-1}|V_i$, it follows that τ_{B_i} has the same sign as τ_{A_i} . Thus B is strongly stable. ■

Here we give another proof that the set of strongly stable maps is open. This new viewpoint leads in a natural way to a new characterization (Theorem 3.3 below) of this set.

Let \mathcal{S} be the set of stable mappings, \mathcal{S}_s the set of strongly stable mappings, and \mathcal{U} the set of real infinitesimally symplectic mappings with at least one eigenvalue with nonzero real part. We then

CLAIM. $\mathcal{S} \setminus \mathcal{S}_s$ is contained in the boundary of \mathcal{U} .

Proof. If $A \in \mathcal{S} \setminus \mathcal{S}_s$, then τ_A is not sign definite on all primary A -invariant subspaces V_i^A of A which are σ -nondegenerate. Using the normal form for A of Corollary 1.1, there is an index i and an A -invariant, σ -nondegenerate subspace $W \subseteq V_i^A$ with symplectic basis $\{e_1, e_2, f_1, f_2\}$ such that the matrix of $A|W$ is

$$\left[\begin{array}{cc|cc} & & -\alpha_i & \\ & & \alpha_i & \\ \hline & & & \\ \alpha_i & & & \\ & & & \\ & & -\alpha_i & \end{array} \right] \quad \text{where } \alpha_i > 0.$$

Let

$$B|W = \left[\begin{array}{cc|cc} 0 & & & \\ 1 & 0 & & \\ \hline & & 0 & -1 \\ & & & 0 \end{array} \right] \quad \text{and} \quad B|W^\sigma = 0.$$

Then $B \in sp(\sigma, \mathbb{R})$. Since W is $A + \epsilon B$ -invariant and $A + \epsilon B \in sp(\sigma, \mathbb{R})$, W^σ is $A + \epsilon B$ -invariant. Therefore the characteristic polynomial $\chi_{A+\epsilon B}(t)$ of $A + \epsilon B$ factors:

$$\chi_{A+\epsilon B}(t) = \chi_{(A+\epsilon B)|W}(t) \chi_{(A+\epsilon B)|W^\sigma}(t).$$

But for $\epsilon \neq 0$

$$\chi_{(A+\epsilon B)|W}(t) = \det \left[\begin{array}{cc|cc} -t & & -\alpha_i & \\ \epsilon & -t & & \alpha_i \\ \hline & & -t & -\epsilon \\ \alpha_i & & & -t \\ & -\alpha_i & & \end{array} \right] = t^4 + 2\alpha_i^2 t^2 + \alpha_i^4 + \epsilon^2 \alpha_i^2$$

has four roots with nonzero real parts. Therefore $A + \epsilon B \in \mathcal{U}$ for all $\epsilon \neq 0$. That is, A is contained in the boundary of \mathcal{U} . ■

Suppose now that \mathcal{S}_s is not open in $sp(\sigma, \mathbb{R})$. Then every open ball \mathcal{B} about $A \in \mathcal{S}_s$ contains $A' \in \mathcal{S} \setminus \mathcal{S}_s$, since $\mathcal{S}_s \subseteq \mathcal{S}$. Let \mathcal{B}' be an open ball about A' with $\mathcal{B}' \subseteq \mathcal{B}$. Then by the claim there is $U \in \mathcal{B}' \cap \mathcal{U}$. Thus every open ball \mathcal{B} about A intersects \mathcal{U} . But by Corollary 3.1, there is an open ball \mathcal{B}'' about A such that $\mathcal{B}'' \subseteq \mathcal{S}$. Since $\mathcal{S} \cap \mathcal{U} = \emptyset$, this is a contradiction. ■

Let \mathcal{T} be the set of real infinitesimally symplectic mappings with eigenvalues which are either zero or purely imaginary. We will show that the set of strongly stable mappings \mathcal{S}_s is the interior of \mathcal{T} . It suffices to show that \mathcal{S}_s is dense in \mathcal{T} , because $\mathcal{S}_s \subseteq \mathcal{T}$. Note that $sp(\sigma, \mathbb{R}) = \mathcal{T} \cup \mathcal{U}$ and \mathcal{U} is open, which implies \mathcal{T} is closed [6, p. 15].

We recall some terminology and results of [6] concerning normal forms of elements of $sp(\sigma, \mathbb{R})$. Consider the pair $(A|W, W)$ where $A \in sp(\sigma, \mathbb{R})$ and W is a σ -nondegenerate, A -invariant subspace of (V, σ) . Two pairs $(A|W, W)$ and $(B|W', W')$ are *equivalent* if and only if there is a real symplectic mapping $P \in Sp(\sigma, \mathbb{R})$ such that $P|W' = W$ and $P^{-1}AP|W' = B|W'$. An equivalence class of pairs is called a *type*. Suppose $(A|W, W)$ is an element of the type Δ . Then writing $A = S + N$ where S is semisimple and N is nilpotent with $SN = NS$, the *height* of Δ is the smallest nonnegative integer m such that $N^{m+1}|W = 0$ while $N^m|W \neq 0$. Suppose that $(A|W, W) \in \Delta$ and there are

proper A -invariant, σ -nondegenerate, σ -orthogonal subspaces W_1 and W_2 such that $W = W_1 \oplus W_2$. Then we write $\Delta = \Delta_1 + \Delta_2$ where $(A|W_i, W_i) \in \Delta_i$ for $i = 1, 2$. A type Δ which cannot be written as the sum of two types is an *indecomposable* type; in other words, if $(A|W, W) \in \Delta$ and W has no proper σ -nondegenerate, A -invariant subspace, then Δ is indecomposable. The main result in [6] is: every type is the sum of indecomposable types which are uniquely determined up to the order of the summands. From the list of representatives of indecomposable types [6, p. 24] only those in list #1 (below) of height m can occur for elements of \mathcal{T} .

LIST 1

Matrix Representations with Respect to Symplectic Bases of Indecomposable
Infinitesimally Symplectic Maps with Zero or Purely Imaginary Eigenvalues

	Matrix	Characteristic polynomial
1.	$\left[\begin{array}{c c} \begin{matrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{matrix} & \\ \hline \begin{matrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{matrix} & \begin{matrix} 0 & -1 \\ & 0 & -1 \\ & & \ddots & \\ & & & 0 & -1 \\ & & & & 0 \end{matrix} \end{array} \right]$	$\frac{m+1}{2}$ t^{m+1}
	$m \text{ odd}, \delta = (-1)^{(m-1)/2}\epsilon,$ $\epsilon = \sigma(e_1, N^m e_1), \epsilon^2 = 1$	
2.	$\left[\begin{array}{c c} \begin{matrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{matrix} & \\ \hline \begin{matrix} 0 & -1 \\ & 0 & -1 \\ & & \ddots & \\ & & & -1 \\ & & & & 0 \end{matrix} \end{array} \right]$	$m+1$ $t^{2(m+1)}$ $m+1$
	$m \text{ even}$	

List 1 continued

LIST 1—Continued

	Matrix	Characteristic polynomial
3.	$\left[\begin{array}{ccc ccc} A & & & & & \\ I & A & & & & \\ & & \ddots & & & \\ & & & I & A & \\ \hline 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \delta I & & \end{array} \right]$	$(t^2 + \alpha^2)^{m+1}$
	$m \text{ odd, } \delta = (-1)^{(m-1)/2} \epsilon, \epsilon = \sigma(e_1, N^m e_1),$ $\epsilon^2 = 1, A = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}, \alpha > 0, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	
4.	$\left[\begin{array}{ccc ccc} 0 & & & & & -\epsilon\alpha \\ 1 & 0 & & & & \epsilon\alpha \\ & 1 & \ddots & & & \\ & & & 1 & 0 & \\ \hline & & & \epsilon\alpha & 0 & -1 \\ & & & & 0 & -1 \\ & & & & & \ddots \\ \epsilon\alpha & & & & & -1 \\ & & & & & 0 \end{array} \right]$	$(t^2 + \alpha^2)^{m+1}$
	$m \text{ even, } \epsilon = \sigma(e_1, SN^m e_1),$ $\epsilon^2 = 1, \alpha > 0$	

The following argument shows that the set \mathcal{P}_{reg} of strongly stable mappings with distinct eigenvalues is dense in \mathcal{T} . Let \mathcal{T}_n be the set of all $\mathcal{O} \in \mathcal{T}$ such that the decomposition of \mathcal{O} into indecomposable types contains no type of height greater than n and at least one type of height n . Clearly $\mathcal{T} = \bigcup_{0 \leq n \leq \dim V} \mathcal{T}_n$. Suppose by induction that \mathcal{P}_{reg} is dense in \mathcal{T}_n for all $n < m$. Let $\mathcal{O} \in \mathcal{T}_m$. Write the type (\mathcal{O}, V) as the sum of two types $(\mathcal{O} | V^i, V^i) \in \Delta^i$ with $i = 1, 2$, where Δ^1 is the sum of indecomposable types $(\mathcal{O} | V_j^1, V_j^1) \in \Delta_j^1$ all with height equal to m and Δ^2 is empty or has height n for some $n < m$. Thus $\mathcal{O} | V^1 \in \mathcal{T}_m$ and $\mathcal{O} | V^2 \in \mathcal{T}_n$ for some $n < m$. Note that Δ^1 is nonempty because $\mathcal{O} \in \mathcal{T}_m$. Choose $(A | W_j, W_j) \in \Delta_j^1$ so that $A | W_j$ is in one of the normal forms given in list #1. Consider the one parameter family of perturbations $A_\lambda | W_j$ of representatives of indecomposable types in \mathcal{T} of height m

which corresponds to $A \mid W_j$ and is given in list #2. From the characteristic polynomial of $A_\lambda \mid W_j$ and the list of indecomposable types in \mathcal{T} of height m given in list #1, we see that if $\lambda \neq 0$ then $A_\lambda \mid W_j \in \mathcal{T}_n$ for some $n < m$, that is,

LIST 2

Infinitesimally Symplectic Perturbations Depending on a Parameter λ of the
Corresponding Matrix in List 1

	Matrix	Characteristic polynomial
1.	$\left[\begin{array}{c c} \begin{matrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & -\lambda^2 \\ & & & 1 & 0 \end{matrix} & \begin{matrix} \\ \\ \\ \\ \end{matrix} \\ \hline \begin{matrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \delta & \end{matrix} & \begin{matrix} 0 & -1 \\ & 0 & -1 \\ & & \ddots & \\ & & & 0 & -1 \\ & & & \lambda^2 & 0 \end{matrix} \end{array} \right]$	$\frac{m+1}{2}$ $t^{m-3}(t^2 + \lambda^2)^2$ $\frac{m+1}{2}$
	m odd, $m \geq 3$, if $m = 1$ then $\begin{bmatrix} 0 & -\delta\lambda^2 \\ \delta & 0 \end{bmatrix}$	$t^2 + \lambda^2$
2.	$\left[\begin{array}{c c} \begin{matrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 1 & 0 \end{matrix} & \begin{matrix} & & & & -\lambda \\ & & & 0 & \\ & & 0 & \ddots & \\ & & & 0 & -\lambda \end{matrix} \\ \hline \begin{matrix} & & & \lambda & \\ & & 0 & & \\ & & & 0 & \\ 0 & \ddots & & & \\ \lambda & & & & 0 \end{matrix} & \begin{matrix} 0 & -1 \\ & 0 & -1 \\ & & \ddots & \\ & & & -1 \\ & & & & 0 \end{matrix} \end{array} \right]$	$m+1$ $t^{2m-2}(t^2 + \lambda^2)^2$ $m+1$
	m even, $m \geq 2$, if $m = 0$ then $\begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix}$	$t^2 + \lambda^2$

List 2 continued

LIST 2—Continued

	Matrix	Characteristic polynomial
3.	$\left[\begin{array}{ccc ccc} B & & & & & \\ I & A & & & & \\ & I & \ddots & & & \\ & & I & A & & \\ \hline 0 & & & & B-I & \\ & \ddots & & & A & -I \\ & & 0 & & & \ddots & -I \\ & & & \delta I & & & A \end{array} \right]_{m+1}$	$(t^2 + \alpha^2)^{m-1}(t^2 + (\alpha + \lambda)^2)$
	$m \text{ odd}, m \geq 3, A = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}, \alpha > 0,$ $B = \begin{bmatrix} 0 & -(\alpha + \lambda) \\ \alpha + \lambda & 0 \end{bmatrix}, \lambda > 0, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$ if $m = 1$ then $\begin{bmatrix} A & -\delta \lambda^2 I \\ \delta I & A \end{bmatrix}$	$(t^2 + (\alpha - \lambda)^2)(t^2 + (\alpha + \lambda)^2)$
4.	$\left[\begin{array}{ccc ccc} 0 & & & & & -\epsilon\beta \\ 1 & 0 & & & & \epsilon\alpha \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & 0 & -\epsilon\beta \\ \hline & & & & \epsilon\beta & -1 \\ & & & & -\epsilon\alpha & 0 \\ & & & & \epsilon\alpha & \\ & & & & \ddots & \\ & & & & -\epsilon\alpha & \\ \epsilon\beta & & & & & -1 \\ & & & & & 0 \end{array} \right]_{m+1}$	$(t^2 + \alpha^2)^{m-1}(t^2 + (\alpha + \lambda)^2)^2$
	$m \text{ even}, m \geq 2, \alpha > 0,$ $\lambda > 0, \beta = \alpha + \lambda,$ if $m = 0$ then $\begin{bmatrix} 0 & -\epsilon\beta \\ \epsilon\beta & 0 \end{bmatrix}$	$t^2 + (\alpha + \lambda)^2$

the type $(A_\lambda | W_j, W_j) \in \Delta_{1,\lambda}^1$ with $\lambda \neq 0$ is a sum of indecomposable types of height less than m . Because $(\mathcal{O} | V_j^1, V_j^1)$ and $(A | W_j, W_j) \in \Delta_j^1$, there is a real symplectic mapping $P_j \in Sp(\sigma, \mathbb{R})$ such that $P_j V_j^1 = W_j$ and $\mathcal{O} | V_j^1 = P_j^{-1} A P_j | V_j^1$. Let

$$\mathcal{O}_\lambda = \begin{cases} P_j^{-1} A_\lambda P_j & \text{on } V_j^1 \\ \mathcal{O} & \text{on } V_j^2 \end{cases}$$

If $\lambda \neq 0$, then $\mathcal{O} \in \mathcal{T}_n$ for some $n < m$ because $(\mathcal{O}_\lambda | V_j^1, V_j^1)$ and $(A_\lambda | W_j, W_j) \in \Delta_j^1$. By the induction hypothesis \mathcal{S}_{reg} is dense in \mathcal{T}_n for all $n < m$. Because for every $\lambda \neq 0$ there is $n < m$ such that $\mathcal{O}_\lambda \in \mathcal{T}_n$, given $\epsilon > 0$ there is $B_\lambda \in \mathcal{S}_{\text{reg}}$ such that $\|\mathcal{O}_\lambda - B_\lambda\| \leq \frac{1}{2}\epsilon$. Since $\mathcal{O}_\lambda \rightarrow \mathcal{O}$ as $\lambda \rightarrow 0$ by construction, for λ sufficiently small and positive $\|\mathcal{O}_\lambda - \mathcal{O}\| \leq \frac{1}{2}\epsilon$. Thus $\|\mathcal{O} - B_\lambda\| \leq \epsilon$, and this proves that \mathcal{S}_{reg} is dense in \mathcal{T}_m .

To complete the induction we have to examine the case $m = 0$. Suppose $\mathcal{O} \in \mathcal{T}_0$. Then \mathcal{O} is semisimple, and its normal form A obtained from list #1 (or a slight generalization of the proof of Corollary 1.1) is $\begin{bmatrix} 0 & -D \\ D & 0 \end{bmatrix}$ where $D = \text{diag}(d_1, \dots, d_{2n})$. Let $\mu = (\mu_1, \dots, \mu_{2n})$ and $A_\mu = \begin{bmatrix} 0 & -D_\mu \\ D_\mu & 0 \end{bmatrix}$ where $D_\mu = \text{diag}(d_1 + \mu_1, \dots, d_{2n} + \mu_{2n})$. Clearly there are μ arbitrarily close to 0 such that A_μ has distinct purely imaginary eigenvalues. Defining $\mathcal{O}_\mu = P^{-1} A_\mu P$ where $\mathcal{O} = P^{-1} A P$ and $P \in Sp(\sigma, \mathbb{R})$, we obtain: given $\epsilon > 0$ there is a μ such that $\mathcal{O}_\mu \in \mathcal{S}_{\text{reg}}$ and $\|\mathcal{O} - \mathcal{O}_\mu\| \leq \epsilon$. Thus \mathcal{S}_{reg} is dense in \mathcal{T}_0 , and we have proved

THEOREM 3.3. *In $sp(\sigma, \mathbb{R})$ the set of strongly stable maps is the interior of the set of maps with zero or purely imaginary eigenvalues.*

In this section we determine the connected components of \mathcal{S}_s , the set of strongly stable matrices in $sp(J_{2m}, \mathbb{R})$.

Let

$$\mathcal{F} = \{(k, r, \epsilon) \mid \text{conditions (i), (ii), (iii) hold}\}$$

- (i) $k \in \mathbb{Z}$ with $1 \leq k \leq m$;
- (ii) $r = (r_1, \dots, r_k) \in \mathbb{Z}^k$ with $1 \leq r_j \leq m$ for $j = 1, \dots, k$ and $r_1 + \dots + r_k = m$;
- (iii) $\epsilon = (\epsilon_1, \dots, \epsilon_k) \in \mathbb{Z}^k$ with $\epsilon_j \in \{-1, 1\}$ for $j = 1, \dots, k$;

and let

$$\mathcal{P}_k = \{\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k \mid 0 < \alpha_1 < \dots < \alpha_k\}.$$

For $\alpha_1 \in \mathcal{P}_1$ and $\epsilon_1 \in \{-1, 1\}$ let $\Delta^{\epsilon_1}(\alpha_1)$ be the type which contains the pair (A, \mathbb{R}^2) where $A = \begin{bmatrix} 0 & -\epsilon_1 \alpha_1 \\ \epsilon_1 \alpha_1 & 0 \end{bmatrix}$ is in $sp(J_2, \mathbb{R})$. For convenience we write $r\Delta^{\epsilon_1}(\alpha_1)$ instead of the sum of $\Delta^{\epsilon_1}(\alpha_1)$ repeated r times. For $(k, r, \epsilon) \in \mathcal{F}$ define

$$\Delta(r, \epsilon, \alpha) = r_1 \Delta^{\epsilon_1}(\alpha_1) + \dots + r_k \Delta^{\epsilon_k}(\alpha_k),$$

$$\mathcal{S}_s(k, r, \epsilon) = \{\mathcal{O} \in \mathcal{S}_s \mid (\mathcal{O}, \mathbb{R}^{2m}) \in \Delta(r, \epsilon, \alpha) \text{ for some } \alpha = (\alpha_1, \dots, \alpha_k) \in \mathcal{P}_k\}.$$

From Corollary 1.1 we see that every element of \mathcal{S}_s is a member of some $\mathcal{S}_s(k, r, \epsilon)$ —in fact a unique member because of the ordering of the components of $\alpha \in \mathcal{P}_k$. Thus

$$\mathcal{S}_s = \bigcup_{(k, r, \epsilon) \in \mathcal{F}} \mathcal{S}_s(k, r, \epsilon)$$

is a finite disjoint union. Consequently the sets $\mathcal{S}_s(k, r, \epsilon)$ are candidates for the connected components of \mathcal{S}_s . For $(k, r, \epsilon) \in \mathcal{F}$ let

$$\mathcal{N}(k, r, \epsilon) = \left\{ A_\alpha \in sp(J_{2m}, \mathbb{R}) \mid A_\alpha = \begin{bmatrix} 0 & -a_\alpha \\ -a_\alpha & 0 \end{bmatrix} \text{ with} \right. \\ \left. a_\alpha = \begin{bmatrix} \epsilon_1 \alpha_1 I_{r_1} & & \\ & \ddots & \\ & & \epsilon_k \alpha_k I_{r_k} \end{bmatrix} \text{ for some } \alpha = (\alpha_1, \dots, \alpha_k) \in \mathcal{P}_k \right\}.$$

Clearly $(A_\alpha, \mathbb{R}^{2m}) \in \Delta(r, \epsilon, \alpha)$ for every $\alpha \in \mathcal{P}_k$. Since

$$\Delta(r, \epsilon, \alpha) = \{P^{-1}A_\alpha P \in sp(J_{2m}, \mathbb{R}) \mid P \in Sp(J_{2m}, \mathbb{R})\},$$

the mapping

$$\psi: Sp(J_{2m}, \mathbb{R}) \times \mathcal{N}(k, r, \epsilon) \rightarrow \mathcal{S}_s(k, r, \epsilon): (P, A_\alpha) \rightarrow P^{-1}A_\alpha P$$

is surjective. The mapping ψ is also continuous because

$$\zeta: Sp(J_{2m}, \mathbb{R}) \times sp(J_{2m}, \mathbb{R}) \rightarrow sp(J_{2m}, \mathbb{R}): (Q, B) \rightarrow Q^{-1}BQ$$

is continuous. Therefore $\mathcal{S}_s(k, r, \epsilon)$ is connected because $Sp(J_{2m}, \mathbb{R})$ is connected [10, p. 92], and $\mathcal{N}(k, r, \epsilon)$ is connected since it is a convex cone in $sp(J_{2m}, \mathbb{R})$.

Actually more can be said about $\mathcal{S}_s(k, r, \epsilon)$. Below we will show that $\mathcal{S}_s(k, r, \epsilon)$ is a simply connected differentiable submanifold of $sp(J_{2m}, \mathbb{R})$ which is diffeomorphic to $(Sp(J_{2m}, \mathbb{R})/U(r_1) \times \cdots \times U(r_k)) \times \mathcal{N}(k, r, \epsilon)$.

First we show that $\mathcal{S}_s(k, r, \epsilon)$ is a submanifold of $sp(J_{2m}, \mathbb{R})$. Let u_r be the Lie algebra of $U_r = U(r_1) \times \cdots \times U(r_k)$. For $A_\alpha \in \mathcal{N}(k, r, \epsilon) \subseteq sp(J_{2m}, \mathbb{R})$ let \mathcal{M} be the image of ad_{A_α} . By Proposition 2.2 u_r is the centralizer of A_α in $sp(J_{2m}, \mathbb{R})$, that is, u_r is the kernel of ad_{A_α} . Therefore $sp(J_{2m}, \mathbb{R}) = \mathcal{M} + u_r$. There is an open set $\mathcal{V} \subseteq \mathcal{M}$ about 0 such that $(\exp \mathcal{V}) U_r = \mathcal{H}$ is an open subset of $Sp(J_{2m}, \mathbb{R})/U_r = M$ containing IU_r [11, p. 350]. Let $Q \in \exp \mathcal{V}$. Then the map

$$s: \mathcal{H} \subseteq M \rightarrow Sp(J_{2m}, \mathbb{R}): QU_r \rightarrow Q$$

is a smooth local section for the fibration $p: Sp(J_{2m}, \mathbb{R}) \rightarrow M$ about IU_r [11, p. 352]. Consider the map

$$\begin{aligned} \varphi_s: \mathcal{H} \times u_r \subseteq M \times u_r &\rightarrow Sp(J_{2m}, \mathbb{R}): (QU_r, u) \\ &\rightarrow P^{-1}s(QU_r)^{-1}(A_\alpha + u)s(QU_r)P \\ &= P^{-1}Q^{-1}(A_\alpha + u)QP. \end{aligned}$$

Since

$$T\varphi_s(IU_r, 0): \mathcal{M} \times u_r \rightarrow sp(J_{2m}, \mathbb{R}): (v, w) \rightarrow P^{-1}([A_\alpha, v] + w)P$$

is an isomorphism (compare the proof of Theorem 3.1), φ_s is a local diffeomorphism with

$$\varphi_s(IU_r, 0) = P^{-1}A_\alpha P = \mathcal{O}.$$

There is a sufficiently small open set $\mathcal{U} \subseteq u_r$ containing 0 such that

$$A + u \in \mathcal{N}(k, r, \epsilon)$$

for all

$$u \in \mathcal{U}' = [(-\mathcal{N}(k, r, \epsilon)) \cup \{0\} \cup \mathcal{N}(k, r, \epsilon)] \cap \mathcal{U},$$

and a sufficiently small open set $\mathcal{H}' \subseteq \mathcal{H} \subseteq M$ containing IU_r such that $\varphi_s|(\mathcal{H}' \times \mathcal{U}')$ is a diffeomorphism onto an open neighborhood of $\mathcal{O} \in \mathcal{S}_s(k, r, \epsilon)$. Thus $\mathcal{S}_s(k, r, \epsilon)$ is a submanifold of $sp(J_{2m}, \mathbb{R})$.

To show that $\mathcal{S}_s(k, r, \epsilon)$ is diffeomorphic to $M \times \mathcal{N}(k, r, \epsilon)$, we first observe that for every $A_\alpha \in \mathcal{N}(k, r, \epsilon)$ the centralizer of A_α in $Sp(J_{2m}, \mathbb{R})$ is

$$\begin{aligned} U_r = \left\{ \begin{bmatrix} \mathcal{O}_1 & & & -B_1 \\ & \ddots & & \\ & & \mathcal{O}_k & -B_k \\ B_1 & & & \mathcal{O}_1 \\ & \ddots & & \\ & & B_k & \mathcal{O}_k \end{bmatrix} \in Sp(J_{2m}, \mathbb{R}) \mid \begin{bmatrix} \mathcal{O}_j & -B_j \\ B_j & \mathcal{O}_j \end{bmatrix} \right. \\ \left. \in Sp(J_{2r_j}, \mathbb{R}), \mathcal{O}_j B_j^t = B_j^t \mathcal{O}_j, \mathcal{O}_j \mathcal{O}_j^t + B_j B_j^t = I_{r_j} \right\} \end{aligned}$$

(compare the proof of Proposition 2.2). Note that U_r does not depend on the choice of $A_\alpha \in \mathcal{N}(k, r, \epsilon)$. Therefore the map

$$\psi: Sp(J_{2m}, \mathbb{R}) \times \mathcal{N}(k, r, \epsilon) \rightarrow \mathcal{S}_s(k, r, \epsilon): (P, A_\alpha) \rightarrow P^{-1}A_\alpha P$$

induces a map

$$\tilde{\psi}: M \times \mathcal{N}(k, r, \epsilon) \rightarrow \mathcal{S}_s(k, r, \epsilon): (QU_r, A_\alpha) \rightarrow Q^{-1}A_\alpha Q.$$

The map $\tilde{\psi}$ is bijective: surjective because ψ is surjective; injective because if $\mathcal{A}, \mathcal{B} \in \mathcal{S}_s(k, r, \epsilon)$, then there are $P, R \in Sp(J_{2m}, \mathbb{R})$ and a unique $A_\alpha \in \mathcal{N}(k, r, \epsilon)$ such that $P^{-1}\mathcal{A}P = A_\alpha = R^{-1}\mathcal{B}R$ which implies $PR^{-1} \in U_r$; thus $PU_r = RU_r$. Because $\tilde{\psi}(QU_r, A_\alpha + B) = \varphi_s(QU_r, B)$ for $(QU_r, B) \in \mathcal{H}' \times \mathcal{W}'$, the map $\tilde{\psi}$ is a local diffeomorphism. Hence $\tilde{\psi}$ is a diffeomorphism.

To show that $\mathcal{S}_s(k, r, \epsilon)$ is simply connected it suffices to show that M is simply connected, because $\mathcal{N}(k, r, \epsilon)$ is convex, which implies that it is simply connected. Because $p: Sp(J_{2m}, \mathbb{R}) \rightarrow M$ is a fibration with connected fiber U_r , we have the following exact sequence of fundamental groups [11, p. 352]

$$\pi_1(U_r) \xrightarrow{-j} \pi_1(Sp(J_{2m}, \mathbb{R})) \xrightarrow{-k} \pi_1(M) \longrightarrow 0.$$

If j is surjective then $\pi_1(M) = 0$. Because $\pi_1(Sp(J_{2m}, \mathbb{R})) = Z$, it suffices to find an element of $\pi_1(U_r)$ which is a generator of $\pi_1(Sp(J_{2m}, \mathbb{R}))$. Let

$$\gamma: S^1 \rightarrow U_r \subseteq Sp(J_{2m}, \mathbb{R}): t \rightarrow \left[\begin{array}{cc|cc} \cos tI_{r_1} & 0 & -\sin tI_{r_1} & 0 \\ 0 & I & 0 & 0 \\ \hline \sin tI_{r_1} & 0 & \cos tI_{r_1} & 0 \\ 0 & 0 & 0 & I \end{array} \right].$$

Then $[\gamma]$, the homotopy class of γ , is the desired element of $\pi_1(U_r)$.

4. A REAL ANALYTIC CONJUGACY THEOREM

In this section we show that a real analytic curve of infinitesimally symplectic mappings passing through a strongly stable mapping is locally real symplectically conjugate to a real analytic curve of strongly stable mappings which are in normal form.

We start by proving a unitary analogue of the desired theorem.

LEMMA 4.1. *Let $h: \mathbb{R} \rightarrow \mathfrak{u}(n)$ be a real analytic curve of skew hermitian mappings where $h(0)$ is diagonal. Then there is an open interval θ containing 0*

and a real analytic curve of unitary mappings $U: \theta \subseteq \mathbb{R} \rightarrow U(n)$ such that

$$H: \theta \subseteq \mathbb{R} \rightarrow u(n): t \rightarrow U(t)^{-1} h(t) U(t)$$

is real analytic and diagonal.

Proof [7]. Since h is real analytic, $h(t) = \sum_{i=0}^{\infty} h_i t^i$ where $h_i \in u(n)$. If $h_j = i\lambda_j I$ for some $\lambda_j \in \mathbb{R}$ and all j , then put $U(t) = I$, thus proving the theorem. Otherwise there is a smallest subscript k such that $h_j = \lambda_j I$ for $0 \leq j \leq k-1$ while $h_k \neq i\lambda I$ for any $\lambda \in \mathbb{R}$. Thus h may be written $h(t) = (\sum_{j=0}^{k-1} i\lambda_j t^j)I + (h_k + \sum_{\ell \geq 1} h_{k+\ell} t^\ell) t^k$. By the spectral theorem there is a unitary mapping U' which conjugates h_k to a diagonal matrix $h'_k \in u(n)$, that is

$$(U')^{-1} h_k U' = h'_k = \begin{bmatrix} i\mu_1 I_{s_1} & & 0 \\ & \ddots & \\ 0 & & i\mu_m I_{s_m} \end{bmatrix} \quad \text{where } \mu_i \in \mathbb{R},$$

$\mu_i \neq \mu_j$ if $i \neq j$, and $s_1 + \cdots + s_m = n$. Since $h_k \neq i\lambda I$ for any $\lambda \in \mathbb{R}$, $h'_k \neq i\lambda I$ for any $\lambda \in \mathbb{R}$. So we may suppose that $s_1 < n$. It is easy to see that the centralizer of h'_k in $u(n)$ is $u(s_1) \times \cdots \times u(s_m)$. Since h'_k is semisimple we may use Theorem 3.1 with $G = U(n)$ and $L_G = u(n)$ to obtain an open interval θ containing 0 and a real analytic curve $U_1: \theta \subseteq \mathbb{R} \rightarrow U(n)$ which conjugates $h'_k + \sum_{\ell \geq 1} h'_{k+\ell} t^\ell$ where $h'_{k+\ell} = (U')^{-1} h_{k+\ell} U'$ into a real analytic curve $\tilde{h}_k: \theta \subseteq \mathbb{R} \rightarrow u(s_1) \times \cdots \times u(s_m)$ with $\tilde{h}_k(0) = h'_k$. Therefore for all $t \in \theta$

$$[U' U_1(t)]^{-1} h(t) [U' U_1(t)] = \left(\sum_{j=0}^{k-1} i\lambda_j t^j \right) I + \tilde{h}_k(t) t^k.$$

Repeating the above argument on each block of \tilde{h}_k gives the desired result after a finite number of repetitions since the block size decreases at each repetition. ■

We now prove

THEOREM 4.1. *Let $\gamma: \mathbb{R} \rightarrow sp(\sigma, \mathbb{R})$ be a real analytic curve with $\gamma(0)$ strongly stable. Then there is an open interval θ containing 0 and a real analytic curve $\beta: \theta \subseteq \mathbb{R} \rightarrow Sp(\sigma, \mathbb{R})$ such that the curve*

$$\Gamma: \theta \subseteq \mathbb{R} \rightarrow sp(\sigma, \mathbb{R}): t \rightarrow \beta(t)^{-1} \gamma(t) \beta(t)$$

is real analytic and $\Gamma(t)$ is in normal form.

Proof. Since $\gamma(0)$ is strongly stable there is a $P \in Sp(\sigma, \mathbb{R})$ such that $P^{-1}\gamma(0)P = A$ is in normal form; that is

$$A = \begin{bmatrix} 0 & -\alpha_1 I_{p_1, n_1} & & \\ & \ddots & \ddots & \\ & & -\alpha_m I_{p_m, n_m} & \\ \alpha_1 I_{p_1, n_1} & & & 0 \\ & \ddots & & \\ & & \alpha_m I_{p_m, n_m} & \end{bmatrix}$$

where $\alpha_i > 0$, $\alpha_i \neq \alpha_j$ if $i \neq j$, $2 \sum_i (p_i + n_i) = \dim V$, either $p_i = 0$ or $n_i = 0$. Define the real analytic curve

$$\tilde{\gamma}: \theta \subseteq \mathbb{R} \rightarrow sp(\sigma, \mathbb{R}): t \rightarrow P^{-1}\gamma(t)P.$$

Then $\tilde{\gamma}(0) = A$. Since A is strongly stable and in normal form, from Theorem 3.1, there is an open interval $\theta' \subseteq \theta$ containing 0 and real analytic curves

$$S: \theta' \subseteq \mathbb{R} \rightarrow Sp(\sigma, \mathbb{R})$$

$$G: \theta' \subseteq \mathbb{R} \rightarrow C(A) \subseteq sp(\sigma, \mathbb{R})$$

with $S(0) = I$ and $G(0) = A$ such that $G(t) = S(t)^{-1} \tilde{\gamma}(t) S(t)$ for all $t \in \theta'$. By the corollary of Proposition 2.2, $C(A)$ the centralizer of A in $sp(\sigma, \mathbb{R})$ consists of real infinitesimally symplectic maps of the form

$$\begin{bmatrix} a_1 & & & -b_1 & & \\ & \ddots & & \ddots & & \\ & & a_m & & -b_m & \\ b_1 & & & -a_1^t & & \\ & \ddots & & \ddots & & \\ & & b_m & & -a_m^t & \end{bmatrix}$$

where $a_\ell, b_\ell \in gl(p_\ell + n_\ell, \mathbb{R})$ and $a_\ell = -a_\ell^t$, $b_\ell = b_\ell^t$. The Lie algebra of the unitary group is imbedded in $sp(J, \mathbb{R})$ by the mapping

$$j: u(n) \rightarrow sp(J, \mathbb{R}): a + ib \rightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

where $a = -a^t$, $b = b^t$. Therefore

$$j^{-1}(C(A)) = \begin{bmatrix} a_1 + ib_1 & & \\ & \ddots & \\ & & a_m + ib_m \end{bmatrix} \in u(p_1 + n_1) \times \cdots \times u(p_m + n_m).$$

Consider the real analytic curve of skew hermitian maps

$$\tilde{G}: \theta' \subseteq \mathbb{R} \rightarrow u(n): t \rightarrow j^{-1}(G(t)).$$

Then by Lemma 4.1 there is an open interval $\theta'' \subseteq \theta'$ containing 0 and real analytic curves

$$U: \theta'' \subseteq \mathbb{R} \rightarrow U(n): t \rightarrow U(t),$$

$$\tilde{I}: \theta'' \subseteq \mathbb{R} \rightarrow u(n): t \rightarrow \begin{bmatrix} i\lambda_1(t) & & \\ & \ddots & \\ & & i\lambda_n(t) \end{bmatrix},$$

with $\tilde{I}(0) = j^{-1}(A)$ such that for all $t \in \theta''$, $U(t)^{-1} \tilde{G}(t) U(t) = \tilde{I}(t)$. The unitary group $U(n)$ is imbedded in the real symplectic group $Sp(J, \mathbb{R})$ by the mapping

$$\mathcal{J}: U(n) \rightarrow Sp(J, \mathbb{R}): a + ib \rightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

where $ab^t = ba^t$ and $aa^t + bb^t = I$. Define the curves

$$\beta: \theta'' \subseteq \mathbb{R} \rightarrow Sp(J, \mathbb{R}): t \rightarrow PS(t) \mathcal{J}(U(t))$$

and

$$\Gamma: \theta'' \subseteq \mathbb{R} \rightarrow sp(J, \mathbb{R}): t \rightarrow j(\tilde{I}(t)).$$

Then β , Γ are real analytic and for all $t \in \theta''$ and $\beta(t)^{-1} \gamma(t) \beta(t) = \Gamma(t)$. ■

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