PEANO BASSO AND PEANO CORTO

ALBERT VISSER

Abstract. In this paper we show that the theories Peano Corto (or: $\mathsf{PA}^{\downarrow\downarrow} := \mathrm{I}(\Sigma_\infty, \Sigma_{1,0})$) and Peano Basso (or: $\mathsf{PA}^{\downarrow} := \mathrm{I}(\Sigma_\infty, \Sigma_{1,1})$), two theories of local induction, are locally cut-interpretable in the basic arithmetic PA^-. We prove a number of theorems about Peano Corto and Peano Basso. We provide some insights that illustrate that these theories are in many respects analogues of full Peano Arithmetic PA.

The theory $\mathsf{PA}^{\downarrow\downarrow}$ extends the theory of parameter-free Π_1-induction, $\mathrm{I}\Pi_1^-$. Hence, $\mathrm{I}\Pi_1^-$ is locally cut-interpretable in PA^-. We will draw a number of consequences of this fact for $\mathrm{I}\Pi_1^-$.

1. Introduction

The induction scheme tells us that any progressive virtual class contains all numbers. There is an extensive literature on restrictions of the induction scheme. Such a restriction is usually realized by considering only progressive virtual classes of a certain prescribed complexity. E.g., if we restrict the classes, say, to Σ_1-definable classes, we get $\mathrm{I}\Sigma_1$ and if we restrict the classes to parameter-free Π_1-definable classes, we get $\mathrm{I}\Pi_1^-$. In this paper we will consider another kind of restriction of induction. it has the form: *if a virtual class is progressive, then it is* large *in some sense.* Such theories were introduced by Andrés Cordón Franco, Alejandro Fernándes Margarit and Felix Lara Martin. We follow these authors in calling such forms of induction *local induction.* (See e.g. [CFL11].)

The specific axiom scheme that we will be looking at in this paper says, roughly, that every progressive virtual class is so large that it has non-empty intersection with every virtual non-empty, parameter-free Σ_1-definable class. This theory proves $\mathrm{I}\Delta_0$. Over $\mathrm{I}\Delta_0$ the scheme takes a very appealing form: it says that every progressive virtual class contains all elements with a (parameter-free) Σ_1-definition. So, we can say that a class is *large* if it contains all *good* numbers, which are in this case the Σ_1-definable ones. Since, in the absence of Σ_1-collection, there turn out to be two relevant notions of Σ_1-definable, two arithmetics result from such a scheme. The stronger one is Peano Basso, which we also call $\mathsf{PA}^{\downarrow}$ or, with a more systematic name, $\mathrm{I}(\Sigma_\infty, \Sigma_{1,1})$. The weaker one is Peano Corto, $\mathsf{PA}^{\downarrow\downarrow}$ or $\mathrm{I}(\Sigma_\infty, \Sigma_{1,0})$.[1]

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[1] In the notation used by Andrés Cordón-Franco, Alejandro Fernández-Margarit and Felix Lara-Martin the theory Peano Corto is called $\mathrm{I}(\Sigma_\infty, \mathcal{K}_1)$.

The central result of this paper is that Peano Basso is *locally weak*. This means that it is *mutually locally interpretable* with a very weak arithmetic PA^-. As we will see more is true: Peano Basso is *mutually locally cut-interpretable* with PA^-. This means that we can choose our interpretations from a very restricted class that guarantees us some extra properties. For example, we cannot interpret (classically) false principles on a cut. We will show that the local cut-interpretability of Peano Basso in PA^- is optimal: we cannot interpret or even model-interpret Peano Basso in PA^-.

An alternative way to understand Peano Corto and Peano Basso is as follows. One can show that Peano Arithmetic is equivalent, over a weak theory like S^1_2, to the full uniform reflection principle for formulas of predicate logic in the signature of arithmetic. Reflection can be interpreted in two ways here: reflection for ordinary provability and reflection for for restricted provability. The result holds for both forms. The theory Peano Corto is equivalent over a suitable weak theory, to wit CFL which will be explained in the paper, to the full *sentential* reflection principle *for restricted provability.* To obtain Peano Basso we have to add Σ_1-collection.

Thus, in a sense, Peano Corto is the solution of the equation:

$$\frac{\mathsf{PA}}{\text{(restricted) uniform reflection}} = \frac{?}{\text{restricted sentential reflection}}.$$

Since Peano Corto proves restricted sentential reflection, it follows that it cannot have a finitely axiomatized extension in the same language. Thus, Peano Corto is not contained in any of the theories $\mathrm{I}\Sigma_n$. So, we have the curious phenomenon of a true, locally weak theory that is incomparable with any of the $\mathrm{I}\Sigma_n$. The theory Peano Basso inherits these properties from Peano Corto.

The main difference between Peano Basso and Peano Corto is that Σ_1-collection is lacking from Peano Corto. The difference between Peano Corto and Peano Basso is due to the fact that, in the absence of Σ_1-collection, the formulas that are Σ_1 when Σ_1-collection is present, split into the $\Sigma_{1,n}$ hierarchy. In this paper we will develop the basic facts concerning this hierarchy.

The theory Peano Corto and, *a fortiori*, Peano Basso extends $\mathrm{I}\Pi_1^-$. Both theories prove the same $\mathsf{bool}(\Sigma_1)$-sentences as $\mathrm{I}\Pi_1^-$ and as Elementary Arithmetic EA.

We will provide a model theoretic characterization of Peano Basso. It is this characterization that shows that, in a sense, Peano Basso is more like PA than Peano Corto.

The theories Peano Basso and Peano Corto share many metamathematical properties not related to strength with PA. For example, both theories fail to have an extension in the same language that is finitely axiomatizable.[2] Every RE sequential theory is mutually locally interpretable with a finite Δ_2-extension of Peano Basso and similarly for Peano Corto, which mirrors the fact that every RE extension of

[2]On the other hand Peano Basso and Peano Corto are locally mutually interpretable with a finitely axiomatized theory, to wit PA^-, but PA is not locally mutually interpretable with any finitely axiomatized theory. We will show that e.g. Peano Basso plus Exp, the axiom saying that exponentiation is total, does share this property with PA.

PA (in the same language) is mutually interpretable with a finite extension of PA (in the same language).

2. Preliminaries

In this section we introduce some basic concepts, notations and some elementary facts.

2.1. **Translations and Interpretations.** We only treat one-dimensional interpretations without parameters for relational signatures. Since our main focus will be a positive result to the effect that one theory is locally cut-interpretable in another theory, the fact that we employ a rather restricted class of interpretations is not a real limitation. For a treatment of the many-dimensional case, see e.g. [Vis12a]. Also, we are mainly interested in sequential theories in this paper. Any multi-dimensional interpretation in a sequential theory is definably and provably isomorphic to a one-dimensional one. So for sequential theories one-dimensional and multi-dimensional interpretability coincide.

We can translate languages with function symbols in their signature into language into languages of relational signature using the well-known term-unraveling algorithm. We always will implicitly assume that this has been done. We note that for translations that are only *relativizations* the presence of terms is no problem.

Consider two relational signatures Σ and Θ. A translation $\tau : \Sigma \to \Theta$ is a quadruple $\langle \Sigma, \delta, \mathcal{F}, \Theta \rangle$, where $\delta(v)$ is a Θ-formula containing just v free and where, for any n-ary predicate P of Σ, $\mathcal{F}(P)$ is a formula $A(v_0, \ldots, v_{n-1})$ in the language of signature Θ, containing just $v_0, \ldots, v_{n-1}$ free. We will often write δ_τ for the domain formula of τ and $P_\tau(v_0, \ldots, v_{n-1})$ for $\mathcal{F}(P)$, where $\mathcal{F}$ is the $\mathcal{F}$ of τ.

We extend the translation τ to the full language as follows:

- $(P(x_0, \ldots, x_{n-1}))^\tau := \mathcal{F}(P)(x_0, \ldots, x_{n-1})$.[3]
- $(\cdot)^\tau$ commutes with the propositional connectives.
- $(\forall x\, A)^\tau := \forall x\, (\delta(x) \to A^\tau)$.
- $(\exists x\, A)^\tau := \exists x\, (\delta(x) \wedge A^\tau)$.

We allow identity to be translated to a formula that is not identity.

We may define the identity translation id_Σ on Σ, the composition $\rho \circ \tau$ of translations τ and ρ, and the disjunctive translation $\tau\langle A \rangle\rho$, that is τ if A and ρ if $\neg A$. Thus, we define:

- $\delta_{\tau\langle A \rangle\rho}(v) :\leftrightarrow (A \wedge \delta_\tau(v)) \vee (\neg A \wedge \delta_\rho(v))$.
- $(P(x_0, \ldots, x_{n-1}))^{\tau\langle A \rangle\rho} :=$
$$(A \wedge P_\tau(x_0, \ldots, x_{n-1})) \vee (\neg A \wedge P_\rho(x_0, \ldots, x_{n-1})).$$

[3] We suppose that some mechanism is chosen to avoid variable-clashes due to the substitution of the x_i for the v_i. We will address this matter in boring detail elsewhere.

A translation is a *relativization* if it sends any predicate P to $P(v_0, \ldots, v_{n-1})$, where the predicates include identity. We will write A^D, for A^τ if A is a relativization with domain formula D. The translations that we will mostly consider in this paper are relativizations to definable cuts. We discuss these in more detail in Subsection 3.4.

A translation is *direct* if its domain is the trivial domain of all objects and if it translates identity as itself. In other words, a translation is *direct* if it is unrelativised and identity preserving.

A translation $\tau : \Sigma \to \Theta$ can be seen as an internal model construction of a model $\tau(\mathcal{M})$ of signature Σ in a model of $\mathcal{M}$ of signature Θ. Note that $\mathcal{M} \mapsto \tau(\mathcal{M})$ is only a partial mapping since we need the condition that $\delta_\tau^{\mathcal{M}}$ is non-empty and that $=_\tau^{\mathcal{M}}$ satisfies the axioms of identity (relative to the signature Σ).

A translation relates signatures; an *interpretation* relates theories. An interpretation $K : V \to U$ is a triple $\langle V, \tau, U \rangle$, where U and V are theories and $\tau : \Sigma_V \to \Sigma_U$. We demand: for all axioms A of V, we have $U \vdash A^\tau$. A theory V is *interpretable* in a theory U if there is an interpretation $K : V \to U$. A theory V is *locally interpretable* in a theory U if every finitely axiomatized subtheory V_0 of V is interpretable in U. Finally, a theory V is *model-interpretable* in a theory U if, for every model $\mathcal{M}$ of U, there is a translation τ such that the internal model $\tau(\mathcal{M})$ is defined and is a model of V. Clearly $U \rhd_{\mathsf{mod}} V$ iff, for every complete extension $U^\star$ of U, we have $U^\star$ interprets V. We write:

- $U \rhd V$ for: V is interpretable in U.
- $U \equiv V$ for $U \rhd V$ and $V \rhd U$.
- $U \rhd_{\mathsf{loc}} V$ for: V is locally interpretable in U.
- $U \equiv_{\mathsf{loc}} V$ for $U \rhd_{\mathsf{loc}} V$ and $V \rhd_{\mathsf{loc}} U$.
- $U \rhd_{\mathsf{mod}} V$ for: V is model-interpretable in U.
- $U \equiv_{\mathsf{mod}} V$ for $U \rhd_{\mathsf{mod}} V$ and $V \rhd_{\mathsf{mod}} U$.
- We use variants like $U \rhd_{\mathsf{mod,loc}} V$ with the obvious meaning.

Here are some further definitions and conventions.

- We often write A^K for A^{τ_K}.
- $\mathsf{ID}_U : U \to U$ is the interpretation $\langle U, \mathsf{id}_{\Sigma_U}, U \rangle$.
- Suppose $K : U \to V$ and $M : V \to W$. Then, $KM := M \circ K : U \to W$ is $\langle U, \tau_M \circ \tau_K, W \rangle$.
- Suppose $K : U \to (V + A)$ and $M : U \to (V + \neg A)$. Then the interpretation $K\langle A \rangle M : U \to V$ is the disjunctive interpretation $\langle U, \tau_K \langle A \rangle \tau_M, V \rangle$. In an appropriate category $K\langle A \rangle M$ is a special case of a product.

We prove a useful lemma. This lemma is exercise 8 of [Hod93], p237.

Lemma 2.1. *Let A be a finitely axiomatized theory. Then $V \rhd A$ iff $V \rhd_{\mathsf{mod}} A$.*

Proof. The direction from left to right is trivial. Suppose every model $\mathcal{M}$ of V has an internal model that satisfies A. Let B be the conjunction of the axioms of identity for the signature of A (including $\exists x\ x = x$) and let $C := (B \wedge A)$. Consider the theory $V + \{\neg C^\tau \mid \tau : \Sigma_A \to \Sigma_V\}$. Clearly, V is inconsistent, since by our assumption, it can have no models. So, V proves a disjunction $\bigvee_{i<n} C^{\tau_i}$. We compress the τ_i into one translation by taking:

- $\tau^\star := \tau_0\langle C^{\tau_0}\rangle(\tau_1\langle C^{\tau_1}\rangle(\tau_2 \ldots (\tau_{n-2}\langle C^{\tau_{n-2}}\rangle\tau_{n-1})\cdots))$.

It is easily seen that $\tau^\star$ gives us an interpretation of A in V. ❑

We note that Lemma 2.1 is very robust w.r.t. the notion of interpretation involved (modulo minor variations in the proof). It works for one-dimensional interpretations with or without parameters, for multi-dimensional interpretations, for unrelativized interpretations, for identity-preserving interpretations, and also for interpretations that are relativizations to definable cuts. The main condition is that the associated class of translations is closed under the operation $(\cdot)\langle\cdot\rangle(\cdot)$.

We note the following consequence.

Theorem 2.2. *We have:*

i. $U \rhd V \Rightarrow U \rhd_{\mathsf{mod}} V$.

ii. $U \rhd_{\mathsf{mod}} V \Rightarrow U \rhd_{\mathsf{loc}} V$.

iii. $U \rhd_{\mathsf{mod,loc}} V \Leftrightarrow U \rhd_{\mathsf{loc}} V$.

Example 2.3. In case we allow piece-wise interpretations the theory INF which is the theory of pure equality EQ plus axioms saying, for every n, 'there are at least n elements' is locally interpretable in EQ. In case we allow only multi-dimensional interpretations INF is interpretable in EQ plus 'there are at least two elements'. Clearly, INF is not an internal model of any finite model of EQ. Regrettably this simplest of examples does not work in the one-dimensional case.

Vaught's Set Theory VS is locally interpretable in the theory of unordered pairing using one-dimensional interpretations. (See e.g. [Vis08].) However, VS is essentially undecidable since it interprets the theory R of Tarski, Mostowski and Robinson. On the other hand, the true theory of Cantor pairing is decidable (see [CR01]). In the standard model for this theory we can build via a translation an internal model for the theory of unordered pairing which inherits the decidability. Thus, the theory VS cannot be an internal model of this model. Hence, VS is not model-interpretable in the theory of unordered pairing.

For some background concerning our next and final example, see Subsections 2.3 and 3.5. Consider any finitely axiomatized sequential theory A. The theory $\mho(A)$ is locally one-dimensionally interpretable in A but it is not model-interpretable, since, by a result of Jan Krajíček, for every interpretation $N : A \rhd \mathsf{PA}^-$, we can find a number m_N such that the theory

$$\mathcal{K}(A) := A + \{\mathsf{incon}^N_{m_N}(A) \mid N : A \rhd \mathsf{PA}^-\}$$

is locally interpretable in A, and hence consistent. See [Kra87]. See also [Vis93] and [Vis05]. Clearly, no model of $\mathcal{K}(A)$ can have an internal model satisfying $\mho(A)$, since we would have $\mathsf{incon}^M_{m_M}(A)$, where M is the corresponding interpretation. So $A \rhd_{\mathsf{loc}} \mho(A)$, but $A \not\rhd_{\mathsf{mod}} \mho(A)$.

We will give an example that separates interpretability and model-interpretability in Subsection 5.5. ❑

2.2. **Sequential and Pair Theories.** A theory is *sequential* iff it directly interprets adjunctive set theory AS. Here AS is the following theory in the language with only one binary relation symbol.

$\mathsf{AS1}$. $\vdash \exists x\, \forall y\, y \notin x$,

$\mathsf{AS2}$. $\vdash \forall x, y\, \exists z\, \forall u\, (u \in z \leftrightarrow (u \in x \vee u = y))$.

So the basic idea is that we can define a predicate $\in^\star$ in U such that $\in^\star$ satisfies a very weak set-theory involving *all* the objects of U. Given this weak set theory, we can develop a theory of sequences for all the objects in U, which again gives us partial truth-predicates, etc. In short, the notion of sequentiality explicates one possible idea of a *theory with coding.*

A theory is *a pair theory* if it directly interprets a weak theory of non-surjective unordered pairing UP. The theory of non-surjective unordered pairing UP is given as follows. It has, apart from identity, one binary symbol $\in$. It has, apart from the axioms of identity, the following axioms.

$\mathsf{UP1}$ $\vdash \exists x\, \forall y\, y \notin x$,

$\mathsf{UP2}$ $\vdash \forall x, y\, \exists z\, \forall u\, (u \in z \leftrightarrow (u = x \vee u = y))$.

The demand that our pairing is non-surjective is, modulo mutual direct interpretability, equivalent to the axiom that there are, provably, at least two objects in the domain. A theory of ordered pairing can developed from our theory of unordered pairing using Kuratowski pairing.

2.3. **Complexity and Satisfaction.** In this subsection we explain the notion of *restricted provability.* An n-proof is a proof from axioms with Gödel number smaller or equal than n only involving formulas of complexity smaller or equal than n. To work conveniently with this notion, a good complexity measure ρ is needed. This should satisfy three conditions. (i) Eliminating terms in favour of a relational formulation should raise the complexity only by a fixed standard number. (ii) Translation of a formula via the translation corresponding to an interpretation K should raise the complexity of the formula by a fixed standard number depending only on K. (iii) The tower of exponents involved in cut-elimination should be of height linear in the complexity of the formulas involved in the proof.

Such a good measure of complexity together with a verification of desideratum (iii) —a form of nesting degree of quantifier alternations— is supplied in the work of Philipp Gerhardy. See [Ger03] and [Ger05]. It is also provided by Samuel Buss in his preliminary draft [Bus11]. Buss also proves that (iii) is fulfilled.

We will use $\mathsf{proof}_{U,n}$ for the proof predicate where only U-axioms with Gödel numbers $\leq n$ are allowed and where the formulas occurring in the proof are in the complexity class Γ_n of all formulas of complexity $\leq n$. Similarly we use $U \vdash_n A$, $\mathsf{con}_n(U)$, $\Box_{U,m} A$, etc. As usual, $\Box_{U,m} A$ is short for $\exists x\, \mathsf{proof}_{U,n}(x, \underline{\ulcorner A \urcorner})$.

In sequential theories we can define partial satisfaction predicates for formulas with complexity below n, for any n. The presence of these predicates has as a consequence that for any sequential theory U and for any n, we can find an interpretation N of a weak arithmetic like Buss' S^1_2 in U such that $U \vdash \mathsf{con}^N_n(U)$. See e.g. [Vis93] for more details.

3. The theory PA^-

In this section we treat the basic theory PA^-

3.1. PA^- **introduced.** The theory PA^- is the theory of discretely ordered commutative semirings with a least element. The theory is mutually interpretable with Robinson's Arithmetic Q. However, PA^- has the additional good property that it is sequential as was shown in [Jeř12].

The theory PA^- is given by the following axioms.

PA^-1 $\quad \vdash x + 0 = x$

PA^-2 $\quad \vdash x + y = y + x$

PA^-3 $\quad \vdash (x + y) + z = x + (y + z)$

PA^-4 $\quad \vdash x \cdot 1 = x$

PA^-5 $\quad \vdash x \cdot y = y \cdot x$

PA^-6 $\quad \vdash (x \cdot y) \cdot z = x \cdot (y \cdot z)$

PA^-7 $\quad \vdash x \cdot (y + z) = x \cdot y + x \cdot z$

PA^-8 $\quad \vdash x \leq y \vee y \leq x$

PA^-9 $\quad \vdash (x \leq y \wedge y \leq z) \to x \leq z$

PA^-10 $\quad \vdash x + 1 \not\leq x$

PA^-11 $\quad \vdash x \leq y \to (x = y \vee x + 1 \leq y)$

PA^-12 $\quad \vdash x \leq y \to x + z \leq y + z$

PA^-13 $\quad \vdash x \leq y \to x \cdot z \leq y \cdot z$

We will consider the subtraction axiom:

sbt $\quad \vdash x \leq y \to \exists z\, x + z = y$

We will call $\mathsf{PA}^-_{\mathsf{sbt}} := \mathsf{PA}^- + \mathsf{sbt}$. The theory $\mathsf{PA}^-_{\mathsf{sbt}}$ extends Robinson's arithmetic Q. In the literature the name "PA^-" is often used for what we call "$\mathsf{PA}^-_{\mathsf{sbt}}$". See e.g. [Kay91].

By Lemma 1 of [Jeř12], the following principles are verifiable in PA^-.

$\mathsf{PA}^-\mathrm{a}$ $\quad \vdash (x \leq y \wedge y \leq x) \to x = y$

$\mathsf{PA}^-\mathrm{b}$ $\quad \vdash x \cdot 0 = 0$

$\mathsf{PA}^-\mathrm{c}$ $\quad \vdash 0 \leq x$

$\mathsf{PA}^-\mathrm{d} \qquad \vdash x \leq y+1 \leftrightarrow (x \leq y \vee x = y+1)$

We note that $\forall x, y\,(x \leq y \to \exists z\; x+z=y)$ is equivalent to

$$\forall x, y\,(x \leq y \to \exists z \leq y\; x+z=y).$$

So the subtraction principle is Π_1.

3.2. **The $\Sigma_{1,n}$-Hierarchy and Collection.**

The facts proved in this subsection about the $\Sigma_{1,n}$-hierarchy are folklore. We work in the arithmetical language with 0, 1, $+$, $\cdot$ and $\leq$.

The class $\Sigma_{1,0}$ consists of formulas of the form $\exists \vec{x}\, S_0(\vec{x}, \vec{y})$, where S_0 is Δ_0. The class $\Sigma_{1,n+1}$ consists of formulas of the form $\exists \vec{x}\; \forall \vec{y} \leq \vec{t}\; S_0(\vec{x}, \vec{y})$, where S_0 is $\Sigma_{1,n}$. Here $\forall \vec{y} \leq \vec{t}$ stands for a block of the form $\forall y_0 \leq t_0 \ldots \forall y_{k-1} \leq t_{k-1}$. Here y_i is not allowed as free variable in t_i. The blocks $\vec{x}$ and $\vec{y} \leq \vec{t}$ are allowed to be empty. The class $\Sigma_{1,\infty}$ is the union of the $\Sigma_{1,n}$. In a similar way we define the formula classes $\Pi_{1,n}$ and $\Pi_{1,\infty}$.

Remark 3.1. It is a bit awkward to pronounce e.g. $\Pi_{1,1}$ as *pi-one-one*, since we can not distinguish this from the customary pronunciation of Π^1_1. Therefore, I propose to pronounce $\Pi_{1,1}$ as: *pi-one-sub-one.* ❏

When we use Σ_1 we will always mean $\Sigma_{1,0}$ and when we use Π_1 we will always mean $\Pi_{1,0}$.

The formula classes $\Sigma_{1,n}$ are inextricably connected to the Σ_1-collection scheme $\mathrm{B}\Sigma_1$. This scheme is given by:

- $\forall a, \vec{z}\,(\forall x \leq a\, \exists y\, A(x,y,\vec{z}\,) \to \exists b\, \forall x \leq a\, \exists y \leq b\, A(x,y,\vec{z}\,))$, where A is Δ_0.

We will write $\mathrm{B}\Sigma_{1,x,y}\{A\}(a,\vec{z}\,)$ for:

$$\forall x \leq a\, \exists y\, A(x,y,\vec{z}\,) \to \exists b\, \forall x \leq a\, \exists y \leq b\, A(x,y,\vec{z}\,),$$

always assuming that the free variables of A are among $x, y, \vec{z}$.

Over $\mathrm{I}\Delta_0$, the scheme $\mathrm{B}\Sigma_1$ can be formalized by $\Pi_{1,1}$ formulas. We define the scheme $\mathrm{B}\Sigma_1^\circ$ by:

- $\forall a, \vec{z}\, \exists u \leq a\, \forall b\, \forall x \leq a\, \exists y \leq b\, (A(u,b,\vec{z}\,) \to A(x,y,\vec{z}\,))$, where A is Δ_0.

Or, in a more readable version:

- $\forall a, \vec{z}\, \exists u \leq a\, \forall b\, (A(u,b,\vec{z}\,) \to \forall x \leq a\, \exists y \leq b\, A(x,y,\vec{z}\,))$ where A is Δ_0.

We will write $\mathrm{B}\Sigma^\circ_{1,x,y}\{A\}(a,\vec{z}\,)$ for:

$$\exists u \leq a\, \forall b\, (A(u,b,\vec{z}\,) \to \forall x \leq a\, \exists y \leq b\, A(x,y,\vec{z}\,)),$$

always assuming that the free variables of A are among $x, y, \vec{z}$. Note that we have: $\mathrm{B}\Sigma^\circ_{1,x,y}\{A\}(a,\vec{z}\,)$ is $\Pi_{1,1}$.

Theorem 3.2. *For any Δ_0-formula A:*

a. $\mathsf{PA}^- + \mathrm{B}\Sigma^\circ_{1,x,y}\{A\}(a,\vec{z}\,) \vdash \mathrm{B}\Sigma_{1,x,y}\{A\}(a,\vec{z}\,)$,

b. $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_{1,x,y}\{A\}(a,\vec{z}\,) \vdash \mathrm{B}\Sigma^\circ_{1,x,y}\{A\}(a,\vec{z}\,)$.

Proof. We fix a Δ_0 formula A and variables $x, y, \vec{z}$ such that the free variables of A are contained in $x, y, \vec{z}$.

We prove (a). We reason in $\mathsf{PA}^- + \mathrm{B}\Sigma^\circ_{1,x,y}\{A\}(a, \vec{z}\,)$. Suppose

$$(1) \quad \forall x \le a\, \exists y\, A(x, y, \vec{z}\,).$$

Pick u as promised by $\mathrm{B}\Sigma^\circ_{1,x,y}\{A\}(a, \vec{z}\,)$. By (1), there is a b with $A(u, b, \vec{z}\,)$. Hence, by $\mathrm{B}\Sigma^\circ_{1,x,y}\{A\}(a, \vec{z}\,)$, we have $\forall x \le a\, \exists y \le b\, A(x, y, \vec{z}\,)$.

We prove (b). We reason in $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_{1,x,y}\{A\}(a, \vec{z}\,)$. In case $\exists x \le a\, \forall y\, \neg A(x, y, \vec{z}\,)$, we can take the u that witnesses $\mathrm{B}\Sigma^\circ_{1,x,y}\{A\}(a, \vec{z})$ as an example of an x such that $x \le a$ and $\forall y\, \neg A(x, y, \vec{z}\,)$. Suppose $\forall x \le a\, \exists y\, A(x, y, \vec{z}\,)$. By $\mathrm{B}\Sigma_{1,x,y}\{A\}(a, \vec{z}\,)$, we can find a b such that $\forall x \le a\, \exists y \le b\, A(x, y, \vec{z}\,)$. Take $b^\star$ the smallest such number (using the Δ_0-minimum principle). In case we would have $\forall x \le a\, \exists y < b^\star\, A(x, y, \vec{z}\,)$, the number $b^\star$ would not be minimal. So, for some $x^\star \le a$ we have $\forall y < b^\star\, \neg A(x^\star, y, \vec{z}\,)$. Clearly, $A(x^\star, b^\star, \vec{z}\,)$. so we can take $u := x^\star$ to witness $\mathrm{B}\Sigma^\circ_{1,x,y}\{A\}(a, \vec{z}\,)$. ❑

Clearly, over $\mathsf{PA}^- + \mathrm{B}\Sigma_1$, the class $\Sigma_{1,\infty}$ collapses modulo provable equivalence to $\Sigma_{1,0}$. We show that in the presence of the negation of an instance of $\mathrm{B}\Sigma_1$ the class $\Sigma_{1,\infty}$ expands to the full arithmetical language in various circumstances.

Lemma 3.3. *We have:*

$$\mathsf{PA}^- + \neg\, \mathrm{B}\Sigma^\circ_{1,x,y}\{C\}(a, \vec{z}) \vdash \forall u\, A(u) \leftrightarrow \forall x \le a\, \exists y\, (C(x, y) \wedge \forall v \le y\, A(v)),$$

where C is in Δ_0 and $x, y, \vec{z}$ includes the free variables of C. The same result holds when we replace $\forall u$ and $\forall v$ by quantifier blocks $\forall \vec{u}$, $\forall \vec{v}$.

Proof. We reason in $\mathsf{PA}^- + \neg\, \mathrm{B}\Sigma^\circ_{1,x,y}\{C\}(a, \vec{z})$. Note that $\neg\, \mathrm{B}\Sigma^\circ_{1,x,y}\{C\}(a, \vec{z}\,)$ is equivalent to: (†) $\forall x \le a\, \exists y\, C(x, y, \vec{z}\,)$ and (‡) $\forall b\, \exists x \le a\, \forall y \le b\, \neg\, C(x, y, \vec{z}\,)$.

The left-to-right direction is immediate by (†). For the right-to-left direction, suppose that (i) $\forall x \le a\, \exists y\, (C(x, y) \wedge \forall u \le y\, A(u))$. Consider any u. By (‡), we can find an $x^* \le a$ such that (ii) $\forall y \le u\, \neg\, C(x^*, y, \vec{z}\,)$ We have, by (i), that (iii) $(C(x^*, y^*) \wedge \forall v \le y^*\, A(v))$, for some y^*. It follows, by (ii) and (iii),that $y^* > u$. Hence, by (iii), we have $A(u)$.

The adaptation of the proof to blocks $\forall \vec{u}$, $\forall \vec{v}$ is obvious. ❑

Theorem 3.4. *Consider any Δ_0-formula $C(x, y, \vec{z}\,)$. We have the following:*

- *i. For any formula $A(\vec{w})$ of the arithmetical language, there is a $\Sigma_{1,\infty}$-formula $A^\star(a, \vec{z}, \vec{w})$, such that:*

$$\mathsf{PA}^- + \neg\, \mathrm{B}\Sigma_{1,x,y}\{C\}(a, \vec{z}) \vdash A(\vec{w}) \leftrightarrow A^\star(a, \vec{z}, \vec{w}).$$

- *ii. For any formula $A(\vec{w})$ of the arithmetical language, there is a formula $\widetilde{A}(\vec{w})$ in $\Sigma_{1,\infty}$ such that $\mathrm{I}\Delta_0 + \neg\, \forall a, \vec{z}\, \mathrm{B}\Sigma_{1,x,y}\{C\}(a, \vec{z}\,) \vdash A(\vec{w}) \leftrightarrow \widetilde{A}(\vec{w})$.*

Proof. We prove (i). We first bring $A(\vec{w}\,)$ in prenex normal form, say this is $A^\circ(\vec{w}\,)$. Then we replace all maximal blocks $\forall \vec{u}$ in A° by $\forall x \le a\, \exists y\, (C(x, y) \wedge \forall \vec{u} \le y\, \ldots)$, obtaining $A^\star(a, \vec{z}, \vec{w}\,)$. By Lemma 3.3, $A^\star$ is equivalent to A. Moreover, $A^\star$ is clearly in $\Sigma_{1,\infty}$.

We prove (ii). From (i), we find, by predicate logic:

$\mathsf{PA}^- + \exists a, \vec{z} \neg \mathrm{B}\Sigma_{1,x,y}\{C\}(a,\vec{z}) \vdash A(\vec{w}\,) \leftrightarrow \exists a, \vec{z}\, (\neg \mathrm{B}\Sigma_{1,x,y}\{C\}(a,\vec{z}) \wedge A^\star(a,\vec{z},\vec{w}\,))$.

By Theorem 3.2, it follows that:

$\mathsf{PA}^- + \exists a, \vec{z} \neg \mathrm{B}\Sigma_{1,x,y}\{C\}(a,\vec{z}) \vdash A(\vec{w}\,) \leftrightarrow \exists a, \vec{z}\, (\neg \mathrm{B}\Sigma^\circ_{1,x,y}\{C\}(a,\vec{z}) \wedge A^\star(a,\vec{z},\vec{w}\,))$.

Thus we can take:

$$\widetilde{A}(\vec{w}\,) := \exists a, \vec{z}\, (\neg \mathrm{B}\Sigma^\circ_{1,x,y}\{C\}(a,\vec{z}) \wedge A^\star(a,\vec{z},\vec{w}\,)).$$

Clearly, $\widetilde{A}$ is in $\Sigma_{1,\infty}$. ❑

So we have the somewhat strange result that in a model of $\mathrm{I}\Delta_0$ the $\Sigma_{1,\infty}$-definable sets are either the $\Sigma_{1,0}$-definable sets or the definable sets *simpliciter*.

3.3. **Initial Embeddings.** We will use a specialization of a theorem due to Dave Marker. See [Mar84]. The proof of this theorem uses methods introduced by J. Schlipf in [Sch78].

Theorem 3.5 (Marker)**.** *Suppose $\mathcal{M}$ and $\mathcal{N}$ are countable models of PA^- that are jointly recursively saturated. Suppose further that, for all sentences S of $\Sigma_{1,\infty}$, we have: if $\mathcal{M} \models S$, then $\mathcal{N} \models S$. Then there is an initial embedding of $\mathcal{M}$ in $\mathcal{N}$.*

An immediate consequence of this theorem is Feferman's Preservation Theorem ([Fef68]).

Corollary 3.6. *Suppose Θ is a theory in the language of arithmetic that extends PA^-. Suppose the set Γ of sentences in the language of arithmetic is preserved under end extensions between models of Θ. Then Γ is axiomatizable by $\Sigma_{1\infty}$-sentences over Θ.*

The proof of the Corollary from the Theorem is again in [Mar84].

Since a model of $\mathrm{I}\Delta_0 + \exists a, \vec{z} \neg \mathrm{B}\Sigma_{1,x,y}\{C\}(a,\vec{z})$ does not have an end-extension, Corollary 3.6 provides an alternative proof that any sentence is provably equivalent to a $\Sigma_{1,\infty}$-sentence over $\mathrm{I}\Delta_0 + \exists a, \vec{z} \neg \mathrm{B}\Sigma_{1,x,y}\{C\}(a,\vec{z})$.

3.4. **Cut-interpretability in PA^-.** A *cut* I is a formula $I(v)$ in the language of PA^- such that PA^- proves that I contains 0 and is closed under $+1$ and is downwards closed w.r.t. $\leq$. An *a-cut* is a cut that is PA^--provably closed under addition and a *am-cut* is a cut that is PA^--provably closed under addition and multiplication. We will treat cuts as virtual classes. We note that on an am-cut we again have PA^-. Moreover, in $\mathsf{PA}^-_{\mathsf{sbt}}$ we will have $\mathsf{PA}^-_{\mathsf{sbt}}$ on an am-cut.

Suppose I is a cut, we define:

- $I_1 := \{x \in I \mid \forall y \in I\; y + x \in I\}$
- $I_2 := \{x \in I_1 \mid \forall y \in I_1\; y \cdot x \in I_1\}$.

It is easily seen that I_1 is an a-cut and I_2 is an am-cut. (This result is an example of Solovay's method of shortening cuts. See [Sol76].) Similarly we can construct a cut that is also closed under ω_1. See e.g. [HP91].

We can easily interpret $\mathsf{PA}^-_{\mathsf{sbt}}$ on an am-cut in PA^-. See also [Jeř12].

Consider extensions U and V of PA^- in the same language. Let Γ be a set of $\Pi_{1,\infty}$-sentences. Suppose $U \rhd_{(\mathsf{loc},)\mathsf{cut}} V$. Then, we have $(U+\Gamma) \rhd_{(\mathsf{loc},)\mathsf{cut}} (V+\Gamma)$. We note that sbt is a $\Pi_{1,0}$-sentence, that $\mathrm{B}\Sigma_1^\circ$ is a $\Pi_{1,1}$-scheme and that $\mathrm{I}\Delta_0+\mathrm{B}\Sigma_1$ is a $\Pi_{1,1}$-theory. So all of these are all preserved under (local) cut-interpretability.

One would expect (local) cut-interpretability to behave in a rather tame way. After all, a true theory can only (locally) cut-interpret a true theory. However, it turns the local cut-interpretable theories are rich and varied.

To make that visible we use an idea due to Solovay in a letter to Nelson.

Lemma 3.7. *In* $\mathrm{I}\Delta_0 + \neg\,\mathsf{Exp}$ *we can prove that there is a unique number* $\mathfrak{s}$ *such that* $\mathsf{supexp}(\mathfrak{s})$ *exists and* $\mathsf{supexp}(\mathfrak{s}+1)$ *does not.*

We leave the simple proof to the reader. We call $\mathfrak{s}$: *Solovay's number.* As we will see '$\mathfrak{s}$' is not a rigid designator and Solovay's number may change its identity when we move to another environment.

Theorem 3.8. *We have* $\mathrm{I}\Delta_0 \rhd_{\mathsf{cut}} (\mathrm{I}\Delta_0 + (\Omega_1 \to \mathsf{Exp}))$.

Proof. We work in $\mathrm{I}\Delta_0$. We construct an am-cut on which we have $\Omega_1 \to \mathsf{Exp}$.

If we have Exp we have $\Omega_1 \to \mathsf{Exp}$ on the identity cut.

Suppose $\neg\,\mathsf{Exp}$. In this case $\mathfrak{s}$ exists. Let $\mathfrak{t} := \mathsf{supexp}(\mathfrak{s})$. Note that $2^{\mathfrak{t}}$ does not exist. Take an a-cut J of numbers x for which both 2^{2^x} and $\mathfrak{t}^x$ exist. Let I be the numbers y such that, for some x in J, we have $y \leq \mathfrak{t}^x$. Then I is an am-cut and $\mathfrak{t}$ is in I. Moreover, $\omega_1(\mathfrak{t})$ does not exist in I, since $|\mathfrak{t}|$ is not in J because $2^{2^{|\mathfrak{t}|}}$ does not exist. Hence I interprets $\neg\,\Omega_1$ and, *a fortiori*, $\Omega_1 \to \mathsf{Exp}$. ❑

The following theorem is a variant of a result due to Solovay which was proved in the letter of Robert Solovay to Edward Nelson that we mentioned above.

Theorem 3.9. *Consider any numbers* $k < n$. *Then* $\mathrm{I}\Delta_0 + (\mathsf{Exp} \vee \mathfrak{s} \equiv k\,(\mathsf{mod}\ n))$ *is cut-interpretable in* $\mathrm{I}\Delta_0$ (*and, hence, in* PA^-).

Proof. We define:

- $\mathsf{itexp}(x,0) := x$, $\mathsf{itexp}(x,n+1) := 2^{\mathsf{itexp}(x,n)}$,
- $\mathsf{Log}_n := \{x \mid \mathsf{itexp}(x,n) \text{ exists}\}$.

We can show that in $\mathrm{I}\Delta_0 + \Omega_n$, the virtual classes Log_j, for $j \leq n$ are am-cuts, in fact we have $\mathsf{Log}_j : \mathrm{I}\Delta_0 + \Omega_n \rhd_{\mathsf{cut}} \mathrm{I}\Delta_0 + \Omega_{n-j}$. (We take $\omega_0(x) := x^2$.)

It is well known that we can cut-interpret $\mathrm{I}\Delta_0 + \Omega_n$ in $\mathrm{I}\Delta_0$. Hence it is sufficient to construct the desired cut in $\mathrm{I}\Delta_0 + \Omega_n$. We work in $\mathrm{I}\Delta_0 + \Omega_n$. If Exp we take the identity cut.

Suppose $\neg\,\mathsf{Exp}$. Consider any $j \leq n$. The class Log_j is a cut on which we have $\neg\,\mathsf{Exp}$ and we have: $(\mathfrak{s} = x)^{\mathsf{Log}_j}$ iff $\mathfrak{s} = x + j$. We define the desired cut as follows:

$$\mathcal{J} := \mathsf{Log}_0\langle \mathfrak{s} \equiv k\,(\mathsf{mod}\ n)\rangle(\mathsf{Log}_1\langle \mathfrak{s} \equiv k+1\,(\mathsf{mod}\ n)\rangle(\ldots \mathsf{Log}_{n-1})\ldots).$$

It is easily seen that $\mathcal{J}$ is a cut interpreting $s \equiv k\,(\mathsf{mod}\ n)$. ❑

For the next theorem we drop our silent assumption that a theory has to have a p-time decidable axiomatization and we allow theories with arbitrary axiom sets.

Theorem 3.10. *There is a class $\mathcal{X}$ of $2^{\aleph_0}$ extensions of* $\mathrm{I}\Delta_0$ *that are locally cut-interpretable in* $\mathrm{I}\Delta_0$*, such that any two elements of $\mathcal{X}$ are incompatible in the sense that their union implies* Exp *and is hence not locally interpretable in* $\mathrm{I}\Delta_0$.

Proof. Consider any binary sequence $\alpha := a_0a_1a_2\ldots$. Using Theorem 3.9 we can locally interpret an extension Θ_α of $\mathrm{I}\Delta_0$ that says: *we have either* Exp *or the binary digits of* $\mathfrak{s}$ *end with* $\ldots a_2a_1a_0$. If $\alpha \neq \beta$ we clearly must have: $\Theta_\alpha \cup \Theta_\beta \vdash \mathsf{Exp}$. ❑

3.5. **The Mho-functor.** An important theoretical notion in the context of the study of sequential theories is the functor $\mho$.[4] In [Vis11a], I defined the functor for the base theory S^1_2. Since PA^- is the central theory of our paper, it seems better to define $\mho$ for the base theory PA^-. We define, for any recursively axiomatized theory U:

- $\mho(U) := \mathsf{PA}^- + \{\mathsf{con}_n(U) \mid n \in \omega\}$.

The central fact about the $\mho$-functor is as follows:

Theorem 3.11. *We have: $U \rhd_{\mathsf{loc}} V \Leftrightarrow \mho(U) \rhd V$. In other words, $\mho$ is the right adjoint of the embedding functor of $\lhd$ considered as a preorder category into $\lhd_{\mathsf{loc}}$ considered as a preorder category.*

For a proof, see [Vis11a].

In Example 2.3, we sketched an argument to show that, for a finitely axiomatized sequential theory A, we have: $A \not\rhd_{\mathsf{mod}} \mho(A)$ and, *a fortiori*, $A \not\rhd \mho(A)$.

4. $\mathrm{I}\Pi_1^-$ AND CFL

In this section we briefly introduce the salient theories $\mathrm{I}\Pi_1^-$ and CFL.

4.1. $\mathrm{I}\Pi_1^-$**.** There is a surprising contrast between the induction scheme *with* parameters and the induction scheme *without* parameters. This is well illustrated by comparing the theory $\mathrm{I}\Pi_1$ of full Π_1-induction and the theory $\mathrm{I}\Pi_1^-$ of parameter-free Π_1-induction. The theory $\mathrm{I}\Pi_1$ is interderivable with $\mathrm{I}\Sigma_1$. See e.g. [HP91, I.2]. However, the proof of equivalence essentially involves a parameter. The theory $\mathrm{I}\Pi_1^-$ is strictly weaker than $\mathrm{I}\Sigma_1^-$, the theory of parameter-free Σ_1-induction. See [KPD88].

We will show, in this paper that $\mathrm{I}\Pi_1^-$ is a subtheory of $\mathsf{PA}^{\downarrow\downarrow}$. This is Theorem 5.1. We will also show that $\mathsf{PA}^{\downarrow}$ is *locally weak*, i.e., it is *locally interpretable* in PA^-.

[4] We pronounce 'mho' is such a way that it rhymes with 'joe'.

This is Theorem 5.11. Hence, the theory $\mathrm{I\Pi}_1^-$ is also locally weak. We will prove that the theory $\mathrm{I\Pi}_1^-$ is *reflexive*. See Subsection 6.3. It follows that the theory $\mathrm{I\Pi}_1^-$ is not *globally interpretable* in PA^-, so the local interpretability result is optimal. In contrast, $\mathrm{I\Pi}_1$ is a finitely axiomatizable sequential theory. This tells us that $\mathrm{I\Pi}_1^-$ and $\mathrm{I\Pi}_1$ have quite different behaviour w.r.t. local and global interpretability, as will be explained in the paper.

The interest of $\mathrm{I\Pi}_1^-$ derives from its deep connections to salient theories like PA^- and EA. We refer the reader to the fundamental papers [KPD88], [Big95] and [CFL11].

We remind the reader of the witness comparison notation. We assume our language has a binary relation symbol $\leq$. Let $x < y :\leftrightarrow (x \leq y \wedge x \neq y)$.

We define, for any $C = \exists x\, C_0(x)$ and $D = \exists y\, D_0(y)$:

- $C \leq D := \exists x\, (C_0(x) \wedge \forall y < x\, \neg D_0(y))$. We note that $C \leq C$ tells us that there is a smallest C_0.
- $C < D := \exists x\, (C_0(x) \wedge \forall y \leq x\, \neg D_0(y))$.
- $(C \leq D)^\perp := (D < C)$, $(C < D)^\perp := (D \leq C)$.

We consider the following theories:

- $\mathrm{I\Pi}_1^-$ is PA^- plus parameter-free Π_1-induction.
- $\mathrm{L\Sigma}_1^\ominus$ is PA^- plus the parameter-free Σ_1-minimum, principle. We note that $\mathrm{L\Sigma}_1^\ominus$ is axiomatized by PA^- plus axioms of the form $S \to S \leq S$, where S is a $\Sigma_{1,0}^+$-sentence. The superscript $+$ indicates that the sentence begins with at least one existential quantifier.
- $\mathrm{L\Sigma}_1^-$ is $\mathrm{L\Sigma}_1^\ominus$ plus the subtraction axiom, i.o.w., it is $\mathsf{PA}^-_{\mathsf{sbt}}$ plus the parameter-free Σ_1-minimum, principle.

We can easily show that $\mathrm{I\Pi}_1^-$ and $\mathrm{L\Sigma}_1^-$ are interderivable. The theory $\mathrm{L\Sigma}_1^\ominus$ is strictly weaker. It is valid on the polynomials in a variable X with coefficients in ω with the obvious interpretation of the operations and the ordering. In this structure e.g. X has no predecessor.

We end this subsection by briefly reflecting on $\mathrm{I\Pi}_1^-$ and the $\Sigma_{1,n}$ -hierarchy. We remind the reader that Π_1 is, in this paper, the formula class $\Pi_{1,0}$. So, the theory $\mathrm{I\Pi}_1^-$ could also be called $\mathrm{I\Pi}_{1,0}^-$. What about $\mathrm{I\Pi}_{1,n}^-$, for $n > 0$?

We note that, by Theorem 3.2, $\mathrm{B\Sigma}_1$ is equivalent with $\mathrm{B\Sigma}_1^\circ$ over $\mathrm{I\Pi}_1^-$, since $\mathrm{I\Pi}_1^-$ extends $\mathrm{I\Delta}_0$. We repeat the $\Pi_{1,1}$-definition of $\mathrm{B\Sigma}_1^\circ$:

- $\forall a, \vec{z}\, \exists u \leq a\, \forall b\, (A(u,b,\vec{z}\,) \to \forall x \leq a\, \exists y \leq b\, A(x,y,\vec{z}\,))$ where A is Δ_0.

It is easily seen that the virtual class

$$\{a \mid \forall \vec{z}\, \exists u \leq a\, \forall b\, (A(u,b,\vec{z}\,) \to \forall x \leq a\, \exists y \leq b\, A(x,y,\vec{z}\,))\}.$$

is progressive. Hence, $\mathrm{I\Pi}_{1,1}^-$ proves $\mathrm{B\Sigma}_1$. Thus, we find that, for all $n \geq 1$, $\mathrm{I\Pi}_{1,n}^-$ is extensionally the same as $\mathrm{I\Pi}_1^- + \mathrm{B\Sigma}_1$. On the other hand $\mathrm{I\Pi}_1^-$ is a subtheory of the theory axiomatized by the true $\Pi_{1,0}$-sentences. The traditional model-theoretical argument (see e.g. [Kay91]) shows that the theory of the true $\Pi_{1,0}$-sentences does not prove $\mathrm{B\Sigma}_1$, so $\mathrm{I\Pi}_{1,0}^- \nvdash \mathrm{B\Sigma}_1$.

4.2. CFL. The theory CFL is introduced by Andrés Cordón Franco, Alejandro Fernándes Margarit and Felix Lara Martin in [CFL11] as an axiomatization of the $\mathsf{bool}(\Sigma_1)$-consequences both of $\mathrm{I}\Pi_1^-$ and of EA.

My preferred axiomatization of CFL is as follows:

- $\mathrm{I}\Delta_0 + \{\exists x\, S_0(x) \to \exists x\, \exists y\, (2^x = y \wedge S_0(x)) \mid S_0 \in \Sigma_1(x)\}$.

Let $\mathcal{K}$ be the $\mathrm{I}\Delta_0$-cut $\{x \mid \exists y\, 2^x = y\}$. We note that CFL is equivalent to:

$$\mathrm{I}\Delta_0 + \{S \to (S < \exists x\, x \notin \mathcal{K}) \mid \ S \text{ is a } \Sigma_1^+\text{-sentence}\}.$$

By an easy argument we can also show that CFL is equivalent to:

$$\mathrm{I}\Delta_0 + \{\exists x\, S_0(x) \to \exists x\, \exists y\, (2^x = y \wedge S_0(x)) \mid S_0 \in \Delta_0(x)\}.$$

In [CFL11], the theory CFL is axiomatized by $\mathrm{I}\Delta_0$ plus the scheme:

$$\begin{aligned}\forall x, y, u, v\, ((S_0(x) \wedge \forall z\, (S_0(z) \to z = x) \wedge \\ S_1(y) \wedge \forall z\, (S_1(z) \to z = y) \wedge \\ u \leq x \wedge v \leq y\ ) \to \exists w\, u^v = w\ ),\end{aligned}$$

where $S_0 \in \Sigma_1(x)$ and $S_1 \in \Sigma_1(y)$.

Let us locally call our version: CFL_0, and the version of [CFL11]: CFL_1. To prove the equivalence of these theories we will need a lemma. The lemma is a special case of Lemma 1.36 of [HP91, Chapter IV]. We reproduce it for the reader's convenience.

Lemma 4.1. *For every $\Sigma_1(x)$-formula S, there is a $\Sigma_1(x)$-formula $S^\star$, such that:*

i. $\mathrm{I}\Delta_0 \vdash \exists x\, S(x) \to \exists y\, S^\star(y)$,

ii. $\mathrm{I}\Delta_0 \vdash \forall x, y\, ((S^\star(x) \wedge S^\star(y)) \to x = y)$,

iii. $\mathrm{I}\Delta_0 \vdash \forall x\, (S^\star(x) \to S(x))$.

In words: every Σ_1-definable element is syntactically Σ_1-definable over $\mathrm{I}\Delta_0$.

Proof. Suppose $S(x)$ is $\exists \vec{y}\, S_0(x, \vec{y})$, where $S_0 \in \Delta_0$. Suppose the sequence $\vec{y}$ has length n. In $\mathrm{I}\Delta_0$, we can build internal sequences of length $n+1$, e.g. by iterating Cantor pairing. So, $\exists x\, S(x)$ is equivalent to $\exists z\, S_0(\pi_0(z), \ldots, \pi_n(z))$. Note that the formula $S_0(\pi_0(z), \ldots, \pi_n(z))$ is Δ_0.

We define $S^\star(x)$ as the formula:

$$\exists z\, (\pi_0(z) = x \wedge S_0(\pi_0(z), \ldots, \pi_n(z)) \wedge \forall w < z\, \neg\, (S_0(\pi_0(w), \ldots, \pi_n(w)))).$$

It is easily seen that, by the Δ_0-minimum principle, the formula $S^\star$ has the desired properties. ❑

The proof of the following theorem is mainly the same as the proof of Theorem 2.3 of [CFL11]. We have:

Theorem 4.2. CFL_0 *and* CFL_1 *are equivalent.*

Proof. We work in CFL_0 and derive the axioms of CFL_1.

Suppose $S_0(x) \wedge \forall z\,(S_0(z) \to z = x)$ and $S_1(y) \wedge \forall z\,(S_1(z) \to z = y)$ and $u \leq x$ and $y \leq v$. Let $S(a) := S_0(\pi_0(a)) \wedge S_1(\pi_1(a))$, where the π_i are the projection functions for the Cantor pairing. We have $S(a)$, for $a := \langle x, y\rangle$. We find that $\forall b\,(S(b) \to a = b)$. By our axiom scheme we find that 2^a exists. We have, $u^v \leq x^y \leq (2^x)^y = 2^{x+y} \leq 2^a$. So, by $\mathrm{I}\Delta_0$-verifiable facts, it follows that u^v exists.

See [HP91, V.3(c)] for the basics of the development of exponentiation in $\mathrm{I}\Delta_0$.

We prove the converse. Consider any formula $S(x)$ in $\Sigma_1(x)$. We work in the formulation of CFL_1 and prove the axioms of CFL_0. Suppose $\exists x\, S(x)$. Let $S^\star$ be as in Lemma 4.1. Let $x^\star$ be the unique x such that $S^\star(x)$. Clearly, 2 is Σ_1-definable, so $2^{x^\star}$ exists. Since $S(x^\star)$, we are done. ❑

The theory CFL is a natural locally weak theory in which we have soundness and completeness of Löb's Logic. See e.g. [Vis12a].

5. Peano Basso and Peano Corto

In Subsection 5.1 we introduce the theories Peano Basso and Peano Corto. We prove the basic facts about them in Subsection 5.2. We prove a number of further results in Subsections 5.3, **??**. Our main result, to wit the local cut-interpretability of $\mathsf{PA}^{\downarrow}$ in PA^{-}, is proved in Subsection 5.4. Finally, in Subsection 5.5, we take a closer look at the series of approximations used in the proof of the main result.

5.1. Introducing Peano Corto and Peano Basso.

We start with the introduction of $\mathsf{PA}^{\downarrow\downarrow}$. First we need some preliminary definitions. We define:

- $\mathsf{prog}_x(A(x)) :\leftrightarrow (A(0) \wedge \forall y\,(A(x) \to A(x+1)))$.
  In case we treat A as a virtual class, we write simple $\mathsf{prog}(A)$.
- $\mathsf{cut}_x(I(x)) :\leftrightarrow I(0) \wedge \forall y\,(I(x) \to I(x+1)) \wedge \forall y, z\,((I(y) \wedge z \leq y) \to I(z))$.
  In case we treat I as a virtual class, we write simply $\mathsf{cut}(I)$. Similarly for e.g. $\mathsf{am\text{-}cut}$.
- $\mathsf{r\text{-}rfn}(\Theta)$ is the principle $\vdash \Box_{\Theta,n} A \to A$, where A is a sentence of the signature Θ and $\Box_{\Theta,n}$ is n-restricted provability in predicate logic of signature Θ. We always assume that $n \geq \rho(A)$, where ρ counts the depth of quantifier alternations. We write $\mathsf{r\text{-}rfn}$ and $\Box_n$ if Θ is the signature of arithmetic.
- $\mathsf{rfn}(\Theta)$ is the principle $\vdash \Box_\Theta A \to A$, where A is a sentence of the signature Θ and $\Box_\Theta$ is provability in predicate logic of signature Θ. We write rfn and $\Box$ if Θ is the signature of arithmetic.
- $\mathsf{r\text{-}RFN}$ is the principle $\vdash \forall \vec{x}\,(\Box_{\mathsf{PA}^-,n} A(\vec{x}) \to A(\vec{x}))$, where A is a formula of the signature of arithmetic and the variables inside of the box are realized using codes of efficient numerals. We always assume that $n \geq \rho(A)$.[5]

[5] The use of PA^--provability is not really necessary in the formulation, but is seems awkward to use the efficient numerals in a context where they do not make sense—especially where they are paraphrased away in a relational version of the language.

- RFN is the principle $\vdash \forall \vec{x}\,(\Box_{\mathsf{PA}^-} A(\vec{x}) \to A(\vec{x}))$, where A is a formula of the signature of arithmetic.

The theory Peano Corto or $\mathsf{PA}^{\downarrow\downarrow}$ or $\mathrm{I}(\Sigma_\infty, \Sigma_{1,0})$ is axiomatized as PA^- plus the scheme:

$$\vdash \forall \vec{z}\,((\mathsf{prog}_x(A(x,\vec{z})) \wedge \exists x\, S_0(x)) \to \exists x\,(A(x,\vec{z}) \wedge S_0(x))).$$

Here, the formulas A has at most $x, \vec{z}$ free and S_0 has at most x free and $S_0 \in \Sigma_{1,0}$.

We note that $\mathsf{PA}^{\downarrow\downarrow}$ extends $\mathrm{I}\Pi_1^-$ and, hence CFL and $\mathrm{I}\Delta_0$.

Theorem 5.1. *We have* $\mathsf{PA}^{\downarrow\downarrow} \vdash \mathrm{I}\Pi_1^-$. *In fact, we even have* $\mathrm{I}\Pi_1^-[\Sigma_{1,0}] \vdash \mathrm{I}\Pi_1^-$.

Proof. Suppose $P(x)$ is $\Pi_{1,0}$ and has only x free. We work in or $\mathrm{I}\Pi_1^-[\Sigma_{1,0}]$.

Suppose $P(x)$ is progressive and $\exists y \neg P(y)$. Then $\exists y\,(\neg P(y) \wedge P(y))$. A contradiction. So, we have $\forall y\, P(y)$. ❑

The theory $\mathsf{PA}^{\downarrow\downarrow}$ has many equivalent formulations. In the next theorem we present a selection of these.

Theorem 5.2. *The following theories are equivalent.*

(1) $\mathsf{PA}^{\downarrow\downarrow}$

(2) PA^- *plus the scheme:*

$$\vdash (\mathsf{prog}_x(A(x)) \wedge \exists x\, S_0(x)) \to \exists x\,(A(x) \wedge S_0(x)).$$

Here, the formulas A has at most x free and S_0 has at most x free and $S_0 \in \Sigma_{1,0}$. We call this system $\mathrm{I}(\Sigma^-_\infty, \Sigma_{1,0})$.

(3) PA^- *plus the rule:*

$$\vdash \forall \vec{z}\, \mathsf{prog}_x(A(x,\vec{z})) \quad \Rightarrow \quad \vdash \forall \vec{z}\,(\exists x\, S_0(x) \to \exists x\,(A(x,\vec{z}) \wedge S_0(x))).$$

Here $A(x,\vec{z})$ has at most $x, \vec{z}$ free and $S_0(x)$ has only x free and S_0 is in Σ_1. We can restrict the antecedent of the rule to PA^--provability.

(4) PA^- *plus the rule:*

$$\vdash \mathsf{prog}_x(A(x)) \quad \Rightarrow \quad \vdash \exists x\, S_0(x) \to \exists x\,(A(x) \wedge S_0(x)).$$

Here $A(x,\vec{z})$ has at most only $x, \vec{z}$ free and $S_0(x)$ has only x free and S_0 is in Σ_1. We can restrict the antecedent of the rule to PA^--provability.

(5) PA^- *plus the scheme:*

$$\vdash \forall \vec{z}\,((\mathsf{am\text{-}cut}_x(J(x,\vec{z})) \wedge S) \to S^{J_{\vec{z}}}).$$

Here $J(x,\vec{z})$ has at most $x, \vec{z}$ free and S is a Σ_1-sentence.

(6) PA^- *plus the scheme:*

$$\vdash (\mathsf{am\text{-}cut}_x(J(x)) \wedge S) \to S^J.$$

Here $J(x)$ has only x free and S is a Σ_1-sentence.

(7) PA^- *plus the rule:*

$$\vdash \forall \vec{z}\, \mathsf{am\text{-}cut}_x(J(x,\vec{z})) \;\;\Rightarrow\; \vdash S \to \forall \vec{z}\, S^{J_{\vec{z}}}.$$

Here $J(x,\vec{z})$ has at most $x, \vec{z}$ free and S is a Σ_1-sentence. We can restrict the antecedent of the rule to PA^--provability.

(8) PA^- *plus the rule:*

$$\vdash \mathsf{am\text{-}cut}_x(J(x)) \;\;\Rightarrow\; \vdash S \to S^J.$$

Here $J(x)$ has only x free and S is a Σ_1-sentence. We can restrict the antecedent of the rule to PA^--provability.

(9) $\mathrm{I}\Delta_0$ *plus the scheme:*

$$\vdash \forall \vec{z}\,(\mathsf{prog}_x(A(x,\vec{z})) \to \forall x\,((S_0(x) \wedge \forall y\,(S_0(y) \to x = y)) \to \;A(x,\vec{z}))).$$

Here $A(x,\vec{z})$ has at most $x, \vec{z}$ free and $S_0(x)$ has only x free and S_0 is in Σ_1. We can prove rule variants and am-cut variants and parameter-free variants of this scheme in the style of the previous items.

(10) $\mathsf{CFL} + \mathsf{r\text{-}rfn}$.

Proof. Clearly: (1) $\Rightarrow$ (2) $\Rightarrow$ (4), and (1) $\Rightarrow$ (3) $\Rightarrow$ (4). In the case of the rule variants the cases where we have $\mathsf{PA}^{\downarrow\downarrow}$-provability in the antecedent imply the cases with PA^--provability in the antecedent. Suppose we have (4). Consider any $A(x,\vec{z})$. Define $A^\star(x) := \forall \vec{z}\,(\mathsf{prog}_x(A(x,\vec{z})) \to A(x,\vec{z}))$. Clearly $\mathsf{PA}^- \vdash \mathsf{prog}_x(A^\star(x))$. So, we find: $\vdash \exists x\, S_0(x) \to \exists x\,(A^\star(x) \wedge S_0(x))$. Ergo,

$$\vdash (\mathsf{prog}_x(A(x,\vec{z})) \wedge \exists x\, S_0(x)) \to \exists x\,(A(x,\vec{z}) \wedge S_0(x)).$$

Thus we have proved the equivalence of (1), (2), (3) and (4).

The equivalence of (5), (6), (7) and (8) is similar.

We show (2) $\Rightarrow$ (6). We reason in $\mathrm{I}(\Sigma^-_\infty, \Sigma_{1,0})$. Consider any am-cut J. Clearly, J is progressive. Let $S := \exists \vec{x}\, S_0(\vec{x})$, where S_0 is in Δ_0 and contains no variables other than $\vec{x}$. We can easily show that S is equivalent to $\exists z\, \exists \vec{x} < z\, S_0(\vec{x})$. We find: $\exists z \in J\, \exists \vec{x} < z\, S_0(\vec{x})$. Hence, S^J.

We show (6) $\Rightarrow$ (2). We reason in the system given in (6). Suppose A is progressive. We can find an am-cut J that is contained in A. Let S be a sentence $\exists x\, S_0(x)$, where $S_0(x)$ is Σ_1. We find S^J, and, *a fortiori*, $\exists x\,(A(x) \wedge S_0(x))$.

By Theorem 5.1, we have $\mathsf{PA}^{\downarrow\downarrow} \vdash \mathrm{I}\Delta_0$. Hence (1) $\Rightarrow$ (9). The direction (9) $\Rightarrow$ (1) is immediate using Lemma 4.1.

We show that (10) $\Rightarrow$ (8). Let S be a Σ_1-sentence and let J be an am-cut of PA^-. We reason in $\mathsf{CFL} + \mathsf{r\text{-}rfn}$. Suppose S. Without loss of generality we may assume that S is of the form $\exists z\, S_0(z)$, where S_0 is in Δ_0. We can find a witness $z^\star$ for $\exists z\, S_0(z)$ such that $2^{2^{z^\star}}$ exists. Since the presence of a number of the size of $2^{2^{z^\star}}$ is sufficient to make the usual proof of Σ_1-completeness work, we find, for a sufficiently large m, that $\Box_m(\bigwedge \mathsf{PA}^- \to S^J)$. Ergo, by reflection, we find S^J.

Finally we show that (1) $\Rightarrow$ (10). By Theorem 5.1, we have CFL. We reason in $\mathsf{PA}^{\downarrow\downarrow}$. Suppose $\Box_m A$, where A is any sentence in the arithmetical language. Since

$\mathsf{PA}^- + A$ is sequential, we have for some am-cut J, $\Box_m^J A \to A$. On the other hand, from $\Box_m A$, we find $\Box_m^J A$. ❑

We have:

Theorem 5.3. $\mathsf{PA}^{\downarrow\downarrow} \nvdash \mathrm{B}\Sigma_1$.

Proof. Characterization (8) of Theorem 5.2, tells us that $\mathsf{PA}^{\downarrow\downarrow}$ is a subtheory of the theory of the true Π_1-sentences. By the usual model-theoretical argument (see [Kay91]), we find that this theory does not prove $\mathrm{B}\Sigma_1$. ❑

We define:

- $\mathrm{I}(\Sigma_\infty, \Sigma_{1,n})$ is PA^- plus the scheme $\vdash \mathsf{am\text{-}cut}(J) \to (S \to S^J)$, where S is a $\Sigma_{1,n}$-sentence.
- The theory Peano Basso or $\mathsf{PA}^{\downarrow}$ is $\mathrm{I}(\Sigma_\infty, \Sigma_{1,1})$

We have overloaded the definition of $\mathrm{I}(\Sigma_\infty, \Sigma_{1,0})$, but this is harmless given the equivalence of the two definitions.

Theorem 5.4. *For $n \geq 1$, the theory $\mathrm{I}(\Sigma_\infty, \Sigma_{1,n})$ is equivalent to $\mathrm{I}(\Sigma_\infty, \Sigma_{1,0}) + \mathrm{B}\Sigma_1$.*

Proof. It is well known that PA^- interprets $\mathrm{B}\Sigma_1$ on an am-cut. See e.g. [Háj93]. Say this cut is J.

We reason in $\mathrm{I}(\Sigma_\infty, \Sigma_{1,1})$. Since $\mathrm{I}(\Sigma_\infty, \Sigma_{1,0})$ proves $\mathrm{I}\Delta_0$, we also have $\mathrm{I}\Delta_0$ on J. Hence we have $(\mathrm{B}\Sigma_1^\circ)^J$. Since $\mathrm{B}\Sigma_1^\circ$ is $\Pi_{1,1}$, we find $\mathrm{B}\Sigma_1^\circ$, and, hence $\mathrm{B}\Sigma_1$.

Since over $\mathsf{PA}^- + \mathrm{B}\Sigma_1$ the $\Sigma_{1,n}$-hierarchy collapses to $\Sigma_{1,0}$, it is immediate that, for $n > 0$, we have $\mathrm{I}(\Sigma_\infty, \Sigma_{1,n})$ is equivalent to $\mathrm{I}(\Sigma_\infty, \Sigma_{1,0}) + \mathrm{B}\Sigma_1$. ❑

We have seen that that our hierarchy $\mathrm{I}(\Sigma_\infty, \Sigma_{1,n})$ produces two theories: $\mathsf{PA}^{\downarrow\downarrow}$ and $\mathsf{PA}^{\downarrow}$ which is equivalent to $\mathsf{PA}^{\downarrow\downarrow} + \mathrm{B}\Sigma_1$. Since $\mathsf{PA}^{\downarrow\downarrow}$ does not prove $\mathrm{B}\Sigma_1$, the theories $\mathsf{PA}^{\downarrow\downarrow}$ and $\mathsf{PA}^{\downarrow}$ are not equivalent.

Remark 5.5. It is rather easy to see how to generalize local $\Sigma_{1,n}$-induction to the full sequential case. Suppose U is sequential and that $N_0 : \mathsf{PA}^- \lhd U$. Then U satisfies local $\Sigma_{1,n}$-induction w.r.t. N_0 iff, for every $N : \mathsf{PA}^- \lhd U$ and every sentence S in $\Sigma_{1,n}$ such that $U \vdash S^N$, we have $U \vdash S^{N_0}$.

It is less clear how to generalize local induction —as defined by Andrés Cordón Franco, Alejandro Fernándes Margarit and Felix Lara Martin— for other formula classes. This is a problem for further research. ❑

5.2. **Basic Facts.** Here are some simple facts about $\mathsf{PA}^{\downarrow\downarrow}$ and $\mathsf{PA}^{\downarrow}$.

Fact 5.6. *Suppose W is an extension of $\mathsf{PA}^{\downarrow\downarrow}$ in the same language and Z is an extension of $\mathsf{PA}^{\downarrow}$ in the same language.*

a. *Suppose* $M : W \rhd U$ *and* $N : U \rhd \mathsf{PA}^-$. *Then, for any* $\Sigma_{1,0}$*-sentence* S*, we have* $M : (W + S) \rhd (U + S^N)$.[6]

b. *Suppose* $M : Z \rhd U$ *and* $N : U \rhd \mathsf{PA}^-$. *Then, for any* $\Sigma_{1,\infty}$*-sentence* S*, we have* $M : (Z + S) \rhd (U + S^N)$.

c. *Suppose* U *is an extension of* PA^- *in the same language and* $J : W \rhd_{\mathsf{cut}} U$. *Then, we have* $J : (W + B) \rhd_{\mathsf{cut}} (U + \mathsf{PA}^{\downarrow\downarrow} + B)$*, for any Boolean combination* B *of* $\Sigma_{1,0}$*-sentences.*

d. *Suppose* U *is an extension of* PA^- *in the same language and* $J : Z \rhd_{\mathsf{cut}} U$. *Then, we have* $J : (Z + B) \rhd_{\mathsf{cut}} (U + \mathsf{PA}^{\downarrow} + B)$*, for any Boolean combination* B *of* $\Sigma_{1,\infty}$*-sentences.*

Proof. Ad (a). Suppose $M : W \rhd U$ and $N : U \rhd \mathsf{PA}^-$. Then, there is a am-cut J of ID that is definably isomorphic to a definable am-cut of NM. If, in W, we have the $\Sigma_{1,0}$-sentence S, we also have S^J, and, hence S^{NM}.

The proof of (b) is analogous to the proof of (a).

Ad (c). Clearly, in W, both $\Sigma_{1,0}$- and $\Pi_{1,0}$-sentences are preserved to definable am-cuts. Hence, ipso facto, Boolean combination of $\Sigma_{1,0}$-sentences are preserved to definable am-cuts. Moreover, suppose that I is a PA^--cut, J is a W-cut and S is a $\Sigma_{1,0}$-sentence. Then IJ is a W-cut. Since,

$$W \vdash (S^J \to S) \wedge (S \to S^{IJ}),$$

we find $W \vdash (S \to S^I)^J$.

The proof of (d) is analogous to the proof of (c). ❑

We have the following corollary.

Corollary 5.7. $\mathsf{PA}^{\downarrow\downarrow} \rhd_{\mathsf{cut}} \mathsf{PA}^{\downarrow}$.

Both for $\mathsf{PA}^{\downarrow\downarrow}$ and $\mathsf{PA}^{\downarrow}$ we have non-finite-axiomatizability results as the following lemma makes clear.

Fact 5.8. *We have the following two insights.*

i. *The theory* $\mathsf{PA}^{\downarrow\downarrow}$ *does not have a consistent finitely axiomatized extension in the same language. Hence,* $\mathsf{PA}^{\downarrow\downarrow}$ *is not contained in any of the* $\mathrm{I}\Sigma_n$. *It does not even have an RE extension that is mutually interpretable with a finitely axiomatized theory,*

ii. *No extension in the same language of* $\mathsf{PA}^{\downarrow\downarrow} + \mathsf{Exp}$ *is mutually locally interpretable with a finitely axiomatized theory.*

Note that we will prove that $\mathsf{PA}^{\downarrow\downarrow}$ and $\mathsf{PA}^{\downarrow}$ are mutually locally cut-interpretable with PA^-, so the second claim of the above fact does not generalize downwards to $\mathsf{PA}^{\downarrow\downarrow}$ and $\mathsf{PA}^{\downarrow}$.

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[6] *Par abus de langage*, we confuse M here with the interpretation $\langle U + S^N, \tau_M, \mathsf{PA}^{\downarrow\downarrow} + S\rangle$.

Proof. Ad (i): Every extension U of $\mathsf{PA}^{\downarrow\downarrow}$ is reflexive, because $U \vdash \mho(U)$. By the results of [Pud85], no reflexive theory can be finitely axiomatizable, so U is not finitely axiomatizable. Moreover, as is easily seen reflexivity is preserved under mutual interpretability, so no theory that is mutually interpretable with U can be finitely axiomatizable either.

Ad (ii): Let U be any extension of $\mathsf{PA}^{\downarrow\downarrow} + \mathsf{Exp}$. Since $U \vdash \mho(U)$, we have $U \vdash (\mho^\star(U))^{\mathcal{J}}$. Here:

- $\mathcal{J}$ is a superlogarithmic cut. We assume that $\mathcal{J}$ satisfies Ω_1.
- $\mho^\star(U) := \mathsf{PA}^- + \{\mathsf{con}^\star_n(U) \mid n \in \omega\}$.
- $\mathsf{con}^\star_n(U)$ is the ordinary consistency of the first n axioms of U.

The fact that $U \vdash (\mho^\star(U))^{\mathcal{J}}$ follows by the formalization of cut-elimination.

Suppose $U \equiv_{\mathsf{loc}} A$, where A is any finitely axiomatized theory. It follows that $U_0 \rhd A$, where U_0 is a finitely axiomized subtheory of U. It follows that $U \vdash \mathsf{con}(A)^{\mathcal{J}}$. Hence $A \rhd (\mathsf{PA}^- + \mathsf{con}(A))$, contradicting the interpretation version of the Second Incompleteness Theorem. ❑

5.3. **Model Theoretic Characterization of Peano Basso.** In some respects $\mathsf{PA}^{\downarrow}$ is more like PA than $\mathsf{PA}^{\downarrow\downarrow}$. The characterization that we demonstrate in this subsection illustrates this fact.

Consider any countable recursively saturated model $\mathcal{M}$ of $\mathsf{PA}^{\downarrow}$. Let $\mathcal{I}$ be the intersection of the definable cuts of $\mathcal{M}$. We add a new symbol J to the language. Consider the following recursively axiomatized theory:

$$U := \mathsf{am\text{-}cut}_v(\mathsf{J}(v)) + \{\mathsf{am\text{-}cut}_v(J) \to \mathsf{J} \subseteq J \mid J \in \mathsf{For}(v)\} + \{S \to S^{\mathsf{J}} \mid S \in \Sigma_{1,\infty}\text{-}\mathsf{sent}\}.$$

Here $\mathsf{For}(v)$ is the set of formulas of the arithmetical language having just v free.

By a simple compactness argument, the theory $\mathsf{Th}(\mathcal{M}) + U$ is consistent. Since $\mathcal{M}$ is countable and recursively saturated, we can expand $\mathcal{M}$ with an am-cut $\mathcal{J}$ such that $\mathcal{M}, \mathcal{J} \models U$ in such a way that $\mathcal{M}, \mathcal{J}$ is again recursively saturated. This uses the insight that countable recursively saturated models are chronically resplendent. The precise result that we use is [Kay91, Theorem 15.8, p252]. We will confuse $\mathcal{J}$ with the restriction of $\mathcal{M}$ to $\mathcal{J}$. Clearly, $\mathcal{J} \models \mathsf{PA}^-$ and $\mathcal{J}$ is an am-cut of $\mathcal{I}$. Moreover, we have: if $\mathcal{M} \models S$, then $\mathcal{J} \models S$, for all $\Sigma_{1,\infty}$-sentences S. Hence, by Theorem 3.5, there is an initial embedding of $\mathcal{M}$ in $\mathcal{J}$. Thus, we have found:

Theorem 5.9. *Let $\mathcal{M}$ be a countable and recursively saturated model of PA^- and let $\mathcal{I}$ be the intersection of the definable cuts of $\mathcal{M}$. Then, $\mathcal{M}$ satisfies $\mathsf{PA}^{\downarrow}$ iff $\mathcal{I}$ contains a, not necessarily definable, cut that is isomorphic to $\mathcal{M}$.*

It is easy to adapt Marker's proof of Theorem 3.5 directly in order to give a theory-free proof of Theorem 5.9.

We note that $\mathcal{I}$ satisfies exponentiation. Thus, if $\mathcal{M}$ does not satisfy exponentiation, then necessarily every isomorphic copy that is an am-cut of $\mathcal{I}$ is not equal to $\mathcal{I}$. Also $\mathsf{PA}^{\downarrow} + \mathsf{incon}(\mathsf{PA}^-)$ is consistent. Suppose $\mathcal{M} \models \mathsf{incon}(\mathsf{PA}^-)$. Since, any superlogarithmic am-cut of $\mathcal{I}$ satisfies $\mathsf{con}(\mathsf{PA}^-)$, it follows that the isomorphic image of $\mathcal{M}$ will be strictly between $\mathcal{I}$ and all superlogarithmic am-cuts of $\mathcal{I}$

We have the following corollary.

Corollary 5.10. *Let $\mathcal{Y}$ be the class of all countable, recursively saturated models $\mathcal{M}$ of PA^- for which there is an initial embedding in the intersection of all definable cuts. Then $\mathsf{PA}^{\downarrow} = \mathsf{Th}(\mathcal{Y})$.*

We note that if the initial embedding of $\mathcal{M}$ into $\mathcal{I}$ is definable, then $\mathcal{M}$ will be a model of PA.

5.4. **Interpretability of Peano Basso in PA^-.** In this subsection we prove the main result of this paper.

Theorem 5.11. *We have $\mathsf{PA}^- \rhd_{\mathsf{loc,cut}} \mathsf{PA}^{\downarrow}$.*

Proof. We verify the local cut-interpretability of $\mathsf{PA}^{\downarrow}$ as given by axiomatization (8) of Theorem 5.2, with PA^--provability in the antecedent. Let $\mathcal{S}$ be a finite set of $\Sigma_{1,\infty}$-sentences with n elements and let $J(x)$ be a formula having only x free. Suppose $\mathsf{PA}^- \vdash \mathsf{am\text{-}cut}(J)$. We interpret in PA^- the sentence:

$$\mathsf{A}_{\mathcal{S},J} := \bigwedge_{S \in \mathcal{S}} (S \to S^J).$$

It is easily seen that this is sufficient since the definable am-cuts of PA^- are closed under finite intersections.

We define: $I := \mathsf{ID}\langle \mathsf{A}_{\mathcal{S},J}\rangle J$. We will show that $\mathsf{PA}^- \vdash (\mathsf{A}_{\mathcal{S},J})^{I^n}$.

Consider any model $\mathcal{M}$ of PA^-. Let $\mathcal{M}_i$ for $i \leq n$ be the model given by $I^i(\mathcal{M})$, so, $\mathcal{M}_0 := \mathcal{M}$, and $\mathcal{M}_{i+1}$ is the model given by $I(\mathcal{M}_i)$. Clearly, for each i, $\mathcal{M}_i \models \mathsf{PA}^-$. It is immediate that, if for no $i' < i$, we have $\mathcal{M}_{i'} \models \mathsf{A}_{\mathcal{S},J}$, then $\mathcal{M}_i$ is given by $J^i(\mathcal{M})$. If for some i, we have $\mathcal{M}_i \models \mathsf{A}_{\mathcal{S},J}$, then for any i' with $i \leq i' \leq n$, we have $\mathcal{M}_{i'} = \mathcal{M}_i$, and, a fortiori, $\mathcal{M}_n \models \mathsf{A}_{\mathcal{S},J}$.

Let k_i be the number of elements of $\mathcal{S}$ that are satisfied in $\mathcal{M}_i$. Suppose $\mathcal{M}_i \not\models \mathsf{A}_{\mathcal{S},J}$. Then, for some S in $\mathcal{S}$, we have $\mathcal{M}_i \models S \wedge \neg S^J$. It follows that $k_{i+1} < k_i$. So, as long as we do not have $\mathsf{A}_{\mathcal{S},J}$, k_i will decrease. Thus, if, at some stage, $k_i = k_{i+1}$, we are done. Otherwise, k_i keeps decreasing. In this case, k_0 must be n and no sentence of $\mathcal{S}$ is true in $\mathcal{M}_n$. But then we must have: $\mathcal{M}_n \models \mathsf{A}_{\mathcal{S},J}$. ❑

There is an alternative proof of Theorem 5.11. This proof has the advantage of being easier —given a lemma—, but it has the disadvantage that it yields its computational content less readily. Emil Jeřábek simplified my simple version to a very simple version. We give his version of the argument.

Second proof of Theorem 5.11. Consider any finite set $\mathcal{S}$ of $\Sigma_{1,\infty}$-sentences. Consider any model $\mathcal{M}$ of PA^-. The set $\mathcal{S}$ splits into $\mathcal{S}_0$ the set of S in $\mathcal{S}$ that are true in all definable $\mathcal{M}$-am-cuts J, and $\mathcal{S}_1$ the set of S in $\mathcal{S}$ such that for some definable $\mathcal{M}$-am-cut J_S we have $\mathcal{M} \models (\neg S)^{J_S}$. Let J^* be the intersection of the J_S for S in

$\mathcal{S}_1$. Then clearly we have $J^*(\mathcal{M}) \models S \to S^J$, for all PA^--verifiable am-cuts J and for all $S \in \mathcal{S}$.

Thus,

$$\mathsf{PA}^- \rhd_{\mathsf{mod,cut}} (\mathsf{PA}^- + \{S \to S^J \mid S \in \mathcal{S} \text{ and } J \text{ is a } \mathsf{PA}^-\text{-verifiable am-cut}\}).$$

By Lemma 2.1 we may conclude that $\mathsf{PA}^- \rhd \mathsf{PA}^- + \mathsf{A}_{\mathcal{S},J}$, for any PA^--am-cut J. Hence $\mathsf{PA}^- \rhd_{\mathsf{loc,cut}} \mathsf{PA}^{\downarrow}$. ❑

The second proof establishes a bit more.

Theorem 5.12. *For any finite set of $\Sigma_{1,\infty}$-sentences $\mathcal{S}$, we have:*

$$\mathsf{PA}^- \rhd_{\mathsf{mod,cut}} (\mathsf{PA}^- + \{(S \to S^J) \mid S \in \mathcal{S} \textit{ and } J \textit{ is a } \mathsf{PA}^-\textit{-verifiable am-cut.}\}).$$

Finally, we worry about where Theorem 5.11 can be verified. The first proof of Theorem 5.11 can evidently be verified in EA. I am not sure that this can be improved. If we do the construction of the cuts I^n naively their size (= number of symbols) will be exponential in n. However, by using the method of writing small formulas (see: [FR79] and [Pud91]) we can make the size of I^n polynomial in n. It seems that the statement of the induction step "that, for $i < n$, either $\mathsf{A}^{I^i}_{\mathcal{S},J}$ or the number of true sentences in $\mathcal{S}$ is $\leq n - i$" involves formulas of exponential size. I leave it as an open problem whether we can get around this.

Remark 5.13. The proofs of Theorem 5.11 were inspired by the proof of the local interpretability of sentential Σ_1-completeness in $\mathrm{I}\Delta_0 + \Omega_1$ of [Vis91a]. See also [Vis12a]. In fact, this earlier result *follows* from Theorem 5.11.

Our proof-method has similarities to the idea of the proof of Goryachev's Theorem. See [Gor86] or [AB04]. In fact, if we drop the demand that our interpretations are relativizations to cuts, we can prove Theorem 5.11 from a Goryachev-style result. See Appendix A. ❑

5.5. **Interpretability, Model Interpretability, Local Interpretability.** We have shown that $\mathsf{PA}^- \rhd_{\mathsf{loc,cut}} \mathsf{PA}^{\downarrow}$. Since, $\mathsf{PA}^{\downarrow}$ extends $\mho(\mathsf{PA}^-)$, it follows that $\mathsf{PA}^- \not\rhd_{\mathsf{mod}} \mathsf{PA}^{\downarrow}$ and, similarly, $\mathsf{PA}^- \not\rhd_{\mathsf{mod}} \mathrm{III}_1^-$ and $\mathsf{PA}^- \not\rhd_{\mathsf{mod}} \mathsf{CFL}$. See Example 2.3.

The approximating sequence of $\mathsf{PA}^{\downarrow}$ that we constructed for local interpretability of $\mathsf{PA}^{\downarrow}$ in PA^- proceeds by finite extensions. Theorem 5.12 shows that for local model interpretability we can make more *greedy* steps: we still treat finitely many $\Sigma_{1,\infty}$-sentences in each step, but we do treat all PA^--verifiable cuts simultaneously. The next theorem shows that these big steps are generally not possible for ordinary local interpretability. We choose the theorem so to illustrate the point even for steps to approximate CFL rather than $\mathsf{PA}^{\downarrow}$.

This gives us the promised example that separates interpretability and model interpretability. The idea of the proof is derived from [Kra87], [Vis93], [Vis05].

Theorem 5.14. *Consider any $\Sigma_{1,0}$-sentence S of the form $\exists x\, S_0(x)$, where $S_0 \in \Delta_0$. Consider any finitely axiomatized sequential theory A. Let*

$$W := \mathrm{I}\Delta_0 + \mathsf{con}_{n_0}(A) + \{S \to \exists x, y\, (\mathsf{itexp}(n, x) = y \wedge S_0(x)) \mid n \in \omega\}.$$

Here $n_0 = \rho(A) + 1$.

We have (i) $A \rhd_{\mathsf{mod}} W$. *On the other hand,* (ii) *suppose* $N : A \rhd W$. *Then, either* S *is true or* $A \vdash \neg S^N$.

Proof. Let A, S, W and N be as in the statement of the theorem.

Claim (i) is immediate since $A \rhd (\mathsf{PA}^- + \mathsf{con}_{n_0}(A))$ and

$$(\mathsf{PA}^- + \mathsf{con}_{n_0}(A)) \ \rhd_{\mathsf{mod}} \\ \mathsf{PA}^- + \mathsf{con}_{n_0}(A) + \{(S \to S^J) \mid J \text{ is a } \mathsf{PA}^-\text{-verifiable am-cut}\}.$$

We turn to (ii). Let R be such that $\mathsf{PA}^- \vdash R \leftrightarrow S \leq \Box_{A,n}(S \to R)^N$.[7] Here n is taken sufficiently large.

Consider any model $\mathcal{M}$ of A. Let $\mathcal{N} := N(\mathcal{M})$. Suppose $\mathcal{N} \models S$. It follows that $R \vee R^\perp$, since we have $\mathrm{I}\Delta_0$ in $\mathcal{N}$. Suppose we have $R^\perp$, i.e. $\Box_{A,n}(S \to R)^N < S$. Let the smallest witness of S be s. Let $\mathcal{N}^*$ be the restriction of $\mathcal{N}$ to the $n \leq \mathsf{itexp}(k, s)$, for some $k \in \omega$. Clearly, $\mathcal{N}^*$ satisfies $\mathsf{EA} + \mathsf{con}_{n_0}(A) + S + R^\perp$. Since we have S we have in $\mathcal{N}^*$, by Σ_1-completeness, $\Box_{A,n} S^N$. Combining this with the fact that we have $\Box_{A,n}(S \to R)^N$, we find that $\mathcal{N}^*$ satisfies (i) $\Box_{A,n} R^N$. On the other hand, $R^\perp$ gives us that $\mathcal{N}^*$ satisfies (ii) $\Box_{A,n}(R^\perp)^N$. By (i) and (ii), we find $\mathcal{N}^* \models \Box_{A,n}\bot$. Since in EA, we have cut-elimination for standard cuts, we may conclude $\mathcal{N}^* \models \Box_{A,n_0}\bot$. Quod non. So, we must have R in $\mathcal{N}$.

By the completeness theorem we find that $A \vdash (S \to R)^N$. Ergo, either R or $R^\perp$ is true. In the first case we have S. In the second case we have $A \vdash (R^\perp)^N$, and, hence, $A \vdash \neg S^N$. ❑

If we take $A := \mathsf{PA}^-$ and, e.g., $S := \mathsf{incon}(\mathsf{PA}^-)$, where we have written $\mathsf{incon}(\mathsf{PA}^-)$ in $\exists\Delta_0$-form, Theorem 5.14 delivers an example of a true theory W such that $\mathsf{PA}^- \rhd_{\mathsf{mod,cut}} W$ but $\mathsf{PA}^- \not\rhd W$. This example involves some theoretical development and there are related ones that similarly involve some theoretical development. So we ask the following question.

Open Question 5.15. Can we find a meaningful but elementary example that separates interpretability from model interpretability, where we allow multi-dimensional interpretations? ❑

6. Consequences

In this section we collect some consequences of Theorem 5.11.

[7] Since we do not have Ω_1, we have to move with some care. We are assuming that a sentence like $\Box_{A,n}(S \to R)^N$ is $\Sigma_{1,0}$ and of the form $\exists x\, C(x)$, where C is Δ_0. However, the usual proof predicate is Δ_1^{b}. See e.g. [Bus86]. So we have to rewrite the more natural version of a provability formula to the desired form.This means specifically that the witness is not really the code of the proof but contains some extra information, etc. Fortunately all this does not affect our argument.

6.1. **Representability of Sequential Theories by a Sentence.** We can exploit the similarity between Peano Corto / Peano Basso and PA to downscale many well-known results about PA. In fact, I suspect that if one takes Per Lindström's great textbook [Lin02], it will turn out that a substantial majority of the results can be downscaled. In this subsection, I will zoom in on a result by Lindström that is one of my favorites. He shows that every RE extensions of PA is finitely axiomatizable over PA modulo mutual interpretability. In other words, each degree of interpretability of the RE extensions of PA contains a finite extension.

The proof given here is a direct adaptation of Lindström's proof.

Let W be either $\mathsf{PA}^{\downarrow\downarrow}$ or $\mathsf{PA}^{\downarrow}$. We consider a sentence Θ such that:

$$\mathsf{PA}^{-} \vdash \Theta \leftrightarrow \forall x\,(\mathsf{con}_x(W+\Theta) \to \mathsf{con}_x(U)).$$

Theorem 6.1. *We have* $(W+\Theta) \equiv_{\mathsf{loc}} U$. *Specifically, we have* $(W+\Theta) \rhd U$ *and* $U \rhd_{\mathsf{loc}} (W+\Theta)$.

Proof. Since, for every n, $W+\Theta \vdash \mathsf{con}_n(W+\Theta)$. It follows that, for every n, $W+\Theta \vdash \mathsf{con}_n(U)$. Ergo, $W+\Theta \vdash \mho(U)$. Hence, $(W+\Theta) \rhd U$.

On the other hand, by Theorem 5.11, $U \rhd_{\mathsf{loc}} (W+\mho(U))$. We trivially have:

$$(W+\mho(U)+\Theta) \rhd (W+\Theta).$$

Consider $W+\mho(U)+\neg\,\Theta$. It is easy to see that, over W, the sentence $\neg\,\Theta$ implies $\forall x\,(\mathsf{con}_x(U) \to \mathsf{con}_x(\Theta))$. Hence, $W+\mho(U)+\neg\,\Theta \vdash \mho(W+\Theta)$. Hence,

$$(W+\mho(U)+\neg\,\Theta) \rhd (W+\Theta).$$

We may conclude that $(W+\mho(U)) \rhd (W+\Theta)$. Thus, $U \rhd_{\mathsf{loc}} (W+\Theta)$. ❑

We note that:

$$\mathrm{I\Pi}_1^- \vdash \Theta \leftrightarrow (\mathsf{con}(U) \vee \exists x\,(\mathsf{incon}_x(W+\Theta) \wedge \forall y < x\,\mathsf{con}_y(U))).$$

So it follows that Θ is Δ_2 over $\mathrm{I\Pi}_1^-$ and, hence, over W.

Remark 6.2. We can get a related result following a different road. Consider any RE pair theory U of finite signature. Remember that a pair theory is a theory that directly interprets the weak theory of unordered pairing UP. The theory U is bi-interpretable with a theory V where the signature just contains a binary predicate symbol. See [Hod93]. Since *being a pair theory* is preserved by bi-interpretabilty, V will be again a pair theory. The improved version of a theorem of Vaught's tells us that V is axiomatizable by a scheme. See [Vau67] and [Vis10]. It follows that V is mutually locally interpretable with a a theory consisting of the first order comprehension scheme plus a single sentence: the result of replacing the schematic variables of the scheme by second order variables and universally quantifying these variables. See [Vis10], Theorem 7.1 and [Vis12b], Section 2.3, for some details. Thus, the first order comprehension scheme over a first-order language with a binary relation symbol is a basic theory, say B, such that, for every RE pair theory U, there is a sentence A such that U is mutually locally interpretable with $\mathsf{B}+A$. In a sense, this last result is better than the result proved above since it has wider scope. Still I think Theorem 6.1 has independent interest since it is very different in flavor and since the sentences it provides have a very specific simple form. ❑

6.2. **Conservativity.** In [KPD88], it was shown that $\mathrm{I}\Pi_1^-$ is Π_2-conservative over EA. We improve this result by showing that $\mathsf{PA}^{\downarrow}$ is Π_2-conservative over EA.

Theorem 6.3. $\mathsf{PA}^{\downarrow}$ *is* Π_2*-conservative over* EA*.*

Proof. Let $A := \forall x\, \exists y\, A_0(x,y)$, where A_0 is Δ_0. Suppose that $\mathsf{PA}^{\downarrow} \vdash A$. By Theorem 5.11 it follows that, for some am-cut J, we have $\mathsf{PA}^- \vdash A^J$. We may conclude $\mathsf{PA}^- \vdash \forall x \in J\, \exists y\, A_0(x,y)$. By Corollary 4.4 of [Vis92], we find: $\mathsf{EA} \vdash A$.[8] ❑

We note that, by the results of [KPD88], the theory EA is Σ_2-conservative over $\mathrm{I}\Pi_1^-$. So, *a fortiori*, EA is Σ_2-conservative over $\mathsf{PA}^{\downarrow}$. It follows that EA and $\mathsf{PA}^{\downarrow}$ prove the same $\mathsf{bool}(\Sigma_{1,0})$-sentences, i.e. the same Boolean combinations of $\Sigma_{1,0}$-sentences. By the results of [CFL11], this class of $\mathsf{bool}(\Sigma_{1,0})$-sentences is also the class of $\mathsf{bool}(\Sigma_{1,0})$-sentences proven by $\mathrm{I}\Pi_1^-$. It is axiomatized by CFL.

6.3. **Reflexivity.** A theory U is *reflexive* iff $U \rhd \mho(U)$.

Theorem 6.4. *The following hold.*

i. Suppose B is a sentence in $\mathsf{bool}(\Sigma_1)$. Then, $\mathsf{CFL} + B$ is reflexive.

ii. Suppose B is a sentence in $\mathsf{bool}(\Sigma_1)$. Then, $\mathrm{I}\Pi_1^- + B$ is reflexive.

iii. Every extension of $\mathsf{PA}^{\downarrow\downarrow}$ in the same language is reflexive.

Proof. Ad (1) and (2): The theories EA and CFL and $\mathrm{I}\Pi_1^-$ prove the same $\mathrm{B}\Sigma_1$-sentences as was shown in [CFL11]. So it is sufficient to show that, for T is CFL or $\mathrm{I}\Pi_1^-$, we have $\mathsf{EA} + B \vdash \mathsf{con}_n(T + B)$. Since T is a subtheory of $\mathsf{PA}^{\downarrow}$, by Theorem 5.11, we can find an PA^--am-cut J and a number m, such that (a) $\mathsf{EA} \vdash \mathsf{con}_m(\mathsf{PA}^- + B^J) \to \mathsf{con}_n(T+B)$. By Lemma 3.4.8 of [Vis92], we have (b) $\mathsf{EA}+B \vdash \mathsf{con}_m(\mathsf{PA}^- + B^J)$. Combining (a) and (b), we are done.

Claim (3) is immediate from Theorem 5.2. ❑

We consider the double structure of recursively enumerable theories with the partial preorders of global interpretability and local interpretability. We zoom in on the theories that are mutually locally interpretable with PA^- ordered by global interpretability. The class of these theories we call $[\mathsf{PA}^-]_{\mathsf{loc}}$. The class $[\mathsf{PA}^-]_{\mathsf{loc}}$ is preordered by global interpretability. Since PA^- is finitely axiomatized, it is in the minimal global degree in $[\mathsf{PA}^-]_{\mathsf{loc}}$.

Here are some examples of theories that are mutually globally interpretable with PA^-:

- Robinson's arithmetic Q.
- Buss' theory S^1_2,
- $\mathrm{I}\Delta_0$. See e.g. [Nel86], [HP91].
- $\mathrm{I}\Delta_0 + \Omega_1$. See e.g. [Nel86], [HP91].

[8] This corollary is a sharpening of a result of [WP87].

- Pudlák's adjunctive set theory AS. See e.g. [MPS90], [Vis11b].
- Grzegorczyk's theory of concatenation TC. See [Grz05], [Šve07], [Šve09], [Vis09].

Reflexive theories U have the property that if $U \rhd_{\mathsf{loc}} V$, then $U \rhd V$. See e.g. [Vis11a]. Thus, the reflexive theories in $[\mathsf{PA}^-]_{\mathsf{loc}}$ are in the maximal global degree of $[\mathsf{PA}^-]_{\mathsf{loc}}$. Here are examples of theories in this maximal degree in the maximal global degree of $[\mathsf{PA}^-]_{\mathsf{loc}}$, that are, thus, all mutually interpretable with each other:

- $\mho(\mathsf{PA}^-)$. See [Vis11a].
- $\mathrm{I}\Delta_0 + \Omega_\infty := \mathrm{I}\Delta_0 + \{\Omega_n \mid n \in \omega\}$. See [Wil86], [Vis93, Claim 3.2.4].
- PA^- plus first-order comprehension. This is a two-sorted theory, but we can rephrase it in a one-sorted format. See [Vis11a].
- CFL.
- $\mathrm{I}\Pi_1^-$.
- $\mathsf{PA}^{\downarrow\downarrow}$.
- $\mathsf{PA}^{\downarrow}$.

We note that $\mathsf{EA} \vdash \mho(\mathsf{PA}^-)$. So it follows that $\mathsf{EA} \rhd \mathsf{PA}^{\downarrow}$.

By the results of [Pud85], the minimal global degree and the maximal global degree of $[\mathsf{PA}^-]_{\mathsf{loc}}$ are distinct. So, CFL, $\mathrm{I}\Pi_1^-$, $\mathsf{PA}^{\downarrow\downarrow}$ and $\mathsf{PA}^{\downarrow}$ are *not* globally interpretable in PA^-. Note that it follows that CFL, $\mathrm{I}\Pi_1^-$, $\mathsf{PA}^{\downarrow\downarrow}$ and $\mathsf{PA}^{\downarrow}$ are not finitely axiomatizable. This last fact for $\mathrm{I}\Pi_1^-$ was first proved in [KPD88]. Note that we have a bit more: CFL, $\mathrm{I}\Pi_1^-$, $\mathsf{PA}^{\downarrow\downarrow}$ and $\mathsf{PA}^{\downarrow}$ cannot be mutually interpretable with a finitely axiomatized theory. On the other hand, in contrast to PA, they are locally mutually interpretable with a finitely axiomatized theory.

Here is a little scheme comparing some theories.

	fin. ax.	$\equiv$ fin. ax.	$\equiv_{\mathsf{loc}}$ fin. ax.
PA^-	+	+	+
S^1_2	+	+	+
$\mathrm{I}\Delta_0$	?	+	+
CFL	-	-	+
$\mathrm{I}\Pi_1^-$	-	-	+
$\mathsf{PA}^{\downarrow\downarrow}$	-	-	+
$\mathsf{PA}^{\downarrow}$	-	-	+
$\mathsf{PA}^{\downarrow} + \mathsf{Exp}$	-	-	-
PA	-	-	-

Remark 6.5. I only know artificial examples of theories U where we *know* that U is mutually interpretable with a finitely axiomatized theory, but not itself finitely axiomatizable. E.g. consider the theory

$$U := \mathsf{EA} + \{\Box^n_{\mathsf{EA}}\bot \to \Box_{\mathsf{EA}}\bot \mid n \geq 1\}.$$

This is a true theory. By Löb's Theorem, it is not finitely axiomatizable. Since it extends EA, surely $U \rhd \mathsf{EA}$. By an interpretation version of the Second Incompleteness Theorem (see [Vis90] or [Fef97]), we have $\mathsf{EA} \rhd (\mathsf{EA} + \Box_{\mathsf{EA}}\bot)$. Since, $\mathsf{EA} + \Box_{\mathsf{EA}}\bot$ extends U, we find $\mathsf{EA} \rhd U$. So, EA and U are mutually interpretable. (By a meta-theorem due to Per Lindström, we even have that EA and U are mutually *faithfully* interpretable. See [Vis05].)

I do not know of a proven example of theories that are mutually am-cut-interpretable such that that one theory is finitely axiomatizable and the other is not. ❑

Open Question 6.6. We list some questions.

1. Does $\mathrm{I}\Pi_1^-$ imply restricted sentential Σ_2-reflection for $\mathrm{I}\Delta_0$?
2. Does $\mathrm{I}\Pi_1^-$ imply restricted sentential Σ_2-reflection for itself?

❑

6.4. **Speed-up.** By the results of [KPD88], EA is Σ_2-conservative over $\mathrm{I}\Pi_1^-$. In contrast to this, we have the following theorems.

Theorem 6.7. EA *is not interpretable in* $\mathsf{PA}^{\downarrow}$.

Theorem 6.8. EA *has superexponential speed-up for* Δ_0*-sentences over* $\mathsf{PA}^{\downarrow}$.

Proof. Since $\mathsf{EA} \vdash \mathsf{PA}^- \rhd_{\mathsf{loc,cut}} \mathsf{PA}^{\downarrow}$, it follows that

$$\mathsf{EA} \vdash \mathsf{con}(\mathsf{PA}^-) \to \mathsf{con}(\mathsf{PA}^{\downarrow}).$$

By a meta-theorem of Wilkie and Paris (see [WP87], see also [Vis92]), it follows that, for some cut I,

$$\mathrm{I}\Delta_0 + \Omega_1 \vdash \mathsf{con}(\mathsf{PA}^-) \to \mathsf{con}^I(\mathsf{PA}^{\downarrow}).$$

Since EA interprets $\mathrm{I}\Delta_0 + \Omega_1 + \mathsf{con}(\mathsf{PA}^-)$ on a superlogarithmic cut J (see [WP87]), we find that EA interprets $\mathrm{I}\Delta_0 + \Omega_1 + \mathsf{con}(\mathsf{PA}^{\downarrow})$ on some cut $J^\star$. The speed-up result now follows by the methods of [Pud85]. ❑

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Appendix A. A Goryachev-style Argument

In this appendix, we provide a Goryachev-style proof of our main result, where we replace local cut-interpretability by local interpretability. For Goryachev's work see [Gor86] or [AB04].

Theorem A.1. $\mathsf{PA}^- + \mathsf{r\text{-}rfn}$ *is locally interpretable in* PA^-.

The proof is a straightforward adaptation of Goryachev's argument. In fact we will establish that $\mathsf{S}^1_2 + \mathsf{r\text{-}rfn}$ is locally interpretable in PA^-.

Proof. We consider:

- $\mathcal{A}$: a finite set of arithmetical sentences,
- n: the cardinality of $\mathcal{A}$.
- k: any standard number such that $k \geq \rho(A)$, for every A in $\mathcal{A}$, and $k > \rho(\mathsf{prov}_y(x)) + 1$, and $k \geq \rho(\mathsf{S}^1_2)$.

Consider any model $\mathcal{K}$ of PA^-. By standard considerations, we can find an internal model $\mathcal{M}$ of $\mathcal{K}$ satisfying $\mathsf{S}^1_2 + \mathsf{con}^n_k(\mathsf{S}^1_2)$. Let:

$$U := \mathsf{S}^1_2 + \mathsf{con}^{n-1}_k(\mathsf{S}^1_2) + \{A \wedge \Box_k A \mid A \in \mathcal{A} \text{ and } \mathcal{M} \models \Box_k A\}$$

By the above observations we find that $\mathcal{M} \models \mathsf{con}_k(U)$. It follows that we can build an $\mathcal{M}$-internal model $\mathcal{M}^\star$ of U.

For any model $\mathcal{N}$ of S^1_2, let $\sigma(\mathcal{N})$ be the number of $A \in \mathcal{A}$ such that $\mathcal{N} \models A \wedge \Box_k A$.

In case, for some $A \in \mathcal{A}$, we have $\mathcal{M} \models \Box_k A$ and $\mathcal{M} \not\models A$, it follows that $\sigma(\mathcal{M}^\star) > \sigma(\mathcal{M})$. If this case obtains, we repeat the procedure on $\mathcal{M}^\star$.

The procedure must stop in at most n steps and hence we find an internal model that satisfies S^1_2 plus $\Box_k A \to A$, for each A in $\mathcal{A}$.

The theorem follows by applying Lemma 2.1. ❑

We note that the theory $\mathsf{PA}^- + \mathsf{r\text{-}rfn}$ has all kinds of interesting properties. For example, no extension of $\mathsf{PA}^- + \mathsf{r\text{-}rfn}$ in the same language is finitely axiomatizable. Moreover, if a theory is locally interpretable in an extension U of $\mathsf{PA}^- + \mathsf{r\text{-}rfn}$ in the same language, then it is globally interpretable.

We show how to interpret $\mathsf{PA}^{\downarrow\downarrow}$ in $\mathsf{PA}^- + \mathsf{r\text{-}rfn}$ on an am-cut. We will use a lemma.

Lemma A.2. *Suppose I is a PA^--cut such that relativization to I interprets S^1_2. Then $\mathsf{PA}^- + \mathsf{r\text{-}rfn}$ proves $\mathsf{S}^1_2 + \mathsf{r\text{-}rfn}$ on I.*

Proof. Suppose I is a PA^--cut and PA^- interprets S^1_2 on I. We reason in $\mathsf{PA}^- + \mathsf{r\text{-}rfn}$.

Suppose $\Box^I_m A$. Then, $\Box^I_{m+\rho(I)+1}(\exists x\, I(x) \to A^I)$, and, hence, it follows that $\Box_{m+\rho(I)+1}(\exists x\, I(x) \to A^I)$. Ergo, A^I. ❑

Theorem A.3. $\mathsf{PA}^{\downarrow\downarrow}$ *is cut-interpretable in* $\mathsf{PA}^- + \mathsf{r\text{-}rfn}$.

Proof. By Lemma A.2, it is sufficient to show that $\mathsf{PA}^{\downarrow\downarrow}$ is cut-interpretable in $\mathsf{S}^1_2 + \mathsf{r\text{-}rfn}$. Let J be S^1_2-cut on which S^1_2 is satisfied and such that, provably in S^1_2, for every $x \in J$, there is a y, such that $2^{2^x} = y$. We show that we have $\mathsf{PA}^{\downarrow\downarrow}$ in J.

By Lemma A.2, the cut J satisfies $\mathsf{S}^1_2 + \mathsf{r\text{-}rfn}$.

Consider any sentence S, in $\Sigma_{1,0}$. Let I be any am-cut of PA^-. We reason in $\mathsf{S}^1_2 + \mathsf{r\text{-}rfn}$.

Suppose S^J. By the usual proof of Σ_1-completeness, we find, for a sufficiently large k, that $\Box_k(\bigwedge \mathsf{PA}^- \to S^{IJ})$. Hence, S^{IJ}. ❑

Combining Corollary 5.7 and Theorems A.1 and A.3, we find:

Theorem A.4. *Both* $\mathsf{PA}^{\downarrow}$ *and* $\mathsf{PA}^{\downarrow\downarrow}$ *are locally interpretable in* PA^-.

We note that, in Theorem A.4, we do not get local cut-interpretability via this proof strategy.

Department of Philosophy, Utrecht University, Janskerkhof 13A, 3512BL Utrecht, The Netherlands

E-mail address: albert.visser@phil.uu.nl