The smoothing transform; the boundary case*

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Abstract

Let $A = (A_1, A_2, A_3, \ldots)$ be a random decreasing sequence of non-negative numbers that are ultimately zero with $E[\sum A_i] = 1$ and $E[\sum A_i \log A_i] = 0$. The uniqueness and properties of the non-negative fixed points of the associated smoothing transform are considered. These are solutions to the functional equation $\Phi(\psi) = E[\prod_i \Phi(\psi A_i)]$, where Φ is the Laplace transform of a non-negative random variable. The study complements existing results on the case when $E[\sum A_i \log A_i] < 0$.

1 Introduction

Let $A = (A_1, A_2, A_3, \ldots)$ be a random decreasing sequence of non-negative numbers that are ultimately zero, so that there is a finite N with $A_N > 0$ and $A_{N+1} = 0$. The sequence is ordered for convenience only; the formulation does not need this property. For any random variable X, let $\{X_i : i\}$ be copies of X, independent of each other and A. A new random variable X^* is obtained as

$$X^* = \sum A_i X_i;$$

unspecified sums and products will always be over i, with i running from 1 to N. Using A in this way to move from X to X^* is called a smoothing transform (presumably because X^* is an 'average' of the copies of X). The random variable W is a fixed point of the smoothing transform when $\sum A_i W_i$ is distributed like W. Here attention is confined to fixed points that are non-negative, that is to $W \geq 0$. This case, though simpler than the one where W is not restricted in this way, still has real difficulties; it is intimately connected to limiting behaviours of associated branching processes. For non-negative W, the distributional equation defining a fixed point is expressed naturally in terms of Laplace transforms; it becomes the functional equation (for Φ)

$$\Phi(\psi) = E\left[\prod_i \Phi(\psi A_i)\right], \tag{1}$$

Keywords and phrases. Smoothing transform; functional equation; branching random walk
MS2000 subject classification. Primary 60J80, secondary 60G42
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where Φ is sought in $\mathcal{L}$, the set of Laplace transforms of finite non-negative random variables with some probability of being non-zero.

There is a very extensive literature on the existence and properties of fixed points, which we do not attempt to review in full; see, for example, Kahane and Peyrière (1976), Biggins (1977), Durrett and Liggett (1983), Pakes (1992), Rösler (1992), Biggins and Kyprianou (1997), Liu (1998) and Liu (2000) — the last two references also contain many others.

The function v, defined by $v(\theta) = \log E\left[\sum A_i^\theta\right]$, is important in describing the different cases that arise. Let Z be the point process with points at $\{-\log A_i : i \leq N\}$ and intensity measure μ then

$$e^{v(\theta)} = E\int e^{-\theta x} Z(dx) = \int e^{-\theta x}\mu(dx).$$

Thus $e^{v(\theta)}$ is the Laplace transform of a positive measure and, in particular, v is convex. Durrett and Liggett (1983) study, fairly exhaustively, the case when N is not random, so that $v(0) = \log N < \infty$, and $v(\gamma) < \infty$ is finite for some $\gamma > 1$; many of their results are extended to cases where N is also random, but with $v(0) = \log EN$ still finite, by Liu (1998). Here, in a similar, but slightly less restrictive, vein, it is assumed throughout that:

A1: $\quad v(0) > 0$, $v(\theta) < \infty$ for some $\theta < 1$ and $E\left[\sum A_i(\log A_i)^2\right] < \infty$.

Note that, although $v(0) = \log EN$ may be infinite, the definition of A includes the assumption that N itself is finite. The last part of A1 implies that $v(1) < \infty$ and that $v'(1)$ makes sense when interpreted as $E\left[\sum A_i \log A_i\right]$. In fact, as can be seen in Durrett and Liggett (1983) and Liu (1998), in considering solutions to (1), there are four regimes, calling for some separate discussion: (i) $v(1) = 0$ and $v'(1) < 0$, (ii) $v(1) = 0$ and $v'(1) = 0$, (iii) $v(1) = 0$ and $v'(1) > 0$, and (iv) $v(1) \neq 0$. Under the side condition that 1 is in the interior of $\{\theta : v(\theta) < \infty\}$, Theorems 1.3 and 1.5 of Biggins and Kyprianou (1997) show that under the first regime there is a solution in $\mathcal{L}$ to the functional equation (1) that is unique up to scaling. The ideas developed there will be important in this discussion, and so the short citation BK97 will be used. Durrett and Liggett (1983) show that in the third and fourth regimes the existence of solutions to (1) is related to the existence of α in $(0,1)$ with $v(\alpha) = 0$ and $v'(\alpha) \leq 0$ and this α links these cases to the first two. The second regime should be regarded as marking the boundary of the first one, or, alternatively, the boundary between the first and the third. This can be seen from their specifications, but there is a further reason. If $v(\theta) < \infty$ for all θ and, for a fixed ϕ, A_i is replaced by $A_i(\phi) = A_i^\phi e^{-v(\phi)}$ then elementary calculus shows that the first regime obtains for $A(\phi)$ when ϕ lies in an open interval, the second obtains at the end points of that interval and the third obtains elsewhere. It is also worth pointing out, but without going into detail, that one continuous analogue of (1) is the equation giving a travelling wave solution of a particular speed for the partial differential equation know as the KPP equation; in that context, the boundary case gives the wave of smallest speed. This last connection, which receives some further comment later, illustrates that case (ii) is likely to be both subtle and important. It is this case that is considered here and so the following is assumed throughout, in addition to A1.

A2: $\quad v(1) = 0$ and $v'(1) = 0$.

With the assumptions laid out, the first main result for this boundary case can now be

stated. It is easy to use weak convergence to establish that (1) has solutions in $\mathcal{L}$, as can be seen in Durrett and Liggett (1983) and Liu (1998). Theorem 1.1 of Liu (1998) covers the case considered here (any many others). The new aspect in the next theorem is uniqueness without any additional conditions.

Theorem 1 *The functional equation (1) has a unique solution in $\mathcal{L}$ up to a scale factor.*

The functional equation relates in a natural way to certain martingales in the branching random walk. This relationship and results for the martingales obtained in Biggins and Kyprianou (2001) lead to rather precise information on the behaviour of a solution to (1) near zero; the main result of this kind, contained in Theorem 5 in the next section, is that, under mild conditions, $(1-\Phi(\psi))/(-\psi\log\psi)$ converges to a finite positive constant as $\psi \downarrow 0$. Kyprianou (1998) and Liu (2000) exploit a result of this type to obtain uniqueness; an approach which necessitates extra conditions on A. The relationship with the branching random walk and the main results stemming from it are described in the next section. An important precursor to the main proofs is that $(1-\Phi(\psi))/\psi$ is slowly varying; this result and some consequences of it are established in Section 3, drawing on results in BK97. The final two sections are devoted to the proof of the main results.

2 The associated branching random walk.

There is a natural (one to one) correspondence, already hinted at, between the framework introduced and the branching random walk, a connection that is the key to some of the proofs. Specifically, let the point process Z (with points at $\{-\log A_i : i \le N\}$) be used to define a branching random walk in the usual way, with independent copies of Z being used to give the positions of each family relative to its parent's position. Ignoring positions gives a Galton-Watson process with (almost surely finite) family size N. People are labelled by their ancestry (the Ulam-Harris labelling) and the generation of u is $|u|$. Let z_u be the position of u, so that $\{z_u : |u| = 1\}$ is a copy of $\{-\log A_i : i \le N\}$. The assumption A2 corresponds to

$$E\left[\sum_{|u|=1} e^{-z_u}\right] = \int e^{-x}\mu(dx) = 1 \quad \text{and} \quad E\left[\sum_{|u|=1} z_u e^{-z_u}\right] = \int e^{-x}\mu(dx) = 0$$

and, in a similar way, A1 corresponds to

$$\int \mu(dx) > 1, \quad \int e^{-\theta x}\mu(dx) < \infty \text{ for some } \theta < 1, \quad \int x^2 e^{-x}\mu(dx) < \infty.$$

Then it is shown in Biggins (1977) that the non-negative martingale

$$W_n = \sum_{|u|=n} e^{-z_u} \tag{2}$$

converges almost surely to zero and that the functional equation (1) has no solutions in $\mathcal{L}$ that have a finite mean. Let $B_n = \inf\{z_u : |u| = n\}$, the position of the leftmost person in the nth generation, which is taken to be infinite when the branching process has already died out by then; then $W_n \to 0$ implies that $B_n \to \infty$ as $n \to \infty$, almost surely. Had

W_n converged to a limit that was positive with positive probability, then splitting on the first generation and letting $n \to \infty$ would show that the transform of that limit would satisfy (1) and be in $\mathcal{L}$. Another related martingale, introduced next, does better in this respect.

Let

$$\partial W_n = \sum_{|u|=n} z_u e^{-z_u};$$

then it is straightforward to check that ∂W_n is a martingale. It is called the derivative martingale because its form can be derived by differentiating $\sum_{|u|=n} e^{-\theta z_u - nv(\theta)}$, which is also a martingale, with respect to θ and then setting θ to one. This martingale has been considered in Kyprianou (1998) and Liu (2000) and its analogue for branching Brownian motion has been discussed by several authors — Neveu (1988) and Harris (1999), for example. In the branching Brownian motion context, travelling wave solutions to the KPP equation are the analogue of solutions to the functional equation and classical theory of ordinary differential equations easily provides existence, uniqueness and aspects of the asymptotic behaviour of these; hence, these properties form part of the starting point in Neveu's study and earlier ones. In contrast, Harris (1999) seeks properties of the solutions through arguments based on associated martingales, which is the approach taken here.

The derivative martingale is one of the main examples in Biggins and Kyprianou (2001), where general results on martingale convergence in branching processes are discussed. The assumptions A1 and A2 are more than enough for the results in the final section there to apply; in particular, Theorem 12 contains the following result.

Theorem 2 *The martingale ∂W_n converges to a finite non-negative limit, Δ, almost surely, and $P(\Delta = 0)$ is either equal to the extinction probability or equal to one.*

Splitting on the first generation shows, in the obvious notation, that

$$\partial W_n = \sum_{|u|=1} \left(z_u e^{-z_u} (W_{n-1})_u + e^{-z_u} (\partial W_{n-1})_u \right).$$

Now, letting n go infinity and using that $W_n \to 0$ almost surely, gives

$$\Delta = \sum_{|u|=1} e^{-z_u} \Delta_u.$$

Hence the transform of Δ satisfies (1) and will have a transform in $\mathcal{L}$ when Δ is not identically zero. Whether the martingale limit Δ is zero or not is related to the behaviour of the solution to (1) near the origin. The precise relationship is formulated in the next theorem, which is the second main result proved in this paper.

Theorem 3 *The limit Δ is not identically zero exactly when one solution to (1) in $\mathcal{L}$ satisfies*

$$\frac{1 - \Phi(\psi)}{-\psi \log \psi} \to 1 \quad \text{as } \psi \downarrow 0; \tag{3}$$

otherwise this limit is infinite for all solutions.

Biggins and Kyprianou (2001) also contains information on when Δ is not zero, and when it is. Rather a lot of notation is needed to state the conditions accurately, which may obscure the main message. That message is that, under fairly mild conditions, the martingale limit is not identically zero, and those conditions are close to necessary in that the limit is certainly zero if they are relaxed a little. Here is the notation. Let

$$X_1 = \sum z_i e^{-z_i} I(z_i > 0), \quad X_2 = \sum e^{-z_i} \quad \text{and} \quad X_3(s) = \sum e^{-z_i} I(z_i > -s).$$

Since $z_i = -\log A_i$, the variables X_1, X_2 and $X_3(s)$ could easily be reformulated in terms of A. Let $\phi(x) = \log\log\log x$, $L_1(x) = (\log x)\phi(x)$, $L_2(x) = (\log x)^2\phi(x)$, $L_3(x) = (\log x)/\phi(x)$ and $L_4(x) = (\log x)^2/\phi(x)$. Theorem 12 of Biggins and Kyprianou (2001) contains the following result.

Theorem 4
(a) If both $E[X_1 L_1(X_1)]$ and $E[X_2 L_2(X_2)]$ are finite then Δ is not identically zero.
(b) If $E[X_1 L_3(X_1)]$ is infinite or, for some s, $E[X_3(s) L_4(X_3(s))]$ is infinite then $\Delta = 0$ almost surely.

This result combines immediately with Theorem 3 to give the next one, which improves the known results about the functional equation and is the third main result of this paper.

Theorem 5 *When the conditions of Theorem 4(a) hold so does (3). When the conditions of Theorem 4(b) hold the limit in (3) is infinite (rather than being 1).*

As has been mentioned already, some results on the relationship between the limiting behaviour in (3), the limit Δ, and the uniqueness of the solution to (1), have been obtained previously, in Kyprianou (1998) and Liu (2000); those studies approach the convergence of ∂W_n and uniqueness through the properties of solutions to the functional equation, specifically, through (3). By Theorem 5, although (3) holds widely it does not always hold, limiting that approach to uniqueness.

3 Multiplicative martingales and slow variation.

Much of the account in BK97 is relevant in next three sections; the presentation here aims to make the discussion and the statements of results self-contained, but accepts that the proofs draw heavily on results and arguments from BK97.

Just as in BK97, any solution to the functional equation (1) corresponds to a bounded martingale in the branching random walk given by

$$\prod_{|u|=n} \Phi\left(\psi e^{-z_u}\right) \quad \text{for } n = 0, 1, 2, 3, \ldots. \tag{4}$$

It is natural to call these multiplicative martingales for they arise by multiplying terms. It turns out that products over other sets of individuals, not just the nth generation ones used in (4), are important. To introduce these, a few definitions are needed first.

For the branching random walk corresponding to A let

$$\mathcal{C}(t) = \{u : z_u > t \text{ but } z_v \le t \text{ for } v < u\}$$

where $v < u$ means v is an ancestor of u. Hence $\mathcal{C}(t)$ identifies the individuals who are the first in their lines of descent to be to the right of t. For simplicity, let $\mathcal{C} = \mathcal{C}(0)$.

Now let $\overline{g}(t)$ be the last generation containing an individual with no ancestor, including itself, in $\mathcal{C}(t)$. Then, $\overline{g}(t) \leq \sup\{n : B_n \leq t\} < \infty$, because, as was noted in the previous section, $B_n \to \infty$. Furthermore, since family sizes are finite, $\mathcal{C}(t)$, which is contained entirely within the first $\overline{g}(t)$ generations, must be finite. These observations combine with Corollary 3.2 and the first part of Theorem 6.2 in BK97 to give the following result, which concerns multiplicative martingales like those defined at (4) but with the products taken over $\mathcal{C}(t)$.

Lemma 1 *Given $\Phi \in \mathcal{L}$ is a solution to (1) let*

$$M_t(\psi) = \prod_{u \in \mathcal{C}(t)} \Phi(\psi e^{-z_u}).$$

Then, for each $\psi \geq 0$, $M_t(\psi)$ is a bounded martingale. In particular,

$$\Phi(\psi) = E\left[\prod_{u \in \mathcal{C}} \Phi(\psi e^{-z_u})\right]. \tag{5}$$

Note that solutions to the functional equation (1) are linked, through this lemma, to the solutions of another functional equation, (5), which has the same form, except that the A defining (5) does not satisfy A2.

If the members of $\mathcal{C}$ are regarded as the children of the initial ancestor, rather than simply descendants, the resulting point process, which is concentrated on $(0, \infty)$ by arrangement, can be used to construct a Crump-Mode-Jagers (CMJ) process from individuals, and their positions, in the branching random walk. The individuals in the branching random walk that occur in the CMJ process are exactly those in $\mathcal{C}(t)$ as $t \geq 0$ varies; furthermore $\mathcal{C}(t)$ is the coming generation for the CMJ process. This is the content of Lemma 8.2 of BK97 when applied in this case. Known results for the CMJ, derived by Nerman (1981), play a central role in proving uniqueness both in BK97 and here.

The next result is known; a rather more involved proof, allowing for solutions that are not Laplace transforms, is given in Kyprianou (1998). The result is also contained in Theorem 1.2 of Liu (1998) when $v(0) < \infty$. This proof exploits the connection between the two functional equations, (1) and (5), derived in Lemma 1.

Lemma 2 *For any solution Φ to (1), let $L(\psi) = (1 - \Phi(\psi))/\psi$. Then L is slowly varying as $\psi \downarrow 0$.*

Proof. Since

$$E\sum_{u \in \mathcal{C}} e^{-z_u} = 1 \quad \text{and} \quad E\sum_{u \in \mathcal{C}} z_u e^{-z_u} > 0,$$

by Theorem 8.3 of BK97 and by definition respectively, Theorem 1.4 of BK97 applies to the functional equation (5), showing L is slowly varying. □

Note that, once they are proved, Theorems 3 and 5 imply that, typically, the slowly varying function L in Lemma 2 is equivalent to the logarithm.

Lemma 3 *Let $M(\psi)$ be the limit of the martingale $M_t(\psi)$ and $W = -\log M(1)$. Then (i) $M(\psi) = e^{-W\psi}$; (ii) W has transform Ψ; and, (iii)*

$$\lim_{t\to\infty} \sum_{u\in\mathcal{C}(t)} e^{-z_u} L(e^{-z_u}) = W. \tag{6}$$

Proof. The first two parts are proved, using Lemma 2, in just the same way as parts (i) and (ii) of Lemma 5.2 in BK97. For the final part, let $\underline{g}(t)$ be the first generation containing a member of $\mathcal{C}(t)$ and let T_n be the rightmost position occupied in the first n generations, which is finite because family sizes are finite. Then $\underline{g}(t) \geq n$ for $t \geq T_n$ so that $\underline{g}(t) \to \infty$ as $t \to \infty$. The final part of the result now follows from the second half of Theorem 6.2 in BK97. □

4 Proof of Theorem 1

The proof is through a sequence of lemmas. The first makes clear that a result similar to, but different from, (6) is enough to yield the desired conclusion.

Lemma 4 *Suppose $\Phi \in \mathcal{L}$ is a solution to (1), $L(\psi) = (1 - \Phi(\psi))/\psi$ and*

$$\lim_{t\to\infty} L(e^{-t}) \sum_{u\in\mathcal{C}(t)} e^{-z_u} = W, \tag{7}$$

where W is the limit, which is defined in Lemma 3, corresponding to Φ. If this holds for every solution in $\mathcal{L}$ to (1) then the solution is unique up to scaling.

Proof. Suppose we take two non-trivial solutions to (1), Ψ_1 and Ψ_2. Both of these can be used to construct martingales as in Lemma 1, with the limits W_1 and W_2 as in Lemma 3; then, using (7),

$$\frac{W_1}{W_2} = \lim_{t\uparrow\infty} \frac{L_1(e^{-t}) \sum_{u\in\mathcal{C}(t)} e^{-z_u}}{L_2(e^{-t}) \sum_{u\in\mathcal{C}(t)} e^{-z_u}} = \lim_{t\uparrow\infty} \frac{L_1(e^{-t})}{L_2(e^{-t})} = c$$

where c must be a constant. Thus $W_1 = cW_2$ and so, using Lemma 3(ii), $\Psi_1(\psi) = \Psi_2(c\psi)$. □

The next result applies to any slowly varying function, a fact which will be important in proving Theorem 3. It identifies a moment condition, (8), that together with (6) ensures (7) holds and hence that Lemma 4 applies.

Lemma 5 *Suppose that for a monotone decreasing, slowly varying L' (not assumed to arise from a solution to (1)) there is a random variable Δ' such that*

$$\lim_{t\to\infty} \sum_{u\in\mathcal{C}(t)} e^{-z_u} L'(e^{-z_u}) = \Delta'$$

and

$$E\left[\sum_{u\in\mathcal{C}} e^{-(1-\epsilon)z_u}\right] < \infty \quad \textit{for some } \epsilon > 0. \tag{8}$$

Then

$$L'(e^{-t}) \sum_{u \in \mathcal{C}(t)} e^{-z_u} \to \Delta'$$

as $t \to \infty$, almost surely.

Proof. The proof of Theorem 8.6 in BK97, with the change that (8) is invoked instead of Theorem 8.5, produces this result. It is this proof that uses that Nerman's (1981) ratio limit theorem. □

In BK97, Theorem 8.5 verified that (8) holds under the conditions there, but it does not apply here. (The assumption, in A2, that $v'(1) = 0$ translates to $EX = 0$ in the notation used in BK97; Theorem 8.5 assumes $-v'(1) = EX > 0$.) The corresponding result under the assumptions in force here is established next.

Lemma 6 *For some $\epsilon > 0$*

$$E\left[\sum_{u \in \mathcal{C}} e^{-(1-\epsilon)z_u}\right] < \infty.$$

Proof. Recall that μ is the intensity measure of Z and that the assumption that $v(1) = 0$ means that $\int e^{-x}\mu(dx) = 1$. Let S_n be the sum of n independent identically distributed variables with law $e^{-x}\mu(dx)$ and let τ be the first index n with $S_n > 0$, so S_τ is the first strict increasing ladder height of the random walk S_n. Then, much as in Lemma 4.1(iii) of BK97,

$$\begin{aligned}
E\left[\sum_{u \in \mathcal{C}} e^{-(1-\epsilon)z_u}\right] &= \sum_n E\left[\sum_{|u|=n} I\{u : z_u > 0 \text{ but } z_v \le 0 \text{ for } v < u\} e^{-(1-\epsilon)z_u}\right] \\
&= \sum_n E\left[\sum_{|u|=n} I\{u : z_u > 0 \text{ but } z_v \le 0 \text{ for } v < u\} e^{\epsilon z_u} e^{-z_u}\right] \\
&= \sum_n E\left[I\{S_n > 0 \text{ but } S_r \le 0 \text{ for } r < n\} e^{\epsilon S_n}\right] \\
&= E e^{\epsilon S_\tau}.
\end{aligned}$$

Thus the required finiteness reduces to the ladder height S_τ having an exponential tail. Now, by A1, for ϵ small enough,

$$\infty > e^{v(1-\epsilon)} = \int e^{-(1-\epsilon)x}\mu(dx) = \int e^{\epsilon x} e^{-x}\mu(dx) = E[e^{\epsilon S_1}]$$

and so the tails of the increment distribution of the random walk decay exponentially. This implies, by standard random walk theory, in particular, XII(3.6a) in Feller (1971), that the same is true of S_τ. □

It is worth noting that this proof also works when assumption A2 is replaced by $v(1) = 0$ and $v'(1) < 0$. This would allow the condition labelled A4 in Biggins and Kyprianou (1997), that 1 is in the interior of $\{\theta : v(\theta) < \infty\}$, to be relaxed in the proof of uniqueness in that context (Theorem 1.5). It can be relaxed to: $v(0) > 0$, $v(\theta) < \infty$ for some $\theta < 1$ and $E\left[\sum A_i \log^+ A_i\right] < \infty$.

Now, Lemmas 3 and 6 show that Lemma 5 applies. Hence Lemma 4 applies and Theorem 1 is proved.

5 Proof of Theorem 3

It is shown in Theorem 2.5 of Liu (2000) that when (3) holds Δ is not identically zero; see also Theorem 3 of Kyprianou (1998). The essence of the argument is that, by using (3) and $B_n \to \infty$, the logarithm of the L^1-convergent martingale (4) is asymptotically equivalent to ∂W_n.

To go the other way, let Δ be the limit of ∂W_n and let

$$\partial W_{\mathcal{C}(t)} = \sum_{u \in \mathcal{C}(t)} z_u e^{-z_u}.$$

Then Corollary 5 of Biggins and Kyprianou (2001) establishes that

$$\partial W_{\mathcal{C}(t)} \to \Delta \quad \text{almost surely}$$

as $t \to \infty$. Hence, taking $L'(x)$ in Lemma 5 to be $-\log x$, Lemmas 5 and 6 together give

$$\lim_{t \to \infty} t \sum_{u \in \mathcal{C}(t)} e^{-z_u} \to \Delta. \tag{9}$$

Now, taking a non-trivial solution to (1),

$$\lim_{t \to \infty} L(e^{-t}) \sum_{u \in \mathcal{C}(t)} e^{-z_u} = W.$$

Hence, just as in the proof of Lemma 4,

$$\frac{\Delta}{W} = \lim_{t \uparrow \infty} \frac{t \sum_{u \in \mathcal{C}(t)} e^{-z_u}}{L(e^{-t}) \sum_{u \in \mathcal{C}(t)} e^{-z_u}} = \lim_{t \uparrow \infty} \frac{t}{L(e^{-t})} = \lim_{t \uparrow \infty} \frac{te^{-t}}{1 - \Phi(e^{-t})},$$

which must be a (non-random) constant. The constant is only zero when Δ is identically zero; otherwise, by scaling the solution to (1), it can be made equal to one. □

The idea that the convergence in (9) produces information on the asymptotics of the functional equation occurs, in the branching Brownian motion context with non-trivial Δ, in Kyprianou (2001). It is also worth noting that (9) is a Seneta-Heyde norming for the Nerman martingale associated with the particular CMJ process arising here. The existence of such a norming in general is covered by Theorem 6.1 of Cohn (1985) and Theorem 7.7 of BK97. The special structure here means that the slowly varying function in the general theorem is the logarithm. Following on from this, in the same spirit as Theorem 1.2 in BK97, it is natural to wonder whether there are constants c_n such that W_n/c_n converges, where W_n was defined at (2). In BK97, the approach to this question, which we have not been able to settle in the present context, needs a 'law of large numbers' which would say, roughly, $W_{n+1}/W_n \to 1$ in probability.

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