A NOTE ON ROTATIONS AND INTERVAL EXCHANGE TRANSFORMATIONS ON 3-INTERVALS

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Abstract. We prove the conjecture that an interval exchange transformation on 3-intervals with corresponding permutation \([1, 2, 3] \rightarrow [3, 2, 1]\), and rationally independent discontinuity points, is never measure theoretically isomorphic to an irrational rotation.

1. Introduction

Interval exchange transformations were first introduced by Keane in [K1], and are defined as follows. Let \(I = [0, 1]\), \(n \geq 2\) and \(a = (a_1, \ldots, a_n)\) a probability vector with \(a_i > 0\). Define \(\beta_0 = 0\) and \(\beta_i = \sum_{k=1}^{i} a_k\), and set \(I_i = [\beta_{i-1}, \beta_i]\). Let \(\tau\) be a permutation of \([1, 2, \ldots, n]\), and consider the probability vector \(a^\tau = (a_{\tau^{-1}(1)}, \ldots, a_{\tau^{-1}(n)})\). Note that \(a_{\tau^{-1}(i)} > 0\) for all \(i\). Let \(\beta_0^\tau = 0\) and \(\beta_i^\tau = \sum_{k=1}^{i} a_{\tau^{-1}(k)}\), and set \(I_i^\tau = [\beta_{i-1}^\tau, \beta_i^\tau]\).

Define \(T: I \rightarrow I\) by

\[
Tx = x - \beta_{i-1} + \beta_{\tau(i)-1}
\]

if \(x \in I_i\). \(T\) is called an \((\alpha, \tau)\) interval exchange transformation on \(n\) intervals. It is clear that \(T\) is invertible, \(T\beta_{i-1} = \beta_{\tau(i)-1}^\tau\) and \(T\) maps \(I_i\) isometrically onto \(I_i^\tau\). Further, \(T\) is continuous except possibly at \(\{\beta_1, \ldots, \beta_{n-1}\}\). At these points \(T\) is right continuous. Note that \(T\) is continuous at \(\beta_i\) if and only if \(\tau(i+1) = \tau(i)+1\). In other words, \(T\) is discontinuous at \(\beta_i\) if and only if \(T\beta_{i-1}, T\beta_i\) do not appear in this order as consecutive terms in the ordered set \(\{\beta_0^\tau, \ldots, \beta_n^\tau\}\). We say \(T\) is in standard form if \(T\) is discontinuous at \(\beta_i\) for all \(i = 1, 2, \ldots, n-1\) or equivalently, if \(\tau(i+1) \neq \tau(i)+1\) for all \(i = 1, 2, \ldots, n-1\). Notice that any interval exchange transformation on \(n\) intervals can be written in standard form as an interval exchange transformation on \(m\) intervals with \(m \leq n\). Since if \(T\) is not in standard form, then \(T\) is continuous at \(\beta_i\) for some \(i\), then \(\tau(i+1) = \tau(i)+1\), and so \(T\) maps the interval \([\beta_{i-1}, \beta_{i+1}]\) isometrically onto \([\beta_{\tau(i)-1}^\tau, \beta_{\tau(i)+1}^\tau]\). Thus, we can redefine \(T\) on intervals with end points

\[
\{\beta_0, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_n\}.
\]

We repeat this process until all the remaining \(\beta\)'s are discontinuity points of \(T\).

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The permutation $\tau$ corresponding to $T$ is said to be irreducible if

$$\tau(\{1, 2, \ldots, k\}) \neq \{1, 2, \ldots, k\}, \text{ for all } k = 0, 1, \ldots, n - 1.$$ 

Note that if $\tau$ is reducible, then $T$ can be decomposed into two interval exchange transformations, one on $[0, \beta_k)$ and the other on $[\beta_k, 1)$. We assume throughout this paper that $T$ is irreducible.

Interval exchange transformations have been studied by several authors. Here we mention few of the known results. In [K1], Keane studied the minimality of such transformations, and in [K2] questions concerning unique ergodicity were investigated. It is easy to see that if $n = 2$, $T$ corresponds to a rotation and if $n = 3$, then $T$ can be seen as an induced transformation of a rotation. Thus, if the $\beta$’s are rationally independent, then in both cases $T$ is uniquely ergodic. Keynes and Newton [KN], and also Keane [K2] gave examples of interval exchange transformations that are not uniquely ergodic. Masur [M], and independently Veech [V1, V2, V3, V4, V5] showed that almost every minimal interval exchange transformation is uniquely ergodic. Later Boshernitzan [B] gave another proof of this result by more elementary means. Some of the spectral properties were studied by Veech in a series of papers [V3, V4, V5]. Oseledets [O] and Goodson [G] constructed ergodic interval exchange transformations with simple spectrum. Recently, Berthé, Chekhova and Ferenzi [BCF] proved that every ergodic interval exchange transformation on three intervals has simple spectrum. The first interval exchange transformation with continuous spectrum was given by Katok and Stepin [KS], their example is also an exchange on three intervals. In [BCF], the authors gave other examples of exchanges on three intervals with continuous spectrum, and they conjectured that no non-trivial exchange on three intervals is measure theoretically isomorphic to an irrational rotation. In section 2 we prove this conjecture as a corollary of a recent result by Simin Li [S], where he gave necessary and sufficient conditions for an interval exchange transformation to be conjugate to an irrational rotation.

2. Non-trivial exchanges on 3-intervals

Let $0 < l < m < 1$ with $1, l, m$ rationally independent. Consider the interval exchange transformation $T$ given by

$$Tx = \begin{cases} 
x + 1 - l & x \in [0, l), \\
x + 1 - l - m & x \in [l, m), \\
x - m & x \in [m, 1).
\end{cases}$$

$T$ corresponds to the permutation $(1, 2, 3) \rightarrow (3, 2, 1)$. Notice that $T$ is the only interval exchange transformation on 3-intervals which is irreducible and in standard form. Moreover, by a result of Keane [K1], $T$ is minimal. We call $T$ a non-trivial exchange transformation on 3-intervals. It is well known that $T$ is an induced transformation of the interval exchange transformation $S$ defined on $[0, 1 - l + m)$ by

$$Sx = \begin{cases} 
x + 1 - l & x \in [0, m), \\
x - m & x \in [m, 1 - l + m).
\end{cases}$$

Since after normalization $S$ is isomorphic to an irrational rotation, $S$ is minimal and uniquely ergodic, and hence so is $T$. 
Let $\alpha = \frac{1}{2-q_n}$ and $\beta = \frac{1}{2-p_n}$. In [KS], the authors proved that if $\alpha$ has unbounded partial quotients and if for some subsequence $q_n$ of denominators of convergents of $\alpha$, we have

$$|\alpha - \frac{p_n}{q_n}| < o\left(\frac{1}{q_n^2}\right), \quad \text{and} \quad |\beta - \frac{r}{q_n}| > \frac{c}{q_n}$$

for all $r$ and some constant $c > 0$, then $T$ is not measure theoretically isomorphic to an irrational rotation. In [BCF], it is proved that when $\alpha$ has bounded partial quotients, and $\beta \in K(\alpha)$ for some Cantor set $K(\alpha)$, then $T$ is not measure theoretically isomorphic to an irrational rotation.

Simin Li [Li] gave recently necessary and sufficient conditions for an interval exchange transformation to be conjugate to an irrational rotation.

**Theorem 1 (Li).** Let $T$ be an interval exchange transformation, and let $d(T^n)$ be the number of discontinuities of $T^n$. Then, $T$ is conjugate to an irrational rotation if and only if (i) $T^n$ is minimal for all $n \geq 1$, (ii) $\{d(T^n)\}$ is bounded by some integer $N > 0$ and (iii) there exists $k > 0$ and $M \geq 2N^2 + 3N^2$ such that $d(T^k) = d(T^{2k}) = \cdots = d(T^{Mk})$.

Since a non-trivial interval exchange on $3$-intervals is uniquely ergodic, to show that it is not measure theoretically isomorphic to an irrational rotation, we prove that $\{d(T^n)\}$ is an unbounded sequence.

**Theorem 2.** Let $T$ be a non-trivial interval exchange transformation on $3$-intervals with rationally independent discontinuity points. Let $D(T^n)$ be the set of discontinuity points of $T^n$, and let $d(T^n)$ denote the cardinality of $D(T^n)$. Then

$$D(T^n) = \{T^{-i}l, T^{-i}m : 0 \leq i, j \leq n - 1\},$$

and hence, $d(T^n) = 2n$.

**Proof:** The proof is done by induction on $n$. The result is true for $n = 1$. Suppose

$$D(T^k) = \{T^{-i}l, T^{-i}m : 0 \leq i, j \leq k - 1\},$$

for $k = 1, 2, \cdots, n$. We prove the result for $k = n + 1$. Let

$$0 < \beta_1 < \beta_2 < \cdots < \beta_{2n} < 1$$

be the discontinuities of $T^n$ written in increasing order. By the induction hypothesis,

$$D(T^n) = \{\beta_i : 1 \leq i \leq 2n\} = \{T^{-i}l, T^{-i}m : 0 \leq i, j \leq n - 1\}.$$

Let $\beta_0 = 0$ and $\beta_{2n+1} = 1$. The underlying partition of $T^n$ is given by

$$\mathcal{P}(T^n) = \{[\beta_i, \beta_{i+1}) : i = 0, 1, \cdots, 2n\}.$$

Let $\tau_n$ be the permutation corresponding to $T^n$ (notice that $T^n$ is an interval exchange transformation). Then,

$$T^n \{\beta_0, \beta_1, \cdots, \beta_{2n}\} = \{\beta^n_{0}, \beta^n_{1}, \cdots, \beta^n_{2n}\}$$

with $\beta_0 = \beta^n_{0} = 0$, and $T^n \beta_i = \beta^n_{i(l+1)}$ for $i = 0, 1, \cdots, 2n$. Furthermore, since $1, l$ and $m$ are rationally independent, and each $\beta^n_i$ is a linear combination of $1, l$ and $m$ with integer coefficients, it follows that $l, m \notin \{\beta^n_{0}, \beta^n_{1}, \cdots, \beta^n_{2n}\}$. Now invertibility of $T$ implies that $T\beta^n_0, \cdots, T\beta^n_{2n}, Tm, Tl$ are all distinct.
Suppose \( l \in (\beta_{n-1}, \beta_n) \), and \( m \in (\beta_{n-1}, \beta_n) \). We consider three cases.

Case 1. If \( r = s \), then \( T^{-n}l, T^{-n}m \in (\beta_{n-1}, \beta_n) \) where \( p = \tau_n^{-1}(r) \). Since \( T \) is an order preserving isometry on \([\beta_{n-1}, \beta_n]\), it follows that \( T^{-n}l < T^{-n}m \). The underlying partition of \( T^{n+1} \) is then given by

\[
P_1(T^{n+1}) = \{[\beta_0, \beta_1), \ldots, [\beta_{n-2}, \beta_{n-1}), [\beta_{n-1}, T^{-n}l), [T^{-n}l, T^{-n}m), [T^{-n}m, \beta_p), (\beta_p, \beta_{p+1}), \ldots, [\beta_{2n}, \beta_{2n+1})\}.
\]

To prove the result, we need to show that

\[
\{\beta_1, \ldots, \beta_{p-1}, T^{-n}l, T^{-n}m, \beta_p, \ldots, \beta_{2n}\}
\]

is the set of discontinuity points of \( T^{n+1} \). Let

\[
D_1 = \{\beta_0, \ldots, \beta_{p-1}, T^{-n}l, T^{-n}m, \beta_p, \ldots, \beta_{2n}\}
\]

and

\[
E_1 = \{\beta_0, \ldots, \beta_{p-1}, l, m, \beta_p, \ldots, \beta_{2n}\},
\]

both considered as ordered sets. Then \( TD_1 = E_1 \), and by discontinuity of \( T^n \) at \( \beta_p \), we have \( T^n\beta_p \neq \beta_p^n \). Further, \( \beta_i^n \in (0, l) \) for \( 1 \leq i \leq r-1 \), and \( \beta_i^n \in (m, l) \) for \( r \leq i \leq 2n \). Hence,

\[
T^{n+1}D_1 = TE_1 = \{Tm = 0, T\beta_1^n, \ldots, T\beta_{2n}^n, Tl = 1-m, T\beta_0^n = 1-l, T\beta_1^n, \ldots, T\beta_{2n}^n\}.
\]

Here, the elements of \( TE_1 \) are listed in increasing order.

We first show that \( T^{n+1} \) is discontinuous at \( \beta_i \) for \( i \neq p \). To do this, we need to prove that \( T^{n+1}\beta_{k-1} \) and \( T^{n+1}\beta_k \) do not appear in this order as consecutive terms in \( TE_1 \). By assumption, \( T^n \) is discontinuous at \( \beta_i \), hence \( T^n\beta_{k-1} \) and \( T^n\beta_k \) do not appear as consecutive terms of the form \( \beta_i^n, \beta_j^n+1 \) in \( E_1 \). Let \( I_0 = [0, l) \), \( I_1 = [l, m) \) and \( I_2 = [m, 1) \). If \( T^n\beta_{k-1}, T^n\beta_k \in I_j \) for some \( j = 0, 2 \), then since \( T \) maps \( I_j \) isometrically onto \( TI_j \), it follows that \( T^{n+1}\beta_{k-1} \) and \( T^{n+1}\beta_k \) cannot appear as consecutive terms in \( TE_1 \). If \( T^n\beta_{k-1} \in I_2 \) and \( T^n\beta_k \in I_j \) for \( j \neq k \), then either \( T^n\beta_{k-1} \in I_0 \) and \( T^n\beta_k \in I_1 \), or \( T^n\beta_{k-1} \in I_2 \) and \( T^n\beta_k \in I_0 \). In the first case we get \( T^{n+1}\beta_{k+1} < 1-m < T^n\beta_{k-1} \), and in the second case, we get \( T^{n+1}\beta_{k-1} < 1-m < T^{n+1}\beta_k \). Hence, \( T^n\beta_{k-1} \) and \( T^n\beta_k \) do not appear as consecutive terms of the form \( \beta_i^n, \beta_j^n+1 \) in \( E_1 \), and so \( T^{n+1} \) is discontinuous at \( \beta_i \).

Now, the discontinuity of \( T^n \) at \( \beta_p \) implies that \( T^{n+1}\beta_p \neq T^n\beta_p^n \), and

\[
Tm = 0 < T\beta_p^n < T^{n+1}\beta_p.
\]

Hence \( T^{n+1}(T^{-n}m) = Tm = 0 \) and \( T^{n+1}\beta_p \) do not appear as consecutive terms in \( TE_1 \). So \( T^{n+1} \) is discontinuous at \( \beta_p \).

The discontinuity of \( T^{n+1} \) at \( T^{-n}l \) follows from the fact that \( T^{n+1}\beta_{p-1} \) is an interior point of \( TI_2 \), while \( T^{n+1}(T^{-n}l) = 1-m \) is the left end-point of \( TI_1 \). Finally, \( T^{n+1}(T^{-n}m) = 0 < T\beta_p^n < 1-m = T^{n+1}(T^{-n}l) \) implies that \( T^{n+1} \) is discontinuous at \( T^{-n}m \). Therefore, \( D_1 = D(T^{n+1}) \).

Case 2: If \( r < s \) and \( p = \tau_m^{-1}r < \tau_m^{-1}k = q \), then \( T^{-n}l \in (\beta_{p-1}, \beta_p) \) and \( T^{-n}m \in (\beta_{q-1}, \beta_q) \). The discontinuity of \( T^n \) at \( \beta_p \) and \( \beta_q \) implies \( T^n\beta_p \neq \beta_p^n \) and \( T^n\beta_q \neq \beta_q^n \). The underlying partition of \( T^{n+1} \) is easily seen to be

\[
P_2(T^{n+1}) = \{[\beta_0, \beta_1), \ldots, [\beta_{p-2}, \beta_{p-1}), [\beta_{p-1}, T^{-n}l), [T^{-n}l, \beta_p), T^{n+1}(T^{-n}m), [T^{-n}m, \beta_{p+1}), \ldots, [\beta_{2n}, \beta_{2n+1})\}.
\]
\[ \{ \beta_p, \beta_{p+1}, \ldots, \beta_{q-1}, T^{-n} m, [T^{-n} m, \beta_q], [\beta_q, \beta_{q+1}], \ldots, [\beta_{2n}, 1] \}. \]

To show the discontinuity of \( T^{n+1} \) at \( \beta_1, \ldots, \beta_{2n}, T^{-n} m \), we consider the ordered sets

\[ D_2 = \{ \beta_1, \ldots, \beta_{p-1}, T^{-n} l, \beta_p, \ldots, \beta_{q-1}, T^{-n} m, \beta_q, \ldots, \beta_{2n} \} \]

and

\[ E_2 = \{ \beta_1^\pi, \ldots, \beta_{p-1}^\pi, \beta_p^\pi, \ldots, \beta_{q-1}^\pi, m, \beta_q^\pi, \ldots, \beta_{2n}^\pi \}. \]

Then, \( T^n D_2 = E_2 \). Notice that \( \beta_1^\pi, \ldots, \beta_{p-1}^\pi \) are interior points of \( I_1 \), \( \beta_p^\pi, \ldots, \beta_{q-1}^\pi \) are interior points of \( I_2 \) and \( \beta_q^\pi, \ldots, \beta_{2n}^\pi \) are interior points of \( I_2 \). Thus,

\[
T^{n+1} D_2 = T E_2 = \{ Tm = 0, T \beta_1^\pi, \ldots, T \beta_{2n}^\pi, Tl = 1 - m, T \beta_1^\pi, \ldots, T \beta_{2n}^\pi = 1 - l, \ldots, T \beta_{2n-1}^\pi \}.
\]

Here, the elements of \( T E_2 \) are listed in increasing order. We first prove that \( T^{n+1} \) is discontinuous at \( \beta_i \) for \( i \neq p, q \). If \( T^n \beta_{i-1}, T^n \beta_i \in I_j \), then since \( T^n \beta_{i-1}, T^n \beta_i \) do not appear as consecutive terms in \( E_1 \) and since \( T \) is an isometry on \( I_j \), we have that \( T^{n+1} \beta_{i-1} \) and \( T^{n+1} \beta_i \) are not consecutive terms of \( E_2 \), and thus \( T^{n+1} \) is discontinuous at \( \beta_i \). If \( T^n \beta_i \in I_j \) and \( T^n \beta_{i-1} \in I_k \) for \( k \neq j \), then we consider several cases,

- If \( T^n \beta_i \in I_2 \) and \( T^n \beta_{i-1} \in I_0 \) or \( I_1 \), then since \( T^n \beta_{i-1} \neq l \) we have \( T^{n+1} \beta_i < 1 - m < T^{n+1} \beta_{i-1} \).
- If \( T^n \beta_i \in I_2 \) and \( T^n \beta_{i-1} \in I_0 \) or \( I_1 \), then since \( T^n \beta_{i-1} \neq l \) it follows that \( T^{n+1} \beta_{i-1} < 1 - m < T^{n+1} \beta_i \).
- If \( T^n \beta_i \in I_1 \) and \( T^n \beta_{i-1} \in I_0 \), then \( T^{n+1} \beta_i < T^{n+1} \beta_{i-1} \).
- If \( T^n \beta_i \in I_0 \) and \( T^n \beta_{i-1} \in I_1 \), then since \( i \neq q \) we have \( T^{n+1} \beta_{i-1} < T \beta_{i-1}^\pi < T^{n+1} \beta_i \).
- If \( T^n \beta_i \in I_0 \) and \( T^n \beta_{i-1} \in I_2 \), then \( T^{n+1} \beta_{i-1} < 1 - m < T^{n+1} \beta_i \).

In all the above cases we see that \( T^{n+1} \) is not continuous at \( \beta_i \).

The discontinuity of \( T^{n+1} \) at \( \beta_{p} \) and \( \beta_{q} \) follows from the fact that \( T^{n+1} \beta_{p} \neq T \beta_{p}^\pi \) and \( T^{n+1} \beta_{q} \neq T \beta_{q}^\pi \), so that neither \( T^{n+1} \beta_{q} \) and \( T^{n+1} (T^{-n} l) \) nor \( T^{n+1} \beta_{q} \) and \( T^{n+1} (T^{-n} m) \) appear as consecutive terms in \( T E_2 \). Finally, from \( T^{n+1} (T^{-n} l) = 1 - m < 1 - l < T \beta_{2n}^\pi \) and \( T^{n+1} (T^{-n} m) = 0 < 1 - m < T \beta_{2n}^\pi \) we have that \( T^{n+1} \) is discontinuous at \( T^{n+1} (T^{-n} l) \) and \( T^{n+1} (T^{-n} m) \). Thus, \( D_3 = D(T^{n+1}) \).

**Case 3:** If \( r < s \) and \( p = r^{-1} r > r^{-1} s = q \), then the underlying partition of \( T^{n+1} \) is given by

\[
P_3(T^{n+1}) = \{[\beta_1, \beta_1], \ldots, [\beta_{q-2}, \beta_{q-1}], [\beta_{q-1}, T^{-n} m], [T^{-n} m, \beta_q], [\beta_q, \beta_{q+1}], \ldots, [\beta_{2n}, 1]\}.
\]

Let

\[ D_3 = \{ \beta_1, \ldots, \beta_{q-1}, T^{-n} m, \beta_q, \ldots, \beta_{p-1}, T^{-n} l, \beta_p, \ldots, \beta_{2n} \} \]

and

\[ E_3 = \{ \beta_1^\pi, \ldots, \beta_{q-1}^\pi, 1, \beta_p^\pi, \ldots, \beta_{2n}^\pi \}. \]

Then,

\[
T^{n+1} D_3 = T E_3 = \{ Tm = 0, T \beta_1^\pi, \ldots, T \beta_{2n}^\pi, Tl = 1 - m, T \beta_1^\pi, \ldots, T \beta_{2n}^\pi = 1 - l, \ldots, T \beta_{2n-1}^\pi \}.
\]
The elements of $D_3$, $E_3$ and $TE_3$ are listed in increasing order. A similar argument as in the above two cases shows that $D_3 = D(T^{n+1})$. Thus, the theorem is proved.

**Theorem 3.** Any non-trivial interval exchange transformation on 3-intervals with rationally independent discontinuity points is not measure theoretically isomorphic to an irrational rotation.

**Proof:** By theorem 2 and unique ergodicity, the result follows from Li’s theorem.

In [BCF], the authors proved that every ergodic interval exchange transformation on three intervals has simple spectrum. Using this result and theorem 3, we have the following corollary.

**Corollary 1.** Every non-trivial interval exchange transformation on three intervals with rationally independent discontinuity points has either rational or continuous spectrum.

**References**


