Preprint No. 1163, Department of Mathematics, University of Utrecht, October 2000. Submitted

A DIRECT SOLVER FOR THE GRADIENT EQUATION

ROB STEVENSON

ABSTRACT. A new finite element discretization of the equation $\mathbf{grad}\, p = \mathbf{g}$ is introduced. This discretization gives rise to an invertible system that can be directly solved, taking a number of operations that is proportional to the number of unknowns. Assuming that g is such that the continuous system has a solution, we obtain an optimal error estimate. We discuss a number of applications related to the Stokes equations.

1. INTRODUCTION AND APPLICATIONS

This paper concerns a finite element discretization of the following problem: On some domain $\Omega \subset I\!R^2$, and for some right-hand side $\mathbf{g} = (g_1, g_2)^T$ with $0 = \operatorname{rot} \mathbf{g}$ $(:= \partial_1 g_2 - \partial_2 g_1)$, find p, with $\int_\Omega p dx = 0$, such that

$$\mathbf{grad}\, p = \mathbf{g}.$$

More precisely, we consider this problem in its variational form: With

$$b(\mathbf{v}, p) := -\int_\Omega p \operatorname{div} v dx,$$

$L_0^2(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}$, and $\mathbf{V} \subset \mathbf{H}(\operatorname{div};\Omega)$ a Hilbert space that will be specified below, given $\mathbf{g} \in \mathbf{V}'$ such that $\mathbf{g}(\mathbf{v}) = 0$ for all $\mathbf{v} \in \mathbf{V}$ for which $b(\mathbf{v}, L_0^2(\Omega)) = 0$, find $p \in L_0^2(\Omega)$, such that

$$b(\mathbf{v}, p) = \mathbf{g}(\mathbf{v}) \qquad (\mathbf{v} \in \mathbf{V}). \tag{1.1}$$

It is well-known that (1.1) has a unique solution p, with $\|p\|_{L^2} \lesssim \|\mathbf{g}\|_{\mathbf{V}'}$, when

$$|b(\mathbf{v}, q)| \lesssim \|\mathbf{v}\|_{\mathbf{V}} \|q\|_{L^2} \qquad (\mathbf{v} \in \mathbf{V}, q \in L_0^2(\Omega)) \tag{1.2}$$

and

$$\|q\|_{L^2} \lesssim \sup_{0 \neq \mathbf{v} \in \mathbf{V}} \frac{|b(\mathbf{v}, q)|}{\|\mathbf{v}\|_{\mathbf{V}}} \qquad (q \in L_0^2(\Omega)). \tag{1.3}$$

(Here and in the sequel, by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.) Assuming that Ω is bounded, connected, and that it is either a polygon, or it has a smooth boundary (sufficient is $\Omega \in C^{0,1}$), it

Date: October 13, 2000.
2000 *Mathematics Subject Classification.* 65N30, 65F05, 42C40, 76D05, 35Q30.
Key words and phrases. LBB-stability, Stokes equations, multiscale bases, direct solver.

is known (see [ASV88], [Neč65]) that (1.3) is valid for $\mathbf{V} = H_0^1(\Omega)^2$, and thus also for $\mathbf{V} = \mathbf{H}(\mathrm{div};\Omega)$. Obviously (1.2) is also valid for both choices.

With $\mathbf{V} = H_0^1(\Omega)^2$ or $\mathbf{V} = \mathbf{H}(\mathrm{div};\Omega)$ respectively, problem (1.1) naturally arises in connection with the (Navier)-Stokes problem or a mixed formulation of the Poisson equation. Our discussion of applications will be focussed on the Stokes problem.

Assuming that Ω is a polygon, starting from some conforming initial triangulation τ_0 of Ω, we consider a sequence of triangulations $(\tau_k)_{k\geq 0}$, where τ_{k+1} is constructed from τ_k by subdividing each triangle from τ_k into four congruent sub-triangles. For each k, we define our trial space Q_k as the space of *piecewise constants* with respect to τ_k with zero mean. In Section 2, we will construct test spaces $\mathbf{V}_k \subset H_0^1(\Omega)^2$ such that

$$\dim \mathbf{V}_k = \dim Q_k, \tag{1.4}$$

and

$$\gamma := \inf_k \inf_{0\neq q_k\in Q_k} \sup_{0\neq \mathbf{v}_k\in\mathbf{V}_k} \frac{|b(\mathbf{v}_k, q_k)|}{\|\mathbf{v}_k\|_{(H^1)^2}\|q_k\|_{L^2}} > 0, \tag{1.5}$$

which latter property is known as the Ladyshenskaja-Babuška-Brezzi (LBB) stability condition.

Because of (1.4) and (1.5), for any $\mathbf{g}_k \in \mathbf{V}_k'$, the problem of finding $p_k \in Q_k$ such that

$$b(\mathbf{v}_k, p_k) = \mathbf{g}_k(\mathbf{v}_k) \qquad (\mathbf{v}_k \in \mathbf{V}_k) \tag{1.6}$$

has a unique solution. Moreover, it will be shown that for common approximations $\mathbf{g}_k$ of the right-hand side $\mathbf{g}$ of (1.1), the square system (1.6) can be constructed and solved in $\mathcal{O}(\dim Q_k)$ operations.

Remark 1.1. In fact the results from this paper can be generalized to certain types of locally refined triangulations and corresponding spaces Q_k. More precisely, those triangulations are covered where

- a triangle from τ_{k+1} is either a triangle from τ_k or it is generated by subdividing a triangle from τ_k into four congruent sub-triangles
- a triangle that is contained in both τ_k and τ_{k+1} is part of τ_ℓ for any $\ell \geq k$,
- two triangles from τ_k that have a non-empty intersection have comparable diameters, uniformly in k.

Yet, since it requires some technicalities to show that in these local refinement cases the resulting system can be constructed (and solved) in $\mathcal{O}(\dim Q_k)$ operations, for ease of presentation in the remainder of this paper we restrict ourselves to the uniform refinement case.

From (1.4), (1.5) we obtain the following optimal error estimate:

Theorem 1.2. *For $\mathbf{g}_k, \mathbf{g} \in H^{-1}(\Omega)^2$, such that $\mathbf{g}(\mathbf{v}) = 0$ for all $\mathbf{v} \in H^1_0(\Omega)^2$ for which $b(\mathbf{v}, L^2_0(\Omega)) = 0$, let $p_k, p \in L^2_0(\Omega)$ be the solutions of*

$$\begin{aligned} b(\mathbf{v}_k, p_k) &= \mathbf{g}_k(\mathbf{v}_k) \qquad (\mathbf{v}_k \in \mathbf{V}_k),\\ b(\mathbf{v}, p) &= \mathbf{g}(\mathbf{v}) \qquad (\mathbf{v} \in H^1_0(\Omega)^2). \end{aligned}$$

Then

$$\|p - p_k\|_{L^2} \leq (1 + \tfrac{\sqrt{2}}{\gamma}) \inf_{q_k \in Q_k} \|p - q_k\|_{L^2} + \tfrac{1}{\gamma}\|\mathbf{g} - \mathbf{g}_k\|_{(H^{-1})^2}.$$

Proof. With $\tilde{p}_k$ being the solution of $b(\mathbf{v}_k, \tilde{p}_k) = \mathbf{g}(\mathbf{v}_k)$ $(\mathbf{v}_k \in \mathbf{V}_k)$, from (1.5) we infer that

$$\|\tilde{p}_k - p_k\|_{L^2} \leq \frac{1}{\gamma} \sup_{0 \neq \mathbf{v}_k \in \mathbf{V}_k} \frac{|\mathbf{g}(\mathbf{v}_k) - \mathbf{g}_k(\mathbf{v}_k)|}{\|\mathbf{v}_k\|_{(H^1)^2}} \leq \frac{1}{\gamma}\|\mathbf{g} - \mathbf{g}_k\|_{(H^{-1})^2}.$$

For arbitrary $q_k \in Q_k$, we have $\|p - \tilde{p}_k\|_{L^2} \leq \|p - q_k\|_{L^2} + \|q_k - \tilde{p}_k\|_{L^2}$, whereas

$$\|q_k - \tilde{p}_k\|_{L^2} \leq \tfrac{1}{\gamma} \sup_{0 \neq \mathbf{v}_k \in \mathbf{V}_k} \frac{|b(\mathbf{v}_k, q_k - \tilde{p}_k)|}{\|\mathbf{v}_k\|_{(H^1)^2}} = \tfrac{1}{\gamma} \sup_{0 \neq \mathbf{v}_k \in \mathbf{V}_k} \frac{|b(\mathbf{v}_k, q_k - p)|}{\|\mathbf{v}_k\|_{(H^1)^2}} \leq \tfrac{\sqrt{2}}{\gamma}\|q_k - p\|_{L^2},$$

so that $\|p - \tilde{p}_k\|_{L^2} \leq (1 + \frac{\sqrt{2}}{\gamma}) \inf_{q_k \in Q_k} \|p - q_k\|_{L^2}$. □

We now discuss some applications of our gradient solver and comment on some existing alternative approaches. Consider the Stokes equations in its primitive variables: Given $\mathbf{f} = (f_1, f_2)^{\mathrm{T}}$, find $\mathbf{u} = (u_1, u_2)^{\mathrm{T}}$ and p, with $\int_\Omega p dx = 0$, satisfying

$$\begin{aligned} -\triangle \mathbf{u} + \mathbf{grad}\, p &= \mathbf{f} && \text{in } \Omega,\\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega,\\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega. \end{aligned}$$

For Ω being a bounded, convex polygon and $\mathbf{f} \in L^2(\Omega)^2$, its known that the unique solution $(\mathbf{u}, p)$ is in $H^2(\Omega)^2 \cap H^1_0(\Omega)^2 \times H^1(\Omega)$ with $\|\mathbf{u}\|_{(H^2)^2} + \|p\|_{H^1} \lesssim \|\mathbf{f}\|_{(L^2)^2}$.

Standard mixed finite element discretizations yield approximations for the velocities which are only *discretely* divergence-free. Approaches to obtain approximate velocities that are exactly divergence-free are based approximating a 'stream-function' ψ, which is a function that satisfies $\mathbf{u} = \mathbf{curl}\, \psi$ $(:= (\partial_2\psi, -\partial_1\psi)^T)$. Indeed, note that an approximation $\psi_k \in H^1(\Omega)$ of ψ yields an approximate velocity vector $\mathbf{u}_k := \mathbf{curl}\, \psi_k \in \mathbf{H}(\mathrm{div}; \Omega)$ with $\operatorname{div} \mathbf{u}_k = 0$.

A computation of an approximation ψ_k of ψ can be based on the biharmonic equation

$$\begin{aligned} \triangle^2 \psi &= \operatorname{rot} \mathbf{f} && \text{in } \Omega,\\ \psi = \partial_{\mathbf{n}}\psi &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Yet, discretizing this equation requires C^1, or in case of non-conforming approximations, 'nearly' C^1 finite elements.

An alternative approach (cf. [GR86, Ch.III §2-3]) is to discretize the problem of finding $\psi \in H_0^1(\Omega)$, $\omega \in H^1(\Omega)$ such that

$$\begin{array}{rcll} (\mathbf{curl}\,\omega, \mathbf{curl}\,\phi)_{(L^2)^2} &=& \mathbf{f}(\mathbf{curl}\,\phi) & (\phi \in H_0^1(\Omega)), \\ (\mathbf{curl}\,\psi, \mathbf{curl}\,\mu)_{(L^2)^2} &=& (\omega, \mu)_{(L^2)} & (\mu \in H^1(\Omega)), \end{array} \tag{1.7}$$

where $\omega = \operatorname{rot} \mathbf{u}$ is called the vorticity.

Instead of solving for ψ and $\omega = \triangle\psi$, it is also possible to set up equations for ψ and all its second derivatives $\partial_{i,j}^2\psi$, leading to the so-called Hellan-Herrmann-Johnson scheme (cf. [GR86, Ch.III §4]).

Above formulations have in common that the pressure p is eliminated, and so a post-processing procedure is needed to obtain approximations of that. Our gradient solver (1.6) can be applied for this goal.

As an example, we consider the stream function-vorticity formulation (1.7). Given some approximation ω_k of ω, based on the equation $\mathbf{grad}\, p = \mathbf{f} - \mathbf{curl}\,\omega$, we can compute $p_k \in Q_k$ from

$$b(p_k, \mathbf{v}_k) = \mathbf{f}(\mathbf{v}_k) - (\mathbf{curl}\,\omega_k, \mathbf{v}_k)_{(L^2)^2} \qquad (\mathbf{v}_k \in \mathbf{V}_k). \tag{1.8}$$

Application of Theorem 1.2 with

$$\mathbf{g}(\mathbf{v}) := \mathbf{f}(\mathbf{v}) - (\mathbf{curl}\,\omega, \mathbf{v})_{(L^2)^2},$$
$$\mathbf{g}_k(\mathbf{v}) := \mathbf{f}(\mathbf{v}) - (\mathbf{curl}\,\omega_k, \mathbf{v})_{(L^2)^2},$$

and so $\|\mathbf{g} - \mathbf{g}_k\|_{(H^{-1})^2} \leq \sqrt{2}\|\omega - \omega_k\|_{L^2}$, shows that

$$\|p - p_k\|_{L^2} \leq (1 + \tfrac{\sqrt{2}}{\gamma}) \inf_{q_k \in Q_k} \|p - q_k\|_{L^2} + \tfrac{\sqrt{2}}{\gamma}\|\omega - \omega_k\|_{L^2}. \tag{1.9}$$

A different approach, for example discussed in [GR86, Ch.III §2], is for some finite element space $\hat{Q}_k \subset H^1(\Omega) \cap L_0^2(\Omega)$, to solve $\hat{p}_k \in \hat{Q}_k$ from

$$(\mathbf{grad}\,\hat{p}_k, \mathbf{grad}\,\hat{q}_k)_{(L^2)^2)} = \mathbf{f}(\mathbf{grad}\,\hat{q}_k) - (\mathbf{curl}\,\omega_k, \mathbf{grad}\,\hat{q}_k)_{(L^2)^2} \qquad (\hat{q}_k \in \hat{Q}_k). \tag{1.10}$$

A disadvantage of this discretized Neumann's problem for the Laplace operator is that it requires an iterative solver. Moreover, without assuming more regularity of p than that it is in $H^1(\Omega)$, a complicated analysis is needed to demonstrate that this method yields convergent approximations, where in any case the error bound is qualitatively not better than (1.9). Necessarily this analysis exploits the special form of the right-hand side of (1.10), where it is needed that ω_k is the second component of the solution (ψ_k, ω_k) of a finite element discretization of (1.7). It is not easily seen what the effect is on the solution $\hat{p}_k$ of *algebraic* error in ω_k, a topic that we will discuss in more detail later on. Note that (1.9) is valid for *any* approximation ω_k of ω.

Another possibility, first proposed in [GR79], is to solve (1.8) using the same trial space Q_k, but with a test space $\hat{\mathbf{V}}_k$ being defined as the lowest order Raviart-Thomas RT0 finite element space with respect to τ_k. For this pair it is known that

$$\inf_k \inf_{0\neq q_k\in Q_k} \sup_{0\neq \hat{\mathbf{v}}_k\in\hat{\mathbf{V}}_k} \frac{|b(\hat{\mathbf{v}}_k, q_k)|}{\|\hat{\mathbf{v}}_k\|_{\mathbf{H}(\mathrm{div})}\|q_k\|_{L^2}} > 0. \tag{1.11}$$

If τ_k contains an internal vertex, then $\dim\hat{\mathbf{V}}_k > \dim Q_k$. Yet, if ω_k is such that $\mathbf{f}(\hat{\mathbf{v}}_k) = (\mathbf{curl}\,\omega_k, \hat{\mathbf{v}}_k)_{(L^2)^2}$ for all $\hat{\mathbf{v}}_k \in \hat{\mathbf{V}}_k$ for which $b(\hat{\mathbf{v}}_k, Q_k) = 0$, then the modified system (1.8) has a unique solution $\hat{p}_k \in Q_k$. However, since $\hat{\mathbf{V}}_k \not\subset H^1(\Omega)^2$, (1.11) with $\|\hat{\mathbf{v}}_k\|_{\mathbf{H}(\mathrm{div})}$ replaced by $\|\hat{\mathbf{v}}_k\|_{(H^1)^2}$ is not valid, and so a bound on $\|p-\hat{p}_k\|_{(L^2)^2}$ similar to (1.9) will depend on a norm of $\omega-\omega_k$ which is stronger than the L^2-norm.

As we have seen, our approach seems to be the most attractive one only assuming that $p \in H^1(\Omega)$. Yet, on the other hand, the approaches based on the discretized Neumann's problem for the Laplace operator or the application of the Raviart-Thomas spaces can be applied with finite element spaces of in principal any degree, which for smooth p may lead to schemes of higher order.

To discuss our final application of our gradient solver, we consider the Stokes equations in the primitive variables written in variational form: Given $\mathbf{f} \in H^{-1}(\Omega)^2$, find $\mathbf{u} \in H_0^1(\Omega)^2$ and $p \in L_0^2(\Omega)$ such that

$$\begin{array}{rcll} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &=& \mathbf{f}(\mathbf{v}) & (\mathbf{v} \in H_0^1(\Omega)^2), \\ b(\mathbf{u},q) &=& 0 & (q \in L_0^2(\Omega)), \end{array} \tag{1.12}$$

where

$$a(\mathbf{w},\mathbf{v}) := \int_\Omega \nabla\mathbf{w} : \nabla\mathbf{v}\,dx.$$

We describe the usual mixed finite element discretization. For $k \in \mathbb{N}$, let $\mathbf{S}_k$ and $\check{Q}_k$ be finite element spaces that serve as, increasingly better, approximations of $H_0^1(\Omega)^2$ and $L_0^2(\Omega)$ respectively. It is no restriction to assume that $\check{Q}_k \subset L_0^2(\Omega)$, but we do allow non-conforming finite element spaces $\mathbf{S}_k$, i.e., $\mathbf{S}_k \not\subset H_0^1(\Omega)^2$. As a consequence, we generally need extensions of $a(\ ,\ )$ and $b(\ ,\ )$ to a scalar product $a_k(\ ,\ )$ and a bilinear form $b_k(\ ,\ )$ on $(H_0^1(\Omega)^2 + \mathbf{S}_k) \times (H_0^1(\Omega)^2 + \mathbf{S}_k)$ and $(H_0^1(\Omega)^2 + \mathbf{S}_k) \times L_0^2(\Omega)$ respectively. We equip $H_0^1(\Omega)^2 + \mathbf{S}_k$ with the energy-norm $\|\cdot\|_{1,k} = \sqrt{a_k(\cdot,\cdot)}$. We assume that $b_k(\ ,\ )$ is uniformly bounded, and that the following LBB-condition is valid:

$$\inf_k \inf_{0\neq q_k\in \check{Q}_k} \sup_{0\neq \mathbf{v}_k\in\mathbf{S}_k} \frac{|b_k(\mathbf{v}_k, q_k)|}{\|\mathbf{v}_k\|_{1,k}\|q_k\|_{L^2}} > 0. \tag{1.13}$$

Assuming that $\mathbf{f} \in \mathbf{S}_k'$, we arrive at the following approximation scheme: Find $\mathbf{u}_k \in \mathbf{S}_k$, $p_k \in \check{Q}_k$ such that

$$\begin{array}{rcll} a_k(\mathbf{u}_k,\mathbf{v}_k) + b_k(\mathbf{v}_k,p_k) &=& \mathbf{f}(\mathbf{v}_k) & (\mathbf{v}_k \in \mathbf{S}_k), \\ b_k(\mathbf{u}_k,q_k) &=& 0 & (q_k \in \check{Q}_k). \end{array} \tag{1.14}$$

Because of (1.13), this system has a unique solution, and depending on the possible consistency error, the approximation properties of $\mathbf{S}_k$ and $\check{Q}_k$, and the regularity of $\mathbf{u}$ and p, appropriate bounds on $\|\mathbf{u}-\mathbf{u}_k\|_{1,k}$ and $\|p-p_k\|_{L^2}$ are known.

Defining the space of discretely divergence-free velocities

$$\mathbf{Z}_k = \{\mathbf{v}_k \in \mathbf{S}_k : b_k(\mathbf{v}_k, q_k) = 0\, \forall q_k \in \check{Q}_k\},$$

the velocity component $\mathbf{u}_k$ of the solution of (1.14) can be characterized as the unique solution of the problem: Find $\mathbf{u}_k \in \mathbf{Z}_k$ such that

$$a_k(\mathbf{u}_k, \mathbf{v}_k) = \mathbf{f}(\mathbf{v}_k) \qquad (\mathbf{v}_k \in \mathbf{Z}_k). \tag{1.15}$$

For some pairs $(\mathbf{S}_k, \check{Q}_k)$ a local basis for $\mathbf{Z}_k$ is known. This opens a way to compute $\mathbf{u}_k$ by solving the elliptic problem (1.15) only, instead of solving the original saddle-point problem (1.14).

An example of a pair for which such a basis is available (cf. [Cro72, Tho81]) is given by the case that, with respect to some conforming triangulation τ_k of Ω, $\mathbf{S}_k = S_k^2$ with S_k being the non-conforming P_1 finite element space, i.e.,

$$S_k = \{v \in \textstyle\prod_{T \in \tau_k} P_1(T),\ v \text{ is continuous at the midpoints } m_e \text{ of the interelement boundaries } e, \text{ and it vanishes at the midpoints } m_e \text{ of edges along } \partial\Omega\},$$

and $\check{Q}_k (= Q_k)$ is the space of piecewise constant functions with zero mean value. In this case

$$a_k(\mathbf{w}_k, \mathbf{v}_k) := \sum_{T \in \tau_k} \int_T \nabla \mathbf{w}_k : \nabla \mathbf{v}_k dx, \quad b_k(\mathbf{v}_k, q_k) := -\sum_{T \in \tau_k} \int_T q_k \operatorname{div} \mathbf{v}_k dx.$$

For this pair, optimal multi-grid, domain decomposition and Cascade multi-level methods for solving $\mathbf{u}_k$ from (1.15) were proposed and analyzed in [Bre90], [Bre96] and [Ste99] respectively.

Other examples of pairs, all involving non-conforming spaces $\mathbf{S}_k$, for which a local basis for the resulting $\mathbf{Z}_k$ has been constructed can be found in [CSS86, Tur94]. Constructions based on wavelets were discussed in [Urb96].

Knowing $\mathbf{u}_j$, we are left with the problem of finding an approximation for the pressure. The obvious approach is to solve $p_k \in \check{Q}_k$ from

$$b_k(\mathbf{v}_k, p_k) = \mathbf{f}(\mathbf{v}_k) - a_k(\mathbf{u}_k, \mathbf{v}_k) \qquad (\mathbf{v}_k \in \mathbf{S}_k). \tag{1.16}$$

Indeed, existence and uniqueness of this p_k are already known, and suitable error estimates are available.

The number of equations in (1.16) exceeds the number of unknowns. However, since (1.16) is trivially valid for $\mathbf{v}_k \in \mathbf{Z}_k$, it is sufficient to satisfy

$$b_k(\mathbf{v}_k, p_k) = \mathbf{f}(\mathbf{v}_k) - a_k(\mathbf{u}_k, \mathbf{v}_k) \qquad (\mathbf{v}_k \in \mathbf{R}_k), \tag{1.17}$$

where $\mathbf{R}_k$ is some subspace of $\mathbf{S}_k$ satisfying $\mathbf{R}_k \cap \mathbf{Z}_k = \{0\}$ and $\dim \mathbf{R}_k = \dim \check{Q}_k$, or equivalently, $\mathbf{S}_k = \mathbf{R}_k \oplus \mathbf{Z}_k$.

In [CSS86, Tur94, Urb96] we find similar-like choices of $\mathbf{R}_k$ which give rise to direct solvers that can be implemented efficiently. In the following, we describe the idea for the non-conforming P_1, piecewise constant finite element pair.

For all pairs $T, \tilde{T} \in \tau_k$, such that $e := T \cap \tilde{T}$ is an edge, let

$$\mathbf{w}_e = |e|^{-1} g_e \mathbf{n}_e \in \mathbf{S}_k,$$

where $\mathbf{n}_e$ is a unit vector normal to e, and $g_e \in S_k$ is the standard basis function defined by $g_e(m_{\tilde{e}}) = \begin{cases} 1 & \text{if } \tilde{e} = e \\ 0 & \text{if } \tilde{e} \neq e \end{cases}$, see Figure 1. It is easily verified that, assuming $\mathbf{n}_e$ points into

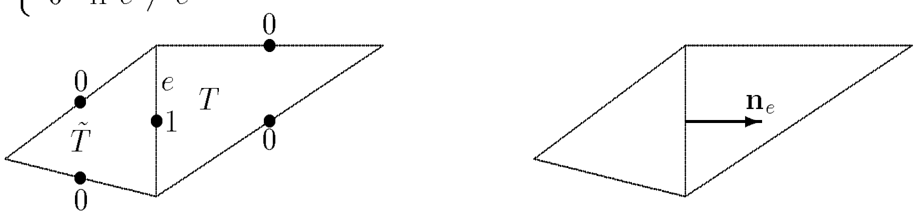

FIGURE 1. g_e and $\mathbf{w}_e$.

T,

$$\operatorname{div} \mathbf{w}_e = \partial_{\mathbf{n}_e} g_e = \begin{cases} -(\operatorname{vol} T)^{-1} & \text{on } T, \\ (\operatorname{vol} \tilde{T})^{-1} & \text{on } \tilde{T}, \\ 0 & \text{elsewhere.} \end{cases}$$

Now on some $T_0 \in \tau_k$, fix p_k, and let $T_1 \in \tau_k$ be such that $e = T_0 \cap T_1$ is an edge. We will call such triangles neighbours. Then (1.16) for $\mathbf{v}_k = \mathbf{w}_e$ determines $p_k|_{T_1}$ uniquely. In this way, by marching from neighbour to a still unvisited neighbour, $p_k \in \prod_{T \in \tau_k} P_0(T)$ can be fixed completely. In the end, by subtracting a suitable constant, p_k is mapped into Q_k. Clearly, this procedure for computing p_k is equivalent to solving (1.17), where $\mathbf{R}_k = \operatorname{span}\{\mathbf{w}_e\}$ with e running over all edges that were crossed in the marching process.

A potential pitfall with the approach of solving (1.17) is that in practice, instead of the exact solution $\mathbf{u}_k$, only an approximation $\tilde{\mathbf{u}}_k \in \mathbf{Z}_k$ will be at ones disposal, since (1.15) will have been solved by an iterative method. With $\mathbf{u}_k$ replaced by $\tilde{\mathbf{u}}_k \neq \mathbf{u}_k$, the system (1.16) does not have a solution, but because of (1.13) and $\mathbf{S}_k = \mathbf{R}_k \oplus \mathbf{Z}_k$, the system (1.17) does, which solution we denote by $\tilde{p}_k$. With $T_k : \mathbf{Z}_k \to \check{Q}_k$ being the linear operator defined by

$$b_k(\mathbf{v}_k, T_k \mathbf{w}_k) = -a_k(\mathbf{w}_k, \mathbf{v}_k) \qquad (\mathbf{v}_k \in \mathbf{R}_k),$$

there holds

$$p_k - \tilde{p}_k = T_k(\mathbf{u}_k - \tilde{\mathbf{u}}_k).$$

Remark 1.3. It is easily verified that $\|T_k\|_{L^2 \leftarrow 1,k} \geq \left(\inf_{0 \neq q_k \in \check{Q}_k} \sup_{0 \neq \mathbf{v}_k \in \mathbf{R}_k} \frac{|b_k(\mathbf{v}_k, q_k)|}{\|\mathbf{v}_k\|_{1,k} \|q_k\|_{L^2}} \right)^{-1}$.

For the non-conforming P_1, piecewise constant finite element pair, we have computed $\|T_k\|_{L^2 \leftarrow 1,k}$ numerically in the following situation: $\Omega = [0,1]^2$, τ_k is a uniform partition of Ω into right-angled isosceles triangles of which the equal sides have length $h_k := 2^{-k}$, and $\mathbf{R}_k = \operatorname{span}\{\mathbf{w}_e\}$, where e runs over all edges corresponding to the dotted lines as indicated in Figure 2 for the case $k = 2$. The results given in Table 1 indicate

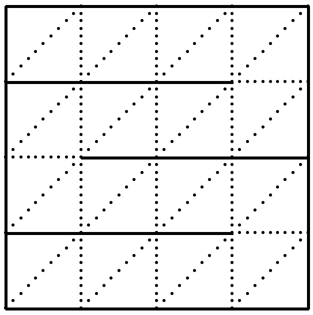

FIGURE 2. Crossed edges in the marching process defining $\mathbf{R}_k$

k	1	2	3	4	5	6	7
$\|T_k\|_{L^2(\Omega)\leftarrow 1,k}$	2.3	10	41	160	650	2600	10000

TABLE 1

that $\|T_k\|_{L^2(\Omega)\leftarrow 1,k} \eqsim h_k^{-2}$. In case of $\mathbf{f} \in L^2(\Omega)^2$, and thus $\|\mathbf{u} - \mathbf{u}_k\|_{1,k} = \mathcal{O}(h_k)$ and $\|p - p_k\|_{L^2} = \mathcal{O}(h_k)$, this means that the algebraic error $\mathbf{u}_k - \tilde{\mathbf{u}}_k$ in $\| \, \|_{1,k}$-norm should be of order h_k^3 to be sure that the resulting algebraic error in the pressure will not dominate the discretization error. That is, thinking of a linearly convergent iterative method, and an initial error that is of order 1, one should triple the number of iterations sufficient for solving the velocities from (1.15), if one wants to use the outcome to compute the pressure afterwards using this marching process.

An alternative approach to solve for the pressure can be based on our gradient solver: Given some approximation $\tilde{\mathbf{u}}_k \in \mathbf{S}_k$ of $\mathbf{u}$, solve $\tilde{p}_k \in Q_k$ from

$$b(\mathbf{v}_k, \tilde{p}_k) = \mathbf{f}(\mathbf{v}_k) - a_k(\tilde{\mathbf{u}}_k, \mathbf{v}_k) \qquad (\mathbf{v}_k \in \mathbf{V}_k).$$

Since $\mathbf{u}$ and p satisfy

$$b(\mathbf{v}, p) = \mathbf{f}(\mathbf{v}) - a_k(\mathbf{u}, \mathbf{v}) \qquad (\mathbf{v} \in H_0^1(\Omega)^2),$$

an application of Theorem 1.2 shows the optimal error estimate

$$\|p - \tilde{p}_k\|_{L^2} \leq (1 + \tfrac{\sqrt{2}}{\gamma}) \inf_{q_k \in Q_k} \|p - q_k\|_{L^2} + \tfrac{1}{\gamma}\|\mathbf{u} - \tilde{\mathbf{u}}_k\|_{1,k},$$

where now $\|p - \tilde{p}_k\|_{L^2}$ and $\|\mathbf{u} - \tilde{\mathbf{u}}_k\|_{1,k}$ are the 'total' errors. In particular, with this approach an algebraic error in the approximate velocities is not blown up.

2. Construction and implementation of the gradient solver

Let τ_0 be some conforming triangulation of a polygon $\Omega \subset I\!R^2$, and for $k \geq 0$, let τ_{k+1} be constructed from τ_k by subdividing each triangle from τ_k into four congruent sub-triangles. For each k, Q_k is defined as the space of *piecewise constants* with respect to τ_k with zero mean. We will construct spaces $\mathbf{V}_k$ satisfying both $\dim \mathbf{V}_k = \dim Q_k$ ((1.4)) and the LBB condition (1.5).

We recall the marching process discussed in §1, which however here will be applied on the coarsest level only. Starting from some $T \in \tau_0$, until we have been in all triangles in τ_0, we travel from already visited triangles to yet unvisited neighbours putting the edges that were crossed between such neighbours in a set called E_0. In case τ_0 does not contain internal vertices and Ω is simply-connected, E_0 will be the set of all internal edges in τ_0, but otherwise E_0 will be a proper subset of that set. In any case the number of elements in E_0 will be equal to $\dim Q_0$. For each $e \in E_0$, $e = T \cap \tilde{T}$ with $T, \tilde{T} \in \tau_0$, let $\mathbf{w}_e \in H^1_0(\Omega)^2$ be some function such that $\int_e \mathbf{w}_e \cdot \mathbf{n}_e ds \neq 0$, where $\mathbf{n}_e$ is a unit vector normal to e. We define $\mathbf{V}_0 = \mathrm{span}\{\mathbf{w}_e : e \in E_0\}$.

For each $0 \neq q \in Q_0$, there exists an $e \in E_0$, $e = T \cap \tilde{T}$ with $T, \tilde{T} \in \tau_0$, such that $q|_T \neq q|_{\tilde{T}}$. From $|b(\mathbf{w}_e, q)| = |(q|_T - q|_{\tilde{T}}) \int_e \mathbf{w}_e \cdot \mathbf{n}_e ds| \neq 0$, we conclude that

$$\inf_{0 \neq q \in Q_0} \sup_{0 \neq \mathbf{v} \in \mathbf{V}_0} \frac{|b(\mathbf{v}, q)|}{\|\mathbf{v}\|_{(H^1)^2} \|q\|_{L^2}} > 0. \tag{2.1}$$

For $m \geq 1$, $m \in I\!N$, we define E_m as the set of *new edges* in τ_m, that is, all edges that were added to refine $T \in \tau_{m-1}$. For each $e \in E_m$, $e = T \cap \tilde{T}$ with $T, \tilde{T} \in \tau_m$, let ℓ be the line connecting both vertices of T and $\tilde{T}$ which are not on e. Since, because of the refinement procedure, $T \cup \tilde{T}$ is a parallelogram, ℓ intersects e at its midpoint m_e. Let $g_e \in H^1_0(T \cup \tilde{T})$ be the function that is 1 at m_e, and that is linear on the four triangles generated by intersecting both T and $\tilde{T}$ along ℓ. For a non-zero vector $\mathbf{s}_e$ in m_e pointing along ℓ, and with $\mathbf{n}_e$ a unit vector normal to e, say pointing into T, we put

$$\mathbf{w}_e = \frac{2 g_e \mathbf{s}_e}{|e| \ \mathbf{s}_e \cdot \mathbf{n}_e} \in H^1_0(T \cup \tilde{T})^2,$$

see Figure 2. By construction, $\mathrm{div}\, \mathbf{w}_e = \frac{2}{|e| \ \mathbf{s}_e \cdot \mathbf{n}_e} \partial_{\mathbf{s}_e} g_e$ is constant on both T and $\tilde{T}$, and in particular

$$\mathrm{div}\, \mathbf{w}_e = \begin{cases} -(\mathrm{vol}\, T)^{-1} & \text{on } T, \\ (\mathrm{vol}\, \tilde{T})^{-1} & \text{on } \tilde{T}, \\ 0 & \text{elsewhere.} \end{cases}$$

We infer that

$$\mathrm{div}\, \mathbf{w}_e \in Q_m, \quad \mathrm{div}\, \mathbf{w}_e \perp_{L^2} Q_{m-1}, \tag{2.2}$$

and, with $\mathbf{W}_k := \mathrm{span} \cup_{m=1}^k \{\mathbf{w}_e : e \in E_m\}$, that

$$\mathrm{div}\, \mathbf{W}_k = Q_k \ominus^{\perp_{L^2}} Q_0. \tag{2.3}$$

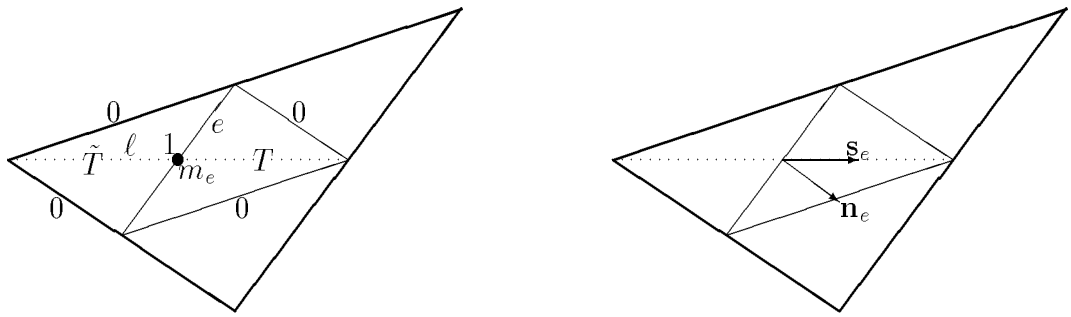

FIGURE 3. g_e and $\mathbf{w}_e$.

Finally, using the fact that $\mathrm{supp}\mathbf{w}_e \cap \mathrm{supp}\mathbf{w}_{\tilde{e}} = \emptyset$ for all $e, \tilde{e} \in E_m$ that are not contained in a common $T \in \tau_{m-1}$, a homogeneity argument shows that

$$2^m \|\cdot\|_{(L^2)^2} \lesssim \|\mathrm{div}\,\cdot\|_{L^2} \quad \text{on } \mathrm{span}\{\mathbf{w}_e : e \in E_m\}. \tag{2.4}$$

Note that the latter relation is valid uniformly in all triangulations τ_0 that satisfy some minimal angle condition.

Defining $\mathbf{V}_k = \mathbf{V}_0 + \mathbf{W}_k$, there holds that

$$\mathrm{dim}\mathbf{V}_k = \#\left(\cup_{m=0}^k E_m\right) = \mathrm{dim}Q_k.$$

In Theorem 2.3 we will prove that

$$\inf_k \inf_{0\neq q\in Q_k \ominus^{\perp_{L^2}} Q_0} \sup_{0\neq \mathbf{v}\in \mathbf{W}_k} \frac{|b(\mathbf{v},q)|}{\|\mathbf{v}\|_{(H^1)^2}\|q\|_{L^2}} > 0. \tag{2.5}$$

Since by (2.2), $b(\mathbf{W}_k, Q_0) = 0$, an application of Lemma 2.1 given below now shows that (2.1) and (2.5) imply that

$$\inf_k \inf_{0\neq q\in Q_k} \sup_{0\neq \mathbf{v}\in \mathbf{V}_k} \frac{|b(\mathbf{v},q)|}{\|\mathbf{v}\|_{(H^1)^2}\|q\|_{L^2}} > 0,$$

i.e, both (1.4) and (1.5) are valid.

Lemma 2.1. *Let $Q = Q_1 + Q_2$ and $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$ be normed linear spaces, b a sesqui-linear form on $\mathbf{V} \times Q$ for which*

$$b(\mathbf{V}_2, Q_1) = 0,$$

and $C, \gamma_1, \gamma_2 > 0$ constants such that

$$\sup_{0\neq q\in Q,\,0\neq v\in \mathbf{V}} \frac{|b(\mathbf{v},q)|}{\|\mathbf{v}\|_{\mathbf{V}}\|q\|_Q} \leq C, \quad \inf_{0\neq q\in Q_i} \sup_{0\neq v\in \mathbf{V}_i} \frac{|b(\mathbf{v},q)|}{\|\mathbf{v}\|_{\mathbf{V}}\|q\|_Q} \geq \gamma_i \quad (i \in \{1,2\}).$$

Then there exists a $\gamma > 0$, only dependent on C, γ_1, γ_2, such that

$$\inf_{0\neq q\in Q} \sup_{0\neq v\in \mathbf{V}} \frac{|b(\mathbf{v},q)|}{\|\mathbf{v}\|_{\mathbf{V}}\|q\|_Q} \geq \gamma.$$

Proof. Let $q = q_1 + q_2$, where $q_1 \in Q_1$, $q_2 \in Q_2$.

In case $\|q_2\|_Q \leq \frac{\gamma_1}{2C}\|q_1\|_Q$, and thus $\|q\|_Q \leq (1+\frac{\gamma_1}{2C})\|q_1\|_Q$, let $\mathbf{v}_1 \in \mathbf{V}_1$ be such that $|b(\mathbf{v}_1, q_1)| \geq \frac{3}{4}\gamma_1\|\mathbf{v}_1\|_{\mathbf{V}}\|q_1\|_Q$. Then

$$\begin{aligned}|b(\mathbf{v}_1, q_1+q_2)| &\geq \tfrac{3}{4}\gamma_1\|\mathbf{v}_1\|_{\mathbf{V}}\|q_1\|_Q - C\|\mathbf{v}_1\|_{\mathbf{V}}\|q_2\|_Q \\ &\geq \tfrac{1}{4}\gamma_1\|\mathbf{v}_1\|_{\mathbf{V}}\|q_1\|_Q \geq \left(\tfrac{1}{4}\gamma_1/(1+\tfrac{\gamma_1}{2C})\right)\|\mathbf{v}_1\|_{\mathbf{V}}\|q\|_Q.\end{aligned}$$

Otherwise, when $\|q_2\|_Q \geq \frac{\gamma_1}{2C}\|q_1\|_Q$, and so $\|q\|_Q \leq (\frac{2C}{\gamma_1}+1)\|q_2\|_Q$, let $\mathbf{v}_2 \in \mathbf{V}_2$ be such that $|b(\mathbf{v}_2, q_2)| \geq \frac{1}{2}\gamma_2\|\mathbf{v}_2\|_{\mathbf{V}}\|q_2\|_Q$. Then

$$|b(q_1+q_2, \mathbf{v}_2)| = |b(q_2, \mathbf{v}_2)| \geq \tfrac{1}{2}\gamma_2\|\mathbf{v}_2\|_{\mathbf{V}}\|q_2\|_Q \geq \left(\tfrac{1}{2}\gamma_2/(\tfrac{2C}{\gamma_1}+1)\right)\|\mathbf{v}_2\|_{\mathbf{V}}\|q\|_Q.$$

□

There remains to prove (2.5). At first for theoretical purposes, but later also for constructing an efficient implementation of the gradient solver, for $m \in \mathbb{N}$, we define $\tilde{S}_m$ as the conforming P_1 finite element space with respect to a refined triangulation $\tilde{\tau}_m$ defined below, i.e., $\tilde{S}_m = C(\Omega) \cap H_0^1(\Omega) \cap \prod_{T\in\tilde{\tau}_m} P_1(T)$. The triangulations $\tilde{\tau}_m$ are constructed from τ_m by subdividing each $T \in \tau_m$ into 6 sub-triangles by connecting the vertices with the midpoints on the opposite edges. The resulting spaces $\tilde{S}_m$ are nested, i.e. $\tilde{S}_m \subset \tilde{S}_{m+1}$,

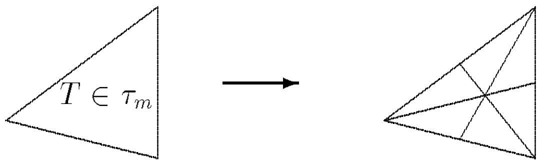

FIGURE 4. Construction of the triangulation $\tilde{\tau}_m$ underlying $\tilde{S}_m$

and for $m \geq 1$,

$$\{\mathbf{w}_e : e \in E_m\} \subset \tilde{\mathbf{S}}_m := \tilde{S}_m^2. \tag{2.6}$$

Remark 2.2. The construction of $\mathbf{w}_e \in H_0^1(T \cup \tilde{T})^2$ directly generalizes to any pair of triangles T, $\tilde{T}$ that share an edge e, and for which $T \cup \tilde{T}$ is convex. Indeed, what is needed is that the line ℓ connecting both vertices of T and $\tilde{T}$ that are not on e intersects e. The reason why we only considered cases where ℓ intersects e at its midpoint is the property (2.6).

Theorem 2.3. *There holds*

$$\inf_k \inf_{0\neq q\in Q_k \ominus^{\perp}{}_{L^2} Q_0} \sup_{0\neq \mathbf{v}\in\mathbf{W}_k} \frac{|b(\mathbf{v}, q)|}{\|\mathbf{v}\|_{(H^1)^2}\|q\|_{L^2}} > 0.$$

Proof. A so-called *strengthened Cauchy-Schwarz inequality* is valid on the sequence $(\tilde{\mathbf{S}}_m)_m$, (cf. e.g. [BY93, Lemma 3.3]), i.e., there exists a $\delta < 1$ such that for $n > m$,

$$|(\mathbf{v}_m, \mathbf{v}_n)_{(H^1)^2}| \lesssim \delta^{n-m}\|\mathbf{v}_m\|_{(H^1)^2} 2^n \|\mathbf{v}_n\|_{(L^2)^2} \qquad (\mathbf{v}_m \in \tilde{\mathbf{S}}_m, \mathbf{v}_n \in \tilde{\mathbf{S}}_n).$$

Combined with the *inverse inequality*,

$$\|\mathbf{v}_m\|_{(H^1)^2} \lesssim 2^m \|\mathbf{v}_m\|_{(L^2)^2} \qquad (\mathbf{v}_m \in \tilde{\mathbf{S}}_m),$$

we derive that

$$\|\sum_{m=0}^{k} \mathbf{v}_m\|_{(H^1)^2}^2 \lesssim \sum_{m=0}^{k} 4^m \|\mathbf{v}_m\|_{(L^2)^2}^2 \qquad (\mathbf{v}_m \in \tilde{\mathbf{S}}_m). \tag{2.7}$$

Substituting $\mathbf{v}_m = \sum_{e \in E_m} c_e \mathbf{w}_e$ in (2.7), by (2.4) and (2.2) we find that

$$\|\sum_{m=1}^{k} \sum_{e \in E_m} c_e \mathbf{w}_e\|_{(H^1)^2}^2 \lesssim \sum_{m=1}^{k} \|\sum_{e \in E_m} c_e \mathrm{div}\, \mathbf{w}_e\|_{L^2}^2 = \|\sum_{m=1}^{k} \sum_{e \in E_m} c_e \mathrm{div}\, \mathbf{w}_e\|_{L^2}^2,$$

or $\|\cdot\|_{(H^1)^2} \eqsim \|\mathrm{div}\, \cdot\|_{L^2}$ on $\mathbf{W}_k$ (uniformly in k).

Since by (2.3) any $q \in Q_k \ominus^{\perp_{L^2}} Q_0$ can be written as $q = \mathrm{div}\, \mathbf{v}$ for some $\mathbf{v} \in \mathbf{W}_k$, we arrive at

$$\|q\|_{L^2} = \frac{|b(\mathbf{v}, q)|}{\|\mathrm{div}\, \mathbf{v}\|_{L^2}} \eqsim \frac{|b(\mathbf{v}, q)|}{\|\mathbf{v}\|_{(H^1)^2}},$$

which completes the proof. □

Finally, for given $g_k \in \mathbf{V}_k'$, we discuss the implementation of setting up and solving a system corresponding to the problem of finding $p_k \in Q_k$ satisfying

$$b(\mathbf{v}_k, p_k) = \mathbf{g}_k(\mathbf{v}_k) \qquad (\mathbf{v}_k \in \mathbf{V}_k). \tag{2.8}$$

Let Φ_0 be some basis on Q_0. If we equip $\mathbf{V}_k$ and Q_k with bottom-to-top level-wise ordered bases $\cup_{m=0}^{k} \{\mathbf{w}_e : e \in E_m\}$ and $\Phi_0 + \cup_{m=1}^{k} \{\mathrm{div}\, \mathbf{w}_e : e \in E_m\}$ respectively, then (2.8) results in a matrix-vector system

$$\mathbf{B}_k \mathbf{P}_k = \mathbf{G}_k,$$

where $\mathbf{P}_k$ is the representation of p_k with respect to above basis of Q_k, $\mathbf{G}_k = [\mathbf{g}_k(\mathbf{w}_e)]_{e \in E_m, 0 \leq m \leq k}$, and $\mathbf{B}_k$ is the matrix having as elements the application of b to all pairs of basis functions from $\mathbf{V}_k$ and Q_k respectively. The multi-level ordering of these bases induces a block partitioning $\mathbf{B}_k = ((\mathbf{B}_k)_{mn})_{0 \leq m,n \leq k}$, with the size of $(\mathbf{B}_k)_{mn}$ being $\#E_m \times \#E_n$. The property (2.2) now implies that $(\mathbf{B}_k)_{mn} = 0$ except for $m = 0$ or $m = n$. Moreover, with respect to a canonical ordering of the basis functions within each level, the matrices $(\mathbf{B}_k)_{mm}$ for $1 \leq m \leq k$ are block diagonal matrices, with blocks of size 3×3. We conclude that $\mathbf{B}_k$ can be inverted in $\mathcal{O}(\dim Q_k)$ operations.

Remark 2.4. If for all $e \in E_0$, $e = T \cap \tilde{T}$ with $T, \tilde{T} \in \tau_0$, the line ℓ connecting both vertices of T and $\tilde{T}$ which are not on e intersects e at its midpoint, then just as on levels> 0, $\mathbf{w}_e$ can be selected in $\tilde{\mathbf{S}}_0$ with $\mathrm{div}\, \mathbf{w}_e \in Q_0$. In this case, $(\mathbf{B}_k)_{0n} = 0$ for $n > 0$, or $\mathbf{B}_k$ will be a block diagonal matrix.

Otherwise, a reasonable approach is to take $\mathbf{w}_e = \frac{2 f_e \mathbf{n}_e}{|e|}$, where $f_e \in H_0^1(T \cup \tilde{T})$ is defined by $f_e(m_e) = 1$, and f_e is linear on all $T \in \tau_1$. Note that $\mathbf{w}_e \in \tilde{\mathbf{S}}_1$, $\int_e \mathbf{w}_e \cdot \mathbf{n}_e ds = 1$, and that $\mathrm{div}\, \mathbf{w}_e \in Q_1$, which means that $(\mathbf{B}_k)_{0n} = 0$ for $n > 1$.

Since the diameters of the supports of the basis functions $\mathbf{w}_e$ of $\mathbf{V}_k$ are not all of order 2^{-k}, but instead range from order 1 to order 2^{-k}, a straightforward computation of $\mathbf{G}_k$, or a sufficiently accurate approximation of this vector involving numerical quadrature, can be expected to demand a number of operations of order $k \dim Q_k$.

Therefore, let us equip $\tilde{S}_k$ with the standard nodal basis $\{\nu_{m,x} : x \in N_k\}$, where N_k is the set of interior vertices of $\tilde{\tau}_k$. Since $\operatorname{diam}(\operatorname{supp}\nu_{k,x}) \eqsim 2^{-k}$, we may expect that $\tilde{\mathbf{G}}_k = [(\mathbf{g}_k^{(1)}(\nu_{k,x}), \mathbf{g}_k^{(2)}(\nu_{k,x}))]_{x\in N_k}$, or a sufficiently accurate approximation of this vector, can be computed in $\mathcal{O}(\dim Q_k)$ operations.

In view of Remark 2.4, we assume that $\mathbf{V}_0 = \operatorname{span}\{\mathbf{w}_e : e \in E_0\} \subset \tilde{\mathbf{S}}_1$, and so $\mathbf{V}_k \subset \tilde{\mathbf{S}}_k$ for $k \geq 1$. For $k \geq 1$, let $\mathbf{I}_k$ be the representation of the embedding of $\mathbf{V}_k$ into $\tilde{\mathbf{S}}_k$. Then there holds $\mathbf{G}_k = \mathbf{I}_k^{\mathrm{T}}\tilde{\mathbf{G}}_k$. With $\operatorname{span}\{\mathbf{w}_e : e \in E_m\}$ being equipped with $\{\mathbf{w}_e : e \in E_m\}$, for $m \geq 1$ let the uniformly sparse matrices $\mathbf{q}_m$ and $\mathbf{p}_{m+1}$ be the representations of the embeddings $\operatorname{span}\{\mathbf{w}_e : e \in E_m\} \to \tilde{\mathbf{S}}_m$ and $\tilde{\mathbf{S}}_m \to \tilde{\mathbf{S}}_{m+1}$ respectively, and let $\check{\mathbf{q}}_0$ be the representation of the embedding $\operatorname{span}\{\mathbf{w}_e : e \in E_0\} \to \tilde{\mathbf{S}}_1$.

With these definitions, the mappings $\mathbf{I}_k$ satisfy

$$\mathbf{I}_{k+1} = \begin{bmatrix} \mathbf{p}_{k+1}\mathbf{I}_k & \mathbf{q}_{k+1} \end{bmatrix} \quad (k \geq 1), \quad \mathbf{I}_1 = \begin{bmatrix} \check{\mathbf{q}}_0 & \mathbf{q}_1 \end{bmatrix}.$$

So for the transpose we get

$$\mathbf{I}_{k+1}^{\mathrm{T}} = \begin{bmatrix} \mathbf{I}_k^{\mathrm{T}}\mathbf{p}_{k+1}^{\mathrm{T}} \\ \mathbf{q}_{k+1}^{\mathrm{T}} \end{bmatrix} \quad (k \geq 1), \quad \mathbf{I}_1^{\mathrm{T}} = \begin{bmatrix} \check{\mathbf{q}}_0^{\mathrm{T}} \\ \mathbf{q}_1^{\mathrm{T}} \end{bmatrix},$$

which induces a top-to-bottom recursive procedure to evaluate $\mathbf{I}_k^{\mathrm{T}}$ times vector, in particular to compute $\mathbf{G}_k = \mathbf{I}_k^{\mathrm{T}}\tilde{\mathbf{G}}_k$, in $\mathcal{O}(\dim Q_k)$ operations.

As a result of the computation described above, one obtains a vector $\mathbf{P}_k$ that represents the solution p_k with respect to the multi-level basis $\Phi_0 + \cup_{m=1}^k\{\operatorname{div}\mathbf{w}_e : e \in E_m\}$ of Q_k. Yet, one often prefers to have a representation, denoted by $\tilde{\mathbf{P}}_k$, of p_k with respect to the canonical basis of $\prod_{T\in\tau_k} P_0(T)$. We follow an analogous procedure as described above. We equip $\prod_{T\in\tau_k} P_0(T)$ with its canonical basis. Let $\hat{\mathbf{I}}_k$ be the representation of the embedding of Q_k into $\prod_{T\in\tau_k} P_0(T)$. Equipping $\operatorname{span}\Phi_0$ with Φ_0, and for $m \geq 1$, $\operatorname{span}\{\operatorname{div}\mathbf{w}_e : e \in E_m\}$ with $\{\operatorname{div}\mathbf{w}_e : e \in E_m\}$, let $\hat{\mathbf{q}}_0$ be the representation of the embedding $\operatorname{span}\Phi_0 \to \prod_{T\in\tau_0} P_0(T)$, and for $m \geq 1$ let the uniformly sparse matrices $\hat{\mathbf{q}}_m$ and $\hat{\mathbf{p}}_m$ be the representations of the embeddings $\operatorname{span}\{\operatorname{div}\mathbf{w}_e : e \in E_m\} \to \prod_{T\in\tau_m} P_0(T)$ and $\prod_{T\in\tau_{m-1}} P_0(T) \to \prod_{T\in\tau_m} P_0(T)$ respectively. Then the mappings $\hat{\mathbf{I}}_k$ satisfy

$$\hat{\mathbf{I}}_{k+1} = \begin{bmatrix} \hat{\mathbf{p}}_{k+1}\hat{\mathbf{I}}_k & \hat{\mathbf{q}}_{k+1} \end{bmatrix} \quad (k \geq 0), \quad \hat{\mathbf{I}}_0 = \hat{\mathbf{q}}_0,$$

which yield a bottom-to-top recursive procedure to evaluate $\hat{\mathbf{I}}_k$ times vector, in particular to compute $\tilde{\mathbf{P}}_k = \hat{\mathbf{I}}_k\mathbf{P}_k$, in $\mathcal{O}(\dim Q_k)$ operations.

Summarizing: Assuming that $\tilde{\mathbf{G}}_k$ is available, we can compute $\tilde{\mathbf{P}}_k = \hat{\mathbf{I}}_k\mathbf{B}_k^{-1}\mathbf{I}_k^{\mathrm{T}}\tilde{\mathbf{G}}_k$ taking $\mathcal{O}(\dim Q_k)$ operations.

References

[ASV88] D.N. Arnold, L.R. Scott, and M. Vogelius. Regular inversion of the divergence operator with Dirichlet boundary conditions on a polygon. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. IV*, 15(2):169–192, 1988.

[Bre90] S.C. Brenner. A nonconforming multigrid method for the stationary Stokes equations. *Math. Comp.*, 55(192):411–437, 1990.

[Bre96] S.C. Brenner. Two-level additive Schwarz preconditioners for nonconforming finite element methods. *Math. Comp.*, 65(215):897–921, 1996.

[BY93] F.A. Bornemann and H. Yserentant. A basic norm equivalence for the theory of multilevel methods. *Numer. Math.*, 64:455–476, 1993.

[Cro72] M. Crouzeix. Résolution numérique des equations de Stokes et Navier-Stokes stationnaires. Séminaire d'analyse numérique, Université de Paris VI, 1972.

[CSS86] C. Cuvelier, A. Segal, and A. van Steenhoven. *Finite Element Methods and Navier-Stokes Equations*. D. Reidel Publishing Company, Dordrecht, 1986.

[GR79] V. Girault and P.A. Raviart. An analysis of a mixed finite element method for the Navier-Stokes equations. *Numer. Math.*, 33:235–271, 1979.

[GR86] V. Girault and P.A. Raviart. *Finite element methods for Navier-Stokes equations, Theory and Algorithms*. Springer-Verlag, Berlin, 1986.

[Neč65] J. Nečas. *Equations aux dérivées partielles*. Presse de l' Université de Montréal, 1965.

[Ste99] R.P. Stevenson. Nonconforming finite elements and the cascadic multi-grid method. Technical Report 1120, University of Utrecht, November 1999. Submitted.

[Tho81] F. Thomasset. *Implementation of Finite Element Methods for Navier-Stokes Equations*. Springer-Verlag, New-York, 1981.

[Tur94] S. Turek. Multigrid techniques for a divergence-free finite element discretization. *East-West J. Numer. Math.*, 2(3):229–255, 1994.

[Urb96] K. Urban. Using divergence free wavelets for the numerical solution of the Stokes problem. In O. Axelsson and B. Polman, editors, *Algebraic Multilevel Iteration Methods with Applications*, pages 261–278, University of Nijmegen, 1996.

Department of Mathematics, Utrecht University, P.O. Box 80.010, NL-3508 TA Utrecht, The Netherlands.

E-mail address: stevenso@math.uu.nl