Abstract: The Bruhat-Tits fixed point theorem states that a group of isometries on a complete metric space with negative curvature possesses a fixed point if it has a bounded orbit. This theorem is extended by a relaxation of the negative curvature condition in terms of the $w$-distance functions introduced by Kada et al. [Non-convex minimization theorems and fixed point theorems in complete metric spaces, Math. Japon. 44 (1996), 381-391].

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1 Introduction

According to the Bruhat-Tits fixed point theorem, a group of isometries on a complete metric space with negative curvature — a complete metric space satisfying a relaxation of the parallelogram law — possesses a fixed point if it has a bounded orbit; see [2] for the original statement, [1] for an extensive overview of the theory on Bruhat-Tits buildings, and [5] for a very accessible treatment of this fixed point theorem.

The result and some of its special cases have wide applicability, witnessing their use in for instance the Bruhat-Tits theory of buildings, group theory (cf. Cartan’s fixed point theorem, [3, Ch. I, Theorem 13.5]), and the theory of trees [6, Section I.4.3, Proposition 19]. Several applications are discussed in [1, Ch. VI]. Perhaps surprisingly, it has also been used in the study of communication to establish that individuals speaking different languages that allow sufficient freedom to express nuances have a common interpretation of at least some phrases in their vocabulary; see [9].

The purpose of this note is to extend the Bruhat-Tits fixed point theorem by stating the negative curvature condition not in terms of distance functions, but using the more general notion of \(w\)-distances, as introduced and studied in a recent sequence of papers by Kada, Suzuki, and Takahashi [4], Suzuki and Takahashi [8], and Suzuki [7].

Section 2 recalls the definition of \(w\)-distances. Generalized Bruhat-Tits spaces are considered in Section 3. Finally, the fixed point theorem is provided in Section 4.

2 Preliminaries

This section settles some standard matters of notation and defines the \(w\)-distance functions introduced in Kada et al. [4]. Denote by \(\mathbb{N}\) the set of positive integers. Let \(X\) be a metric space with distance \(d\). Following [4, p. 381], we call a function \(p : X \times X \to [0, \infty)\) a \(w\)-distance on \(X\) if the following conditions hold:

- \(p\) satisfies the triangle inequality, i.e., \(\forall x, y, z \in X : p(x, z) \leq p(x, y) + p(y, z)\);
- \(p(x, \cdot) : X \to [0, \infty)\) is lower semicontinuous for every \(x \in X\), i.e., if a sequence \((y_m)_{m \in \mathbb{N}}\) in \(X\) converges to \(y \in X\), then \(p(x, y) \leq \lim \inf_{m \to \infty} p(x, y_m)\);
- for every \(\varepsilon > 0\) there exists a \(\delta > 0\) such that for each \(x, y, z \in X\): if \(p(z, x) \leq \delta\) and \(p(z, y) \leq \delta\), then \(d(x, y) \leq \varepsilon\).
The metric $d$ is a $w$-distance. Examples of many other $w$-distances are found in [4] and [8, Lemma 1]. Kada et al. [4, Lemma 1] prove:

**Lemma 2.1** Let $(X, d)$ be a metric space and let $p$ be a $w$-distance on $X$. Consider points $x, y, z \in X$, a sequence $(x_n)_{n \in \mathbb{N}}$ in $X$, and a sequence $(\alpha_n)_{n \in \mathbb{N}}$ in $[0, \infty)$ converging to zero. The following claims hold:

(a) If $p(x_n, x_m) \leq \alpha_n$ for all $m, n \in \mathbb{N}$ with $m > n$, then $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(X, d)$.

(b) If $p(x, y) = p(x, z) = 0$, then $y = z$.

Balls are defined with respect to the $w$-distance $p$. The ball around $x \in X$ with radius $r \geq 0$ is the set

$$B(x, r) := \{y \in X \mid p(y, x) \leq r\}.$$ 

A set $S \subseteq X$ is $p$-bounded if there exists a ball $B(x, r)$ such that $S \subseteq B(x, r)$.

The composition of two functions $g, h : X \rightarrow X$ is denoted by $gh : x \mapsto g(h(x))$. For $S \subseteq X$, write $g(S) = \{g(s) \mid s \in S\}$.

## 3 Generalized Bruhat-Tits spaces

A **generalized Bruhat-Tits space** $(X, d, p)$ is a complete metric space $(X, d)$ with a $w$-distance $p$ satisfying the following property: for any two points $x_1, x_2 \in X$ there is a point $z \in X$ such that for all $y \in X$,

$$p(x_1, x_2)^2 + 4p(y, z)^2 \leq 2p(y, x_1)^2 + 2p(y, x_2)^2. \tag{1}$$

This is a generalization in terms of the $w$-distance $p$ of the well-known negative curvature inequality, sometimes referred to as the semi-parallelogram law [cf. 5]. For a brief motivation, we resort to planar geometry. Consider Figure 1. The parallelogram law states that, using the Euclidean distance $d_2$, the sum of the squared lengths of the diagonals equals the sum of the squared lengths of the sides of the parallelogram:

$$d_2(x_1, x_2)^2 + d_2(y, x_3)^2 = 2d_2(y, x_1)^2 + 2d_2(y, x_2)^2.$$ 

Let $z$ be the midpoint between $x_1$ and $x_2$. Since $2d(y, z) = d(y, x_3)$, substitution in the parallelogram law yields:

$$d_2(x_1, x_2)^2 + 4d_2(y, z)^2 = 2d_2(y, x_1)^2 + 2d_2(y, x_2)^2.$$
Generalizing this to an arbitrary metric space \((X, d)\) and allowing for a weak inequality, rather than equality, \((X, d)\) is said to satisfy the semi-parallelogram law if for any two points \(x_1, x_2 \in X\) there exists a point \(z \in X\) such that for all \(y \in X\),

\[
d(x_1, x_2)^2 + 4d(y, z)^2 \leq 2d(y, x_1)^2 + 2d(y, x_2)^2.
\]

Replacing the distance function \(d\) with the more general notion of a \(w\)-distance \(p\), one finds condition (1).

**Theorem 3.1** Let \((X, d, p)\) be a generalized Bruhat-Tits space and let \(S \subseteq X\) be \(p\)-bounded. There exists a unique ball of minimum radius containing \(S\).

**Proof.**

**Existence:** Since \(S\) is \(p\)-bounded, the set \(B = \{B(x, r) \mid S \subseteq B(x, r)\}\) of balls containing \(S\) is nonempty. Set \(r = \inf_{B(x, r) \in B} r'\) and let \(B(x_n, r_n)\) be a sequence in \(B\) such that \((r_n)_{n \in \mathbb{N}}\) is a non-increasing sequence converging to \(r\). We proceed to show that \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence in \((X, d)\). Let \(m, n \in \mathbb{N}\) with \(m > n\). Then (1) implies the existence of a point \(z_{mn} \in X\) such that

\[
\forall y \in X : \quad p(x_n, x_m)^2 + 4p(y, z_{mn})^2 \leq 2p(y, x_n)^2 + 2p(y, x_m)^2. \tag{2}
\]

By definition of \(r\), there exists a point \(y \in S\) with \(p(y, z_{mn}) \geq r^2 - \frac{1}{n}\). Substituting this in (2) and using that \(y \in B(x_k, r_k)\) for all \(k \in \mathbb{N}\) and \((r_k)_{k \in \mathbb{N}}\) is a non-increasing sequence, yields:

\[
p(x_n, x_m)^2 \leq 2p(y, x_n)^2 + 2p(y, x_m)^2 - 4p(y, z_{mn})^2 \\
\leq 2r_n^2 + 2r_m^2 - 4(r^2 - \frac{1}{n}) \\
\leq 2r_n^2 + 2r_m^2 - 4(r^2 - \frac{1}{n}) \\
= 4(r_n^2 - r^2) + \frac{4}{n}.
\]
Lemma 2.1(a) with \( a_n = \sqrt{4(r_n^2 - r^2) + \frac{4}{n}} \) implies that \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence in \((X, d)\).

Since \((X, d)\) is a complete metric space, the Cauchy sequence \((x_n)_{n \in \mathbb{N}}\) has a limit \(x \in X\). We proceed to show that the ball \(B(x, r)\) is a ball of minimum radius containing \(S\). The definition of \(r\) implies that the radius is indeed minimal. Let \(y \in S\). The triangle inequality and the lower semicontinuity property of \(d\) imply that for each \(n \in \mathbb{N}\):

\[
p(y, x) \leq p(y, x_n) + p(x_n, x) \\
\leq r_n + \liminf_{m \to \infty} p(x_n, x_m) \\
\leq r_n + a_n,
\]

where the last inequality follows from the fact that \(p(x_n, x_m) \leq a_n\) for each \(m > n\), as shown earlier in the proof. Taking limits as \(n \to \infty\) yields that \(p(y, x) \leq r\), i.e., \(y \in B(x, r)\). Hence \(S \subseteq B(x, r)\).

**Uniqueness:** Suppose there are two balls of minimum radius containing \(S\): \(B(x_1, r)\) and \(B(x_2, r)\). Then (1) implies the existence of a point \(z \in X\) such that for each \(y \in X\):

\[
p(x_1, x_2)^2 + 4p(y, z)^2 \leq 2p(y, x_1)^2 + 2p(y, x_2)^2. \tag{3}
\]

Since \(r\) is the minimum radius of a ball containing \(S\), it follows that for each \(\varepsilon > 0\) there exists a \(y \in S\) such that \(p(y, z) \geq r - \varepsilon\). Substituting this in (3) and using the fact that \(y \in S \subseteq B(x_i, r)\) for \(i = 1, 2\), yields that \(\forall \varepsilon \in (0, r)\):

\[
p(x_1, x_2)^2 \leq 2p(y, x_1)^2 + 2p(y, x_2)^2 - 4p(y, z)^2 \\
\leq 2r^2 + 2r^2 - 4(r - \varepsilon)^2 \\
= 4r^2 - 4(r - \varepsilon)^2.
\]

Letting \(\varepsilon \to 0\), this implies that \(p(x_1, x_2)^2 \leq 0\). Consequently, \(p(x_1, x_2) = 0\). Similarly, \(p(x_2, x_1) = 0\). The triangle inequality implies that \(0 \leq p(x_1, x_1) \leq p(x_1, x_2) + p(x_2, x_1) = 0\), so \(p(x_1, x_1) = 0\). Lemma 2.1(b) and \(p(x_1, x_1) = p(x_1, x_2) = 0\) imply that \(x_1 = x_2\): the two balls are identical. \(\square\)

4 Fixed point theorem

Let \((X, d, p)\) be a generalized Bruhat-Tits space. A \textit{p-isometry} of \(X\) is a bijection \(g : X \to X\) such that \(g\) preserves \(p\)-distances:

\[
\forall x_1, x_2 \in X : \quad p(g(x_1), g(x_2)) = p(x_1, x_2).
\]
Note that if \( g \) and \( h \) are \( p \)-isometries, so is their composition \( gh \) and the inverse \( g^{-1} \). Moreover, the identity function \( \text{id} : x \mapsto x \) is a \( p \)-isometry. The straightforward proofs of these claims are left to the reader. A group \( G \) of \( p \)-isometries is a set of \( p \)-isometries such that

(a) the identity function \( \text{id} : x \mapsto x \) is an element of \( G \);

(b) \( G \) is closed under inversion: if \( g \in G \), then \( g^{-1} \in G \);

(c) \( G \) is closed under composition: if \( g_1, g_2 \in G \), then \( g_1g_2 \in G \).

Let \( G \) be a group of isometries and \( x \in X \). The orbit \( O(x) \) of \( x \) is defined to be the collection of images \( g(x) \) with \( g \in G \):

\[
O(x) := \{ g(x) \mid g \in G \}.
\]

**Lemma 4.1** Let \( x \in X, g \in G \). Then \( g(O(x)) = O(x) \).

**Proof.**

\( \subseteq \): Let \( y \in g(O(x)) \), i.e., there is a \( h \in G \) such that \( y = gh(x) \). Since \( G \) is closed under composition: \( gh \in G \), so \( y \in O(x) \).

\( \supseteq \): Let \( y \in O(x) \), i.e., there is a \( h \in G \) such that \( y = h(x) \). Since \( g \in G \), it follows that \( g^{-1} \in G \) and \( g^{-1}h \in G \). Consequently, \( y = h(x) = (gg^{-1})h(x) = g(g^{-1}h(x)) \in g(O(x)) \). \( \Box \)

Let \( G \) be a group of \( p \)-isometries in a generalized Bruhat-Tits space \((X, d, p)\). A fixed point of \( G \) is a point \( x \in X \) such that \( g(x) = x \) for all \( g \in G \), or equivalently, if \( x \in X \) with \( O(x) = \{x \} \).

If there is a point with a \( p \)-bounded orbit, then \( G \) has a fixed point.

**Theorem 4.2** Let \((X, d, p)\) be a generalized Bruhat-Tits space, \( G \) a group of \( p \)-isometries, and \( y \in X \) such that \( O(y) \) is \( p \)-bounded. Then \( G \) has a fixed point.

**Proof.** By Theorem 3.1 there is a unique ball \( B(x, r) \) of minimum radius \( r \) containing \( O(y) \).

The point \( x \in X \) is shown to be a fixed point of \( G \). Let \( g \in G \). For each \( z \in B(x, r) \) it holds that \( p(g(z), g(x)) = p(z, x) \leq r \), so \( g(z) \in B(g(x), r) \). Hence \( g(B(x, r)) \subseteq B(g(x), r) \). Conversely, for every \( z \in B(g(x), r) \), it holds that \( p(g^{-1}(z), x) = p(z, g(x)) \leq r \), so \( g^{-1}(z) \in B(x, r) \), so \( z \in B(g(x), r) \). Consequently, \( B(g(x), r) \subseteq g(B(x, r)) \). Conclude that \( g(B(x, r)) = B(g(x), r) \).

But then Lemma 4.1 and the inclusion \( O(y) \subseteq B(x, r) \) imply that \( O(y) = g(O(y)) \subseteq g(B(x, r)) = B(g(x), r) \), i.e., \( B(g(x), r) \) is also a ball of minimum radius containing \( O(y) \). By uniqueness (Theorem 3.1), it follows that \( g(x) = x \). \( \Box \)
The original Bruhat-Tits fixed point theorem is recovered by restricting attention to the space $(X, d, d)$, i.e., by simply considering the $\omega$-distance $p = d$.

References


