Automatica, Vol. 20, No. 1, pp. 79–91, 1984
Printed in Great Britain

0005-1098/84 $3.00 + 0.00
Pergamon Press Ltd.

Singularly Perturbed Systems of Diffusion Type and Feedback Control*

A. VAN HARTEN†

Asymptotic analysis yields new insight about the behaviour and stability of controlled diffusion processes, and it is useful for the determination of optimal feedback loops.

Key Words—Singular perturbations; feedback control; partial differential equations; stability criteria; optimization.

Abstract—Asymptotic approximations describing the behaviour of linear systems of diffusion type (convective or non-convective) with a small diffusivity, to which a feedback control of distributed or boundary type based on point sensors is applied, are constructed and proven to be correct. As a consequence one can find a near-optimal feedback control for a cost minimization problem with a quadratic performance index measuring the deviation of the stationary state from an ideal state, under the restriction of a prescribed exponential degree of stability of the stationary state.

1. INTRODUCTION

IN THIS paper linear systems of diffusion-type subject to a certain feedback control mechanism in a situation, where the diffusion constant is a small parameter will be considered. Such controlled diffusion systems can be found for example in the context of heating problems (Curtain and Pritchard, 1978; van Harten, 1979a) or chemical or nuclear reactor design (Owens, 1980). For the feedback control there are many possibilities: feedback without or with memory, with distributed input or input through the boundary, etc., while it also depends on the number and kind of observations (Curtain and Pritchard, 1978; Schumacher, 1981; Triggiani, 1979, 1980). Here we shall consider distributed as well as boundary control, but always on the basis of an instantaneous feedback coupling using observations of the state in a finite number of points $y_1, \ldots, y_p$ in the interior of the domain D. In the case of Dirichlet boundary conditions the evolution of the state is described by one of the following problems:

$$\frac{\partial u}{\partial t} = L_\varepsilon u + \tilde{\Pi}_d u + h, \quad u = s \text{ on } \partial D, \; u(\cdot, 0) = \psi. \tag{1$_d$}$$

$$\frac{\partial u}{\partial t} = L_\varepsilon u + h, \quad u = \tilde{\Pi}_b u + s \text{ on } \partial D, \quad u(\cdot, 0) = \psi. \tag{1$_b$}$$

Here L_ε, $\tilde{\Pi}_d$, $\tilde{\Pi}_b$ are of the following form

$$L_\varepsilon = \varepsilon L_2 - L_r \tag{2}$$

with

$$L_2 = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j}, \quad L_r = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i} + \gamma$$

$$\tilde{\Pi}_d u = c_0 + \sum_{i=1}^{p} c_i(\delta_{y_i} u - I_i), \quad c_0, c_i \in C^\infty(\bar{D}) \tag{3$_d$}$$

$$\tilde{\Pi}_b u = b_0 + \sum_{i=1}^{p} b_i(\delta_{y_i} u - I_i), \quad b_0, b_i \in C^\infty(\partial D). \tag{3$_b$}$$

We suppose, that $\bar{D} \subset \mathbb{R}^n$ is a compact set with a smooth boundary ∂D. The coefficients a_{ij}, v_i, γ are also supposed to be smooth. Further we assume, that L_2 is uniformly elliptic. With δ_{y_i} we denote the continuous linear functional on $C(\bar{D})$, which maps $u \to u(y_i)$. Note, that the feedback control consists of a part independent of the observations $\delta_{y_i} u$ and a part proportional to the difference between the $\delta_{y_i} u$s and certain ideal values L_i. We shall always assume that the observation points y_i have an $O(1)$ distance to the boundary ∂D.

Because of the small parameter ε in front of the highest order derivatives, the problems $(1)_{b,d}$ have a singular perturbation character. The stationary, uncontrolled problem corresponding to $(1)_{b,d}$ has been thoroughly analysed (Eckhaus, 1979; Fife, 1974; van Harten, 1975, 1978; de Groen, 1976). It was understood that for the behaviour of the solution for $\varepsilon \downarrow 0$ it makes a big difference whether there is convection: $v \not\equiv 0$, or not: $v \equiv 0$. If there is convection the structure of the velocity field plays an

* Received 28 October 1982; revised 6 June 1983. The original version of this paper was presented at the IFAC/CISM Workshop on Singular Perturbation in Systems and Control, which was held in Udine, Italy, during July 1982. The published proceedings of this IFAC meeting may be ordered from Pergamon Press Limited, Headington Hill Hall, Oxford OX3 0BW, U.K. This paper was recommended for publication in revised form by associate editor V. Vitkin under the direction of editor H. Kwakernaak.

† Mathematical Institute, State University of Utrecht, 3508 TA Utrecht, The Netherlands.

important role especially the presence of turning points, cycles or tangency points at the boundary. If $v \equiv 0$ the sign of the coefficient γ is very relevant. Here we shall consider the following two cases:

$$r = 0: L_0 u = \gamma u, \quad \gamma > 0 \tag{4a}$$

and a domain as in Fig. 1

$$r = 1: L_1 u = v.\nabla u + \gamma u, \quad \|v\| > 0 \tag{4b}$$

with a domain and velocity field as in Fig. 1. In the case $r = 1$ we have

$$v.n < 0 \text{ on } \partial D_e; \quad v.n > 0 \text{ on } \partial D_0 \tag{6}$$

n = the outer normal on ∂D. Further, let $z(t; x)$ parametrize the characteristic through x, i.e.

$$\frac{dz}{dt} = v(z); \quad z(0; x) = x. \tag{7}$$

Then, we assume that $\forall x \in \bar{D} \exists t_e(x) \leqslant 0$ such that

$$z(t_e(x); x) \triangleq x_e \in \partial D_e. \tag{8}$$

Note, that the conditions on the velocity field in (4b)–(8) are such that turning points or cycles are excluded and at each point of ∂D the field is transverse. Further, using the theory of ordinary differential equations it is clear, that $t_e(x)$ and x_e are uniquely defined, smooth functions of x.

As for the behaviour for $\varepsilon \downarrow 0$ of the solution of the dynamic, uncontrolled singular perturbation problem corresponding to $(1)_{b,d}$, there are only a few references (Lions, 1973; Besjes, 1974; van Harten, 1979a). The asymptotic theory for solutions of such parabolic problems is somewhat less developed than for elliptic problems. In this respect Section 3 contains some new contributions for the case of a first order unperturbed operator L_1 as in (4b).

When now, for a moment, we forget the point of view of asymptotics and take $\varepsilon = \varepsilon_0$ = fixed, there are a large number of results from infinite dimensional control theory, which are applicable. They concern, for example, the following subjects: well posedness of the controlled problem, generation of semi-groups by the controlled diffusion operator and stabilizability of the system (Curtain and Pritchard, 1978; Schumacher, 1981; van Harten and Schumacher, 1980; van Harten, 1979b; Triggiani, 1975, 1980; Balas, 1979). Some of these results will be useful in the sequel and sometimes it will be nice to compare our results found by asymptotic calculations with predictions valid for the general case, see Section 4 for the topic of stabilizability.

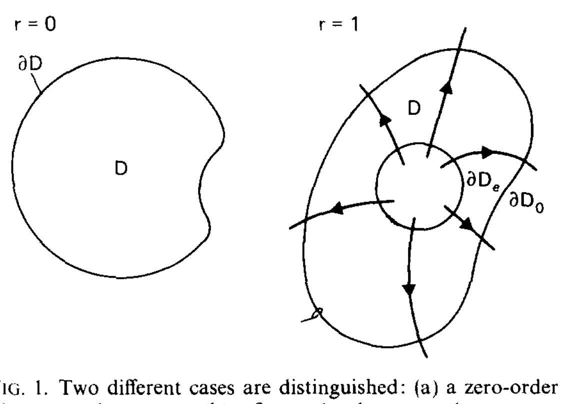

FIG. 1. Two different cases are distinguished: (a) a zero-order degenerated operator; (b) a first-order degenerated operator.

Our purpose is to use singular perturbation techniques to analyse the behaviour of the solution of the controlled problem $(1)_{d,b}$ asymptotically for $\varepsilon \downarrow 0$. In Section 2 this is done for the corresponding stationary problem and in Section 3 for the dynamic problem. As a result we obtain explicit formal asymptotic approximations as $\varepsilon \downarrow 0$ for the effect of the control and also for the spectrum of the controlled operator. In Section 4 it is sketched, why the results of the previous sections are rigorous. In Section 5 we use the results found before to construct a near optimal control with respect to a certain cost-functional. In that optimization problem the number of sensors is one and all parameters: c_0, c (the input functions), the observation point y_1 and the reference value l, are varied in the optimization. Finally, we remark that asymptotic methods are used more often in control theory, but for problems where the small parameter ε enters in a different way. For example, the asymptotics as considered here is from a completely different type than in Balas (1982) where the small parameter is in front of the $\partial/\partial t$ term. Other examples of different asymptotics can be found in Lions (1973).

2. THE STATIONARY, CONTROLLED PROBLEM

Without loss of generality we can restrict ourselves to the following problems with homogeneous boundary conditions

$$(L_\varepsilon + \Pi_d - \lambda)u^\varepsilon + f = 0, \quad u^\varepsilon = 0 \text{ on } \partial D \tag{9}_d$$

$$(L_\varepsilon - \lambda)u^\varepsilon + f = 0, \quad u^\varepsilon = \Pi_b u^\varepsilon \text{ on } \partial D \tag{9}_b$$

with

$$\Pi_d = \sum_{i=1}^{p} c_i \delta_{y_i}, \quad \Pi_b = \sum_{i=1}^{p} b_i \delta_{y_i}.$$

Note, that we have introduced a spectral parameter in λ in $(9)_{d,b}$. The trick to solve these problems is well known from Weinstein–Aronszajn's theory. If $\lambda \notin \sigma(L_\varepsilon)$ we can rewrite $(9)_{d,b}$ as follows:

$$u^\varepsilon = -F^\varepsilon - \sum_{i=1}^{p} \xi_i^\varepsilon C_i^\varepsilon \tag{10}_d$$

$$u^\varepsilon = -F^\varepsilon + \sum_{i=1}^{p} \xi_i^\varepsilon B_i^\varepsilon \tag{10}_b$$

with $\xi_i^\varepsilon = \delta_{y_i} u^\varepsilon$, $F^\varepsilon = (L_\varepsilon - \lambda)^{-1} f$, $C_i^\varepsilon = (L_\varepsilon - \lambda)^{-1} c_i^\varepsilon$ and B_i is the solution of $(L_\varepsilon - \lambda)B_i^\varepsilon = 0$, $B_i^\varepsilon = b_i$ on

∂D. Substitution of $x = y_k$ in (10) provides us with a linear system of equations for $\xi^\varepsilon \in \mathbb{C}^p$

$$[I + \Omega_d^\varepsilon]\xi^\varepsilon = -\eta^\varepsilon \qquad (11)_d$$

$$[I + \Omega_b^\varepsilon]\xi^\varepsilon = -\eta^\varepsilon \qquad (11)_b$$

with $\eta_k^\varepsilon = \delta_{y_k}F^\varepsilon$, $[\Omega_d^\varepsilon]_{k,i} = \delta_{y_k}C_i^\varepsilon$, $[\Omega_b^\varepsilon]_{k,i} = \delta_{y_k}B_i^\varepsilon$. If $I + \Omega_d^\varepsilon$, $I - \Omega_b^\varepsilon$ are invertible, we only have to put the solution ξ^ε of $(11)_{d,b}$ into $(10)_{d,b}$ and the solution u^ε of $(9)_{d,b}$ is known. Note, that then the effect of the control is given by the following expressions

$$\text{e.d.c.} = \langle (I + \Omega_d^\varepsilon)^{-1}\eta^\varepsilon, C^\varepsilon \rangle \qquad (12)_d$$

$$\text{e.b.c.} = -\langle (I - \Omega_b^\varepsilon)^{-1}\eta^\varepsilon, B^\varepsilon \rangle \qquad (12)_b$$

with for $e, e' \in \mathbb{C}^p$

$$\langle e, e' \rangle = \sum_{i=1}^{p} e_i e_i'$$

and C^ε, B^ε the vectors with components C_i^ε, B_i^ε.

Of course the points, where $I + \Omega_d^\varepsilon$, $I - \Omega_b^\varepsilon$ are singular, belong to the spectrum of the controlled operator.

It is now clear, that in order to construct approximations of the solution of the controlled stationary problem for $\varepsilon \downarrow 0$ it is sufficient to have asymptotic approximations for the functions F^ε, C_i^ε and B_i^ε. These functions are determined by uncontrolled problems of the following type

$$(L_\varepsilon - \lambda)C^\varepsilon = c, \quad C^\varepsilon = 0 \text{ on } \partial D \qquad (13)_d$$

$$(L_\varepsilon - \lambda)B^\varepsilon = 0, \quad B^\varepsilon = b \text{ on } \partial D. \qquad (13)_b$$

Using the method of matched asymptotic expansions approximations C, B of C^ε and B^ε are easily found (Eckhaus, 1979; Fife, 1974; van Harten, 1975, 1978). Thus we are lead to approximations $\Omega_{d,b}$ of $\Omega_{d,b}^\varepsilon$, η of η^ε and ξ of ξ^ε, where ξ satisfies the approximate version of $(11)_{d,b}$, i.e. $(1 + \Omega_d)\xi = -\eta$, $(1 - \Omega_b)\xi = -\eta$ and if $1 + \Omega_d$, $1 - \Omega_b$ are invertible we end up with an approximation u of u^ε.

In the case of a zero-order unperturbed operator as in (9) the approximations consist of a regular expansion in the interior of D and a boundary layer of width $\sqrt{\varepsilon}$ along all of ∂D

$$C^\varepsilon = \hat{C}_0(x) + \hat{G}_0(\zeta, \phi)H(x) + O(\sqrt{\varepsilon}) \quad (14)_d$$

$$B^\varepsilon = \bar{G}_0(\zeta, \phi)H(x) + O(\sqrt{\varepsilon}) \qquad (14)_b$$

where ζ is the distance to $\partial D/\sqrt{\varepsilon}$, ϕ, a (local) parametrization of ∂D and $H(x)$ a suitably chosen C^∞ cutoff function. Note, that in the case of $(13)_b$ the regular expansion is $\equiv 0$ and the approximation is completely of layer type. The functions $\hat{C}_0$, $\hat{G}_0$ and $\bar{G}_0$ are found as the solutions of the following problems

$$-(\gamma + \lambda)\hat{C}_0 = c \qquad (15)$$

$$\left(a\frac{d^2}{d\zeta^2} - \bar{\gamma} - \lambda\right)G_0 = 0,$$

$$\hat{G}_0|_{\zeta=0} = -\hat{C}_{0|\partial D}, \quad \lim_{\zeta\to\infty} \hat{G}_0 = 0 \qquad (16)_d$$

$$\left(a\frac{d^2}{d\zeta^2} - \bar{\gamma} - \lambda\right)\bar{G}_0 = 0,$$

$$\bar{G}_{0|\zeta=0} = b, \quad \lim_{\zeta\to\infty} \bar{G}_0 = 0 \qquad (16)_b$$

with $a = \sum n_i(a_{ij})_{|\partial D} n_j > 0$, $\bar{\gamma} = \gamma_{|\partial D} > 0$. Hence

$$\hat{C}_0 = -c/(\gamma + \lambda), \hat{G}_0 = -\hat{C}_{0|\partial D}\exp(-\mu\zeta) \quad (17a)$$

$$\bar{G}_0 = b\exp(-\mu\zeta) \qquad (17b)$$

with $\mu = \sqrt{[(\bar{\gamma} + \lambda)/a]}$. In order to be able to divide by $\gamma + \lambda$ and to have exponential decay of the boundary layer terms we must have

$$\lambda \notin (-\infty, -\hat{\gamma}] \text{ with } \hat{\gamma} = \min_{x\in D} \gamma(x) > 0. \qquad (18)$$

Now using the approximations as found in (14)–(17) we find

$$\Omega_d^\varepsilon = \Omega_d + O(\sqrt{\varepsilon}) \qquad (19)_d$$

$$\Omega_b^\varepsilon = \Omega_b + O(\sqrt{\varepsilon}) \qquad (19)_b$$

with

$$[\Omega_d]_{k,i} = -\frac{c_i(y_k)}{\lambda + \gamma(y_k)}, \quad \Omega_b = 0.$$

The conclusion is, that in the case of boundary control (i) the effect of the control is only noticeable in a layer of width $\sqrt{\varepsilon}$ along ∂D and (ii) the spectrum of the controlled operator is contained in a set, which shrinks with $\varepsilon \downarrow 0$ to $(-\infty, -\hat{\gamma}]$. For distributed control the spectrum is contained in a set, which shrinks with $\varepsilon \downarrow 0$ to $(-\infty\ -\hat{\gamma}] \cup \{\lambda_1, \ldots, \lambda_q\}$ with $q \leqslant p$ and $\lambda_1, \ldots, \lambda_q$ the eigenvalues of the matrix Λ, where

$$\Lambda_{k,i} = -\gamma(y_k) + c_i(y_k). \qquad (20)_d$$

More precise information on the set, which contains the spectrum is given in Section 4.

Let us now consider *the case of a first order unperturbed operator* as in (4b). Then the approximations of the solutions of $(13)_{d,b}$ have a different structure. They consist of a regular expansion valid up to ∂D_e and a boundary layer of width ε, along ∂D_0.

In order to describe these approximations it is simpler to introduce the following coordinates

$$s = -t_e(x); \quad \phi = x_e. \qquad (21)$$

Note, that in these coordinates $\partial D_e = \{s = 0\}$ and $v.\nabla = \partial/\partial s$. The other part of the boundary, ∂D_0, can be given as $\{(s, \phi) | s = T(\phi)\}$, where the interpretation of $T(\phi)$ is the time it takes to travel along the characteristic through $(0, \phi)$ from $(0, \phi)$ to ∂D_e. Now our approximation will have the following form

$$C^\varepsilon = \hat{C}_0(s, \phi) + \hat{G}_0^0(\zeta, \phi)H(T(\phi) - s) + O(\varepsilon) \quad (22)_d$$

$$B^\varepsilon = \bar{B}_0(s, \phi) + \bar{G}_0^0(\zeta, \phi)H(T(\phi) - s) + O(\varepsilon) \quad (22)_b$$

with $\zeta = (T(\phi) - s)/\varepsilon$ and H a suitably chosen C^∞ cutoff function. The functions $\hat{C}_0$, $\hat{G}_0$, $\bar{B}_0^0$, $\bar{G}_0^0$ are found as the solutions of

$$\left(-\frac{\partial}{\partial s} - \gamma - \lambda\right)\hat{C}_0 = c, \quad \hat{C}_0(0,\phi) = 0 \quad (23)_d$$

$$\left(-\frac{\partial}{\partial s} - \gamma - \lambda\right)\bar{B}_0 = 0, \quad \bar{B}_0(0,\phi) = b_e(\phi) \quad (23)_b$$

$$\left(A^0\frac{\partial^2}{\partial\zeta^2} + \frac{\partial}{\partial\zeta}\right)\hat{G}_0^0 = 0,$$

$$\hat{G}_0(0,\phi) = -\hat{C}_0(T(\phi),\phi), \lim_{\zeta\to\infty} \hat{G}_0(\zeta,\phi) = 0 \quad (24)_d$$

$$\left(A^0\frac{\partial^2}{\partial\zeta^2} + \frac{\partial}{\partial\zeta}\right)\bar{G}_0^0 = 0,$$

$$\bar{G}_0(0,\phi) = b_0(\phi) - \bar{B}_0(T(\phi),\phi), \lim_{\zeta\to\infty} \bar{G}_0(\zeta,\phi) = 0 \quad (24)_b$$

with

$$A^0 = \sum \left(\frac{\partial t_e}{\partial x_i}a_{ij}\frac{\partial t_e}{\partial x_j}\right)\Bigg|_{\partial D_0}$$

b_e and b_0 are the values of b on ∂D_e and ∂D_0, respectively. It is easy to check, that $\hat{C}_0$, $\hat{G}_0^0$, $\bar{B}_0$ and $\bar{G}_0^0$ are given by

$$\hat{C}_0(s,\phi) = -\int_0^s c(\bar{s},\phi)\exp\left[-\int_{\bar{s}}^s \gamma(s',\phi)\,ds' - \lambda(s-\bar{s})\right]d\bar{s} \quad (25)$$

$$\hat{G}_0^0(\zeta,\phi) = -\hat{C}_0(T(\phi),\phi)\exp(-\zeta/A^0)$$

$$\bar{B}_0(s,\phi) = b_e(\phi)\exp\left[-\int_0^s \gamma(s',\phi)\,ds' - \lambda s\right] \quad (25)_b$$

$$\bar{G}_0^0(\zeta,\phi) = [b_0(\phi) - \bar{B}_0(T(\phi),\phi)]\exp(-\zeta/A^0).$$

Now, using the approximations as found in (22)–(25) we obtain

$$\Omega_d^\varepsilon = \Omega_d(\lambda) + O(\varepsilon) \quad (26)_d$$

with

$$[\Omega_d(\lambda)]_{k,i} = -\int_0^{s_k} c_i(\bar{s},\phi_k)\exp\left[-\int_{\bar{s}}^{s_k} [\gamma(s',\phi_k)\,ds' - \lambda(s_k-\bar{s})]\,d\bar{s}\right]$$

where (s_k, ϕ_k) denotes the point y_k in (s,ϕ) coordinates, i.e. $s_k = -t_e(y_k)$; $\phi_k = (y_k)_e$.

$$\Omega_b = \Omega_b(\lambda) + O(\varepsilon) \quad (26)_b$$

with

$$[\Omega_b(\lambda)]_{k,i} = b_{i,e}(\phi_k)\exp\left[-\int_0^{s_k} \gamma(s',\phi_k)\,ds' - \lambda s_k\right].$$

Hence, the spectrum of the controlled operator is contained in a set which shrinks to the zeros of a holomorphic function $\omega_d(\lambda)$, $\omega_b(\lambda)$ in the respective cases of distributed control and boundary control with

$$\omega_d(\lambda) = \det\{I + \Omega_d(\lambda)] \quad (27)_d$$

$$\omega_b(\lambda) = \det[I - \Omega_b(\lambda)]. \quad (27)_b$$

Of course, it would be interesting to have a rough idea about the location of the zeros of these holomorphic functions. Using integration by parts it is not difficult to show that in the case of distributed control $\forall A \in \mathbb{R}\, \exists B > 0$ such that for all λ with $\mathrm{Re}\,\lambda \geqslant A: [\Omega_d(\lambda)]_{i,j} \leqslant B/(1+|\lambda|)$. Then an application of Gershgorin's theorem (Wilkinson, 1965) shows that the zeros $\{\lambda_k; k \in \mathbb{N}\}$ of $\omega_d(\lambda)$ can be numbered in such a way that $\mathrm{Re}\,\lambda_k \downarrow -\infty$ for $k \uparrow \infty$. However, in the case of boundary control the situation is quite different. It is easy to verify, that

$$\begin{aligned}\omega_b(\lambda) &= \exp(-(\lambda-\lambda_0)\,\mathrm{tr}(S))\\ &\quad \det[e^{(\lambda-\lambda_0)S} - \Omega_b(\lambda_0)]\\ &= \exp(-(\lambda-\lambda_0)\,\mathrm{tr}(S))\\ &\quad \det[e^{(\lambda-\lambda_0)S}\Omega_b^{-1}(\lambda_0) - I]\\ &\quad \det[\Omega_b(\lambda_0)].\end{aligned}$$

Here S denotes the diagonal matrix with $S_{k,k} = s_k$ and the only requirement for λ_0 is, that $\Omega_b(\lambda_0)$ is non-singular. Using again Gershgorin's theorem we see, that the zero's of $\omega_b(\lambda)$ lie in a strip $\{\lambda | \alpha < \mathrm{Re}\,\lambda < \beta\}$.

More detailed information on the location of the spectrum of the controlled operator can again be found in Section 4. As for the effect of the control notice that in the case of boundary control, the control input on ∂D_0 is only noticeable in the boundary layer of width ε along ∂D_0.

On the basis of the results derived for the asymptotic location of the spectrum of the controlled operator, it is expected that an approximation $\bar{u}$ of the solution of the dynamic problem [see $(29)_{d,b}$], will grow not faster than

$$|\bar{u}(x,t)| \leqslant C(\nu,\varepsilon)e^{\nu t}. \quad (28)$$

In this estimate we can presumably take $\nu \in \mathbb{R}$ and $\nu > \nu_d^r$ in the case of distributed control with $\nu_d^0 = \max(-\hat{\gamma}, \mathrm{Re}\,\lambda_1, \ldots, \mathrm{Re}\,\lambda_q)$, $\nu_d^1 = \max\{\mathrm{Re}\,\lambda | \omega_d(\lambda) = 0\}$ and in the case of boundary control $\nu \in \mathbb{R}$, $\nu > \nu_b^r$ with $\nu_b^0 = -\hat{\gamma}$, $\nu_b^1 = \sup\{\mathrm{Re}\,\lambda | \omega_b(\lambda) = 0\}$. In the next section we shall see, that an estimate as in (28) indeed holds and in addition we shall find how the constant $C(\nu,\varepsilon)$ in (28) depends on ε.

3. THE DYNAMIC, CONTROLLED PROBLEM

Here we shall consider the time evolution of the state, when the equation and the boundary conditions are homogeneous:

$$\frac{\partial \bar{u}^\varepsilon}{\partial t} = (L_\varepsilon + \Pi_d)\bar{u}^\varepsilon$$

$$\bar{u}^\varepsilon = 0 \text{ on } \partial D$$

$$\bar{u}^\varepsilon(\cdot, 0) = \psi \in C^\infty(\bar{D})$$

$$\psi = 0 \text{ on } \partial D \qquad (29)_d$$

$$\frac{\partial \bar{u}^\varepsilon}{\partial t} = L_\varepsilon \bar{u}^\varepsilon$$

$$\bar{u}^\varepsilon = \Pi_b \bar{u}^\varepsilon \text{ on } \partial D$$

$$\bar{u}^\varepsilon(\cdot, 0) = \psi \in C^\infty(\bar{D})$$

$$\psi = \Pi_b \psi \text{ on } \partial D. \qquad (29)_b$$

In order to solve these problems we denote the observations $\delta_{y_k}\bar{u}^\varepsilon(\cdot, t)$ by $\xi_k^\varepsilon(t)$. The solutions of $(29)_{d,b}$ can then be expressed in the following way:

$$\bar{u}^\varepsilon(\cdot, t) = e^{L_\varepsilon t}\psi + \sum_{i=1}^{p} \int_0^t \xi_i^\varepsilon(t-\tau) e^{L_\varepsilon \tau} c_i \, d\tau \qquad (30)_d$$

$$\bar{u}^\varepsilon(\cdot, t) = e^{L_\varepsilon t}\psi + \frac{\partial}{\partial t}\left\{\sum_{i=1}^{p} \int_0^t \xi_i(t-\tau) \times (1 - e^{L_\varepsilon \tau}) B_i^\varepsilon \, d\tau\right\}. \qquad (30)_b$$

Here B_i^ε denotes the solution of the uncontrolled, stationary problem: $LB_i^\varepsilon = 0$, $B_i^\varepsilon = b_i$ on $\partial D \cdot B_i^\varepsilon$ is well defined, since $0 \notin \sigma(L_\varepsilon)$, see Section 4.

By $v(\cdot, t) = e^{L_\varepsilon t}\chi$ we denote the solution of the uncontrolled, dynamic problem starting at $t = 0$ in χ

$$\frac{\partial v^\varepsilon}{\partial t} = L_\varepsilon v^\varepsilon$$

$$v^\varepsilon = 0 \text{ on } \partial D \qquad (31)$$

$$v^\varepsilon(\cdot, 0) = \chi.$$

Substitution of $x = y_k$ in $(30)_{d,b}$ yields the following Volterra equations for $\xi^\varepsilon(t)$

$$\xi^\varepsilon(t) = \eta^\varepsilon(t) + \int_0^t K_d^\varepsilon(\tau)\xi^\varepsilon(t-\tau)\, d\tau \qquad (32)_d$$

$$\xi^\varepsilon(t) = \eta^\varepsilon(t) + \frac{d}{dt}\left\{\int_0^t K_b^\varepsilon(\tau)\xi^\varepsilon(t-\tau)\, d\tau\right\} \qquad (32)_b$$

with $\eta_k^\varepsilon(t) = \delta_{y_k} e^{L_\varepsilon t}\psi$, $[K_d^\varepsilon(\tau)]_{k,i} = \delta_{y_k} e^{L_\varepsilon t} c_i$, $[K_b^\varepsilon(\tau)]_{k,i} = \delta_{y_k}(1 - e^{L_\varepsilon \tau})B_i^\varepsilon$. Once the solutions $\xi^\varepsilon(t)$ of these Volterra equations $(32)_{d,b}$ are known we find the solutions $\bar{u}^\varepsilon$ of $(29)_{d,b}$ simply by substitution of $\xi^\varepsilon(t)$ in $(30)_{d,b}$. Let us now consider our task, the construction of asymptotic approximation for $\varepsilon \downarrow 0$ of the solutions of $(29)_{d,b}$. This task really reduces to finding an asymptotic approximation v for $\varepsilon \downarrow 0$ of the solution v^ε of the uncontrolled problem (31). In this respect it is important to notice that an asymptotic approximation of B_i^ε is already known, see Section 2, $(13)_b$.

Once such an approximation is available we also have approximations $K_{d,b}$ of $K_{d,b}^\varepsilon$, η of η^ε and ξ of ξ^ε, where ξ is found as the solution of the approximate version of $(31)_{d,b}$. Next an approximation $\bar{u}$ of $\bar{u}^\varepsilon$ is found by substituting all approximations of the r.h.s. of $(30)_{d,b}$.

In *the case of an unperturbed operator of zero order* as in (4a) the approximation consists of a regular expansion corrected by a boundary layer of width $\sqrt{\varepsilon}$ along all of ∂D, just as in the stationary case. However, the various terms in the approximation now satisfy dynamic equations

$$v^\varepsilon = V_0(x, t) + \tilde{P}_0(-)H(x) + O(e^{-\gamma t}\sqrt{\varepsilon}) \qquad (33)$$

$$\frac{\partial V_0}{\partial t} = -\gamma V_0 \qquad (34)$$

$$V_0(\cdot, 0) = \chi$$

$\tilde{P}_0$ is the zero order boundary layer term and H is a suitable C^∞ cutoff function. In order to construct $\tilde{P}_0$ we introduce rather special coordinates (y, θ) near ∂D, such that: $\partial D = \{y = 0\}$ and

$$L_2 = \tilde{a}\frac{\partial^2}{\partial y^2} + \sum_{i,j=1}^{n-1} \tilde{a}_{ij}\frac{\partial^2}{\partial\theta_i\partial\theta_j} + \tilde{a}_0\frac{\partial}{\partial y} + \sum_{i=1}^{n-1}\tilde{a}_i\frac{\partial}{\partial\theta_i}. \qquad (35)$$

This can be done by first introducing (y, ϕ)-coordinates with y the distance from x to ∂D and ϕ the parameters of the point on ∂D nearest to x. Next, we define $\theta_i = \phi_i + g_i(y, \phi)$ with g_i the solution of $\partial g_i/\partial y = -\frac{1}{2}\hat{a}_{0,i}(y, \phi)$, $g_i(0, \phi) = 0$, where $\hat{a}_{0,i}$ denotes the coefficient in front of $\partial^2/\partial y\partial\phi_i$ in L_2. The reason for the introduction of these special coordinates is to avoid singularities at ∂D, $t = 0$ in $[(\partial/\partial t) - L_\varepsilon]\tilde{P}_0$, which are produced by the differentiations $\partial^2/\partial y\partial\phi_i$. For the same reason we want to annihilate the effect of the differentiation $\tilde{a}_0$ $\partial/\partial y$ on $\tilde{P}_0$ and therefore we write

$$\tilde{P}_0(-) = \exp\left(-\tfrac{1}{2}\int_0^y \frac{\tilde{a}_0(s,\theta)}{\tilde{a}(s,\theta)}\, ds\right) P_0(\zeta, \theta, t) \qquad (36)$$

with $\zeta = y/\sqrt{\varepsilon}$ and P_0 the solution of

$$\frac{\partial P_0}{\partial t} = a\frac{\partial P_0}{\partial \zeta^2} - \bar{\gamma}P_0$$

$$P_{0|\zeta=0} = -V_{0|y=0}, \lim_{\zeta\to\infty} P_0(\zeta, \theta, t) = 0$$

$$P_{0|t=0} = 0 \qquad (37)$$

where a and $\bar{\gamma}$ have the same meaning as in (16), but with ϕ replaced by θ. The solutions of the problems (34) and (37) are

$$V_0(x,t) = \chi(x)\exp(-\gamma(x)t)$$
$$P_0(\zeta,\theta,t) = -\chi|_{\partial D}(\theta)\exp(-\bar{\gamma}(\theta)t)\frac{2}{\sqrt{\pi}}\int_{\zeta/\sqrt{(4a(\theta)t)}}^{\infty} e^{-\tau^2}\,d\tau. \tag{38}$$

In the case $\chi = B_i^\varepsilon$ we proceed in a slightly different way, because B_i^ε consists only of a boundary layer term along ∂D. We now take $V_0(\cdot,0) \equiv 0$ and $P_{0|y=0} \equiv 0$, $P_{0|t=0} = b_i(\theta)\exp(-\mu(\theta)\zeta)$ with $\mu = \sqrt{(\bar{\gamma}/a)}$, see $(17)_b$. This leads us to

$$(\chi = B_i^\varepsilon)V_0 \equiv 0$$
$$P_0(\zeta,\theta,t) = b_i(\theta)e^{-\bar{\gamma}t}\int_0^\infty G(\zeta,\eta,t,\theta)e^{-\mu\eta}\,d\eta \tag{38'}$$

with

$$G(\zeta,\eta,t,\theta) = \frac{1}{\sqrt{(4\pi at)}}\left\{\exp\left(-\frac{(\zeta-\eta)^2}{4at}\right) - \exp\left(-\frac{(\zeta+\eta)^2}{4at}\right)\right\}.$$

It is a nice exercise in the use of the maximum principle for a parabolic equation (Friedman, 1976; Protter and Weinberger, 1967) to show that $|P_0| \leqslant |b_i| \max(e^{-\bar{\gamma}t}, e^{-\mu\zeta})$. Note, that in (33) we can indeed take $\hat{\gamma} = \min \gamma(x)$. Let us now exploit these results to find asymptotic approximations of the kernels K_d^ε and K_b^ε in $(32)_{d,b}$

$$K_d^\varepsilon = K_d + O(\sqrt{\varepsilon}e^{-\hat{\gamma}t}) \tag{39}$$
$$K_b^\varepsilon = K_b + O(\sqrt{\varepsilon}e^{-\hat{\gamma}t}) \tag{39$_b$}$$

with $[K_d](t) = e^{-\Gamma t}Z$, $Z_{k,i} = c_i(y_k)$, Γ = the diagonal matrix with $\Gamma_{k,k} = \gamma(y_k)$ and $K_b \equiv 0$. Using these approximations for the kernels the equations for $\xi^\varepsilon(t)$ reduce to

$$\xi(t) = e^{-\Gamma t}\eta_0 + \int_0^t e^{-\Gamma(t-\tau)}Z\xi(\tau)\,d\tau; \tag{40$_d$}$$
$$\xi = e^{-\Gamma t}\eta_0. \tag{40$_b$}$$

We note, that $(40)_d$ is equivalent to a system of ODEs $\dot{\xi} = \Lambda\xi$, $\xi(0) = \eta_0$ with η_0 the vector with components $\psi(y_k)$ and $\Lambda = -\Gamma + Z$, as in $(20)_d$. Hence, the solution of $(40)_d$ is

$$\xi(t) = e^{\Lambda t}\eta_0. \tag{41^d}$$

Substitution of these approximations in $(30)_{d,b}$ provides us with an approximation $\bar{u}$ of the solution $\bar{u}^\varepsilon$ of $(29)_{d,b}$. For the growth of $\bar{u}$ for $t \to \infty$ we find the following estimates

$$|\bar{u}(x,t)| \leqslant C(v)e^{vt}, \quad v > \max(-\hat{\gamma}, \operatorname{Re}\sigma(\Lambda)) = v_d^0 \tag{42$_d$}$$
$$|\bar{u}(x,t)| \leqslant C(v)e^{vt}, \quad v > -\hat{\gamma} = v_b^0 \tag{42$_b$}$$

with constants $C(v)$ independent of ε. Note that this is in nice agreement with the results on the location of the spectrum of the controlled operator, compare (28).

In *the case of a first order unperturbed operator* as in $(4b)_1$, the structure of an approximation of the solution of (31) is rather complicated. Figure 2 shows where the various layers are found.

Our approximation has the following form:

$$\tilde{v}^\varepsilon = P_0^\varepsilon H^e + \tilde{P}_0^0 H^0 + (Q_0 + YH)H_-^c H_+^c(1 - H^e) + (V_0 + G_0H)(1 - H^e)(1 - H_-^c)(1 - H^0) + O(\varepsilon^{1/5}e^{-\omega t}). \tag{43}$$

Here H^e, H^0, H and H_+^c, H_-^c denote suitable C^∞ cut-off functions. Let $I(\alpha)$ be $\equiv 1$ for $\alpha \leqslant 1$ and $\equiv 0$ for $\alpha \geqslant 2$. Then we choose $H = I(T(\phi) - s)$, $H^e = I(s/\delta)\ I(t/\delta)$, $H^0 = I((T(\phi) - s)/\delta)$. $I(t/\delta)$, $H_+^c = I((t-s)/\delta^2)\,H_-^c = I((s-t)/\delta^2)$ with $\delta = \varepsilon^{1/5}$. Of course, $T(\phi)$ and s have the same meaning as in Section 2. In the order of the error of the approximation $\omega \in \mathbb{R}$ can be taken arbitrary, as we shall see in the next section. The zero order terms in (43) are found in the following way

$$\frac{\partial V_0}{\partial t} = -\frac{\partial V_0}{\partial s} - \gamma V_0; \qquad V_{0|t=0} = \chi \tag{44}$$

$$\frac{\partial Q_0}{\partial s} = A\frac{\partial^2 Q_0}{\partial \tau^2} - Q_0 \tag{45}$$

$$\lim_{\tau\to\infty} Q_0(s,\phi,\tau) = 0, \quad \lim_{\tau\to-\infty} Q_0(s,\phi,\tau) = V_0(s,\phi,s)$$

with $\tau = (t-s)/\sqrt{\varepsilon}$, $A = \sum_{j=1}^{n} \frac{\partial t_e}{\partial x_i} a_{ij} \frac{\partial t_e}{\partial x_i}$

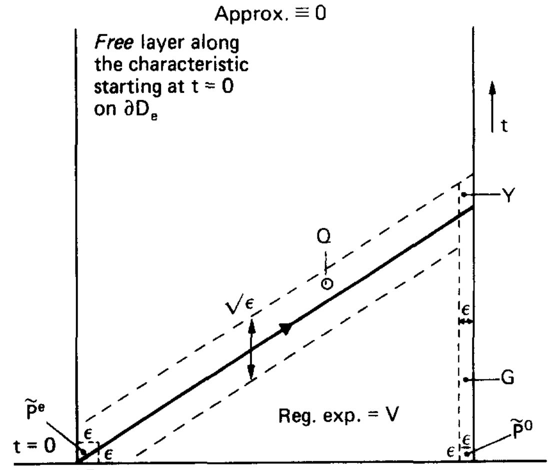

FIG. 2. The approximation consists of a regular part combined with several layers.

$$A^0\frac{\partial^2 Y_0}{\partial\zeta^2}-\frac{\partial Y_0}{\partial\zeta}=0 \tag{46}$$

$$Y_0(0,\phi,\tau)=-Q_0(T(\phi),\phi,\tau)$$

$$\lim_{\zeta\to\infty} Y_0(\zeta,\phi,\tau)=0$$

with τ as above, $\zeta=(T(\phi)-s)/\varepsilon$ and $A^0=A|_{\partial D_0}$. For the calculation of $\tilde{p}^e$ we introduce special coordinates near ∂D_e in order to avoid singularities for $t\downarrow 0$ in $L_e\tilde{P}^e$. Instead of (s,ϕ) we work with (s,θ) defined in such a way, that $\partial D_e=\{s=0\}$ and

$$L_2=\bar{A}\frac{\partial^2}{\partial s^2}+\sum_{i,j=1}^{n-1}\bar{A}_{i,j}\frac{\partial^2}{\partial\theta_i\partial\theta_j}+\bar{A}_0\frac{\partial}{\partial s}+\sum_{i=1}^{n-1}\bar{A}_i\frac{\partial}{\partial\theta_i}.$$

Such coordinates can be found by a procedure as in (35). Then, we define

$$\tilde{P}_0^e=\exp\left(-\tfrac{1}{2}\int_0^s\frac{\bar{A}_0(y,\theta)}{\bar{A}(y,\theta)}\,dy\right)P_0^e(\eta,\theta,\tau_1) \tag{47}$$

with $\eta=s/\varepsilon$, $\tau_1=t/\varepsilon$ and P_0^e the solution of

$$\frac{\partial P_0^e}{\partial\tau_1}=A^e\frac{\partial^2P_0^e}{\partial\eta^2}-\frac{\partial P_0^e}{\partial\eta} \tag{48}$$

$$P_0^e(\eta,\theta,0)=\chi|_{\partial D_e},\ P_0^e(0,\theta,\tau)=0$$

and $A^e=\bar{A}|_{\partial D_e}=A|_{\partial D_e}(\theta)$. In a completely analogous way we can introduce coordinates $s_0=T(\phi)-s$ and $\bar{\theta}$ such that $\partial D_0=\{s_0=0\}$ and

$$L_2=\bar{A}'\frac{\partial^2}{\partial s_0^2}+\sum_{i,j=0}^{n-1}\bar{A}'_{i,j}\frac{\partial^2}{\partial\bar{\theta}_i\partial\bar{\theta}_j}+\bar{A}'_0\frac{\partial}{\partial s_0}+\sum_{i=1}^{n-1}\bar{A}'_i\frac{\partial}{\partial\bar{\theta}_i}.$$

We put

$$\tilde{P}_0^0=\exp\left(-\tfrac{1}{2}\int_0^{s_0}\frac{\bar{A}'_0(y_0,\bar{\theta})}{\bar{A}'(y_0,\bar{\theta})}\,dy_0\right)P_0^0(\zeta,\bar{\theta},\tau_1)$$

with $\zeta=s_0/\varepsilon$ and P_0^0 the solution of

$$\frac{\partial P_0^0}{\partial\tau_1}=A^e\frac{\partial^2P_0^0}{\partial\zeta^2}+\frac{\partial P_0^0}{\partial\zeta} \tag{50}$$

$$P_0^0(\zeta,\bar{\theta},0)=\chi|_{\partial D_0};\ P_0^0(0,\bar{\theta},\tau_1)=0$$

with $A=A|_{\partial D_0}(\bar{\theta})$. In the case $\chi=B_i^e$ the initial values for P_0^0 are taken as the zero order term of the boundary layer expansion of B_i^e, i.e. $\bar{G}_0(\zeta,\bar{\theta})+\bar{B}_{0|\partial D_0}(\bar{\theta})$ as given in $(25)_b$ with $b_0=b_{i|\partial D_0}$ and $b_e=b_{i|\partial D_e}$. The solutions of these problems can easily be calculated

$$V_0(s,\phi,t)=\chi(s-t,\phi)\exp\left(-\int_{s-t}^s\gamma(s',\phi)\,ds'\right) \tag{51}$$

$$Q_0(s,\phi,\tau)=\exp\left(-\int_0^s\gamma(s',\phi)ds').\ q_0(r,\phi,\tau)\right) \tag{52}$$

$$q_0(r,\phi,\tau)=\chi|_{\partial D_e}(\phi)\frac{1}{\sqrt{\pi}}\int_{\tau/\sqrt{(4r)}}^{\infty}e^{-t^2}\,dt$$

with

$$r=\int_0^s A(s',\phi)\,ds'$$

$$Y_0(\zeta,\phi,\tau)=-Q_0(T(\phi),\phi,\tau)\exp(-\zeta/A^0) \tag{53}$$

$$P_0^e(\eta,\theta,\tau_1)=\chi|_{\partial D_e}(\theta)e^{(2\eta-\tau_1)/(4A^e)}\int_0^\infty G^e(\xi,\eta,\tau_1,\theta)e^{-\xi/(2A^e)}\,d\xi \tag{54}$$

with

$$G^e(\xi,\eta,\tau_1,\theta)=\frac{1}{\sqrt{(4\pi A^e\tau_1)}}\left\{\exp\left(-\frac{(\xi-\eta)^2}{4A^e\tau_1}\right)-\exp\left(-\frac{(\xi+\eta)^2}{4A^e\tau_1}\right)\right\}.$$

$$P_0^0(\zeta,\bar{\theta},\tau_1)=e^{-(2\zeta+\tau_1)/(4A^0)}\times\int_0^\infty G^0(\eta,\zeta,\bar{\theta},\tau_1)e^{-\eta/(2A^0)}P_0^0(\eta,\bar{\theta},0)\,d\eta \tag{55}$$

with

$$G^0(\eta,\zeta,\bar{\theta},\tau_1)=\frac{1}{\sqrt{(4\pi A^0\tau_1)}}\left\{\exp\left(-\frac{\zeta-\eta)^2}{4A^0\tau_1}\right)-\exp\left(-\frac{(\zeta+\eta)^2}{4A^0\tau_1}\right)\right\}.$$

Using the results as found in (43)–(55) we obtain the following approximations for the equations given in $(32)_{d,b}$

$$\xi(t)=\eta(t)+\int_0^t K_d(\tau)\xi(t-\tau)\,d\tau \tag{56$_d$}$$

$$\xi(t)=\eta(t)+\frac{d}{dt}\left\{\int_0^t K_b(\tau)\xi(t-\tau)\,dt\right\} \tag{56$_b$}$$

with

$$[K_d]_{k,i}(t)=c_i(s_k-t,\phi_k)\exp\left(-\int_{s_k-t}^{s_k}\gamma(s',\phi_k)\,ds'\right)\quad\text{for } t<s_k$$
$$=0 \text{ for } t>s_k$$

$$[K_b]_{k,i}(t)=0 \text{ for } t<s_k$$
$$=b_{i,e}(\phi_k)\exp\left(-\int_0^{s_k}\gamma(s',\phi_k)\,ds'\right),\quad\text{for } t>s_k$$

and

$$\eta_k(t)=\psi(s_k-t,\phi_k)\exp\left(-\int_{s_k-t}^{s_k}\gamma(s',\phi_k)\right),\quad\text{for } t<s_k$$
$$=0 \text{ for } t>s_k.$$

Here (s_k, ϕ_k) represents the observation points y_k in (s, ϕ) coordinates. The errors in these approximations will be discussed in the next section. We observe, that for $t > s_k$ the kth component of (56) yields

$$\xi_k(t) = \int_0^{s_k} [K_d(\tau)\xi(t-\tau)]_k \, d\tau.$$

It is also easy to check, that $(56)_b$ is equivalent to $\xi_k(t) = \eta_k(t) + [K_b(\infty)\xi(t - s_k)\check{H}(t - s_k)]_k$ with $\check{H}$ the Heaviside function. This shows that $(56)_{d,b}$ are equations with a finite *retardation*. Further, using the conditions which ψ satisfies on ∂D, it is easy to check that η and ξ are continuous functions with bounded derivatives. With the exception of $t = s_k$ for $(56)_d$ and $t = ns_k$ for $(56)_b$, $\dot{\xi}$ and $\dot{\eta}$ are continuous. The equations $(56)_{d,b}$ can easily be solved using Laplace transformation. Let us denote the Laplace transform of f by Lf with

$$(Lf)(\lambda) = \int_0^{\infty} f(t) e^{-\lambda t} \, dt.$$

Because of the convolution structure of $(56)_{d,b}$ the equations for $L\xi$ are very simple

$$(1 + \Omega_d(\lambda))L\xi = L\eta \qquad (57)$$

$$(1 - \Omega_b(\lambda))L\xi = L\eta \qquad (57)_b$$

with Ω_d, Ω_b as in $(26)_{d,b}$. Hence

$$\xi = L^{-1}[(1 + \Omega_d)^{-1} L\eta] \qquad (58)_d$$

$$\xi = L^{-1}[(1 - \Omega_b)^{-1} L\eta]. \qquad (3.30)_b$$

The inverse Laplace transformation is given by

$$(L^{-1}g)(t) = e^{\nu t}(2\pi)^{-1} \int_{-\infty}^{\infty} g(\nu + i\eta) e^{i\eta t} \, d\eta$$

(Schwartz, 1965). Here we can use any $\nu > \nu_d^1 = \max\{\operatorname{Re}\lambda | \omega_d(\lambda) = 0\}$, $\nu > \nu_b^1 = \max\{\operatorname{Re}\lambda | \omega_b(\lambda) = 0\}$ in $(58)_{d,b}$, respectively.

It is not difficult to verify that the substitution of ξ in $(3.2)_{d,b}$ provides us with approximations $\bar{u}$ of the solutions $\bar{u}^\varepsilon$ which satisfy the following estimates

$$|\bar{u}(x,t)| \leqslant C(\nu) e^{\nu t} \quad \nu > \nu_d^1 \qquad (59)_d$$

$$|\bar{u}(x,t)| \leqslant C(\nu) e^{\nu t} \quad \nu > \nu_b^1. \qquad (59)_b$$

This is in good agreement with the asymptotic location of the spectrum as determined in Section 2, [cf. (28)].

4. ON THE ASYMPTOTIC VALIDITY OF THE FORMAL APPROXIMATIONS

Here we shall derive some results on the correctness of the approximations of the solutions of the stationary problems, as found in Section 2, and we shall also demonstrate that the asymptotic location of the spectrum of the controlled operator given in that section is correct. Furthermore, we shall discuss the validity of our approximations of the solutions of the dynamic problems found in Section 3.

Let us first consider the case, *where the unperturbed operator* L_1 *is of first order*. The following result will be very useful.

Theorem. Let L_1 be as in (4b) and let w be the solution of the dynamic problem

$$\frac{\partial w}{\partial t} = L_\varepsilon w + R \qquad (60)$$

$$w = 0 \text{ on } \partial D$$

$$w(\cdot, 0) = \psi \in C_0(\bar{D}) = \{\chi \in C(\bar{D}) | \chi = 0 \text{ on } \partial D\}$$

with $R \in C(\bar{D} \times (0, \infty))$ and bounded for $t \downarrow 0$. Then $w \in C(D \times [0, \infty))$ and $\exists \varepsilon_0, \beta, C, T > 0 \forall \varepsilon \in (0, \varepsilon_0]$ $\forall \omega \in [0, \beta/\varepsilon] \forall x \in \bar{D}, t \geqslant 0$

$$|w(x,t)| \leqslant C e^{-\omega(t-T)} \max\left\{|\psi|_0, \sup_{[0,t]} |R(\cdot,\tau) e^{\omega\tau}|_0\right\} \qquad (61)$$

where $|\cdot|_0$ denotes the sup-norm for the domain D.

Proof. We first take ψ, R smooth with $\operatorname{supp}(\psi) \subset D$ and $\operatorname{supp}(R) \subset \{t > 0\}$. Then (Ladyzenskaja and co-workers, 1968), there exists a solution w of (60) in $C^\infty(\bar{D} \times [0, \infty))$. The function $\tilde{w}$ defined by $w = \tilde{w} \exp(-\omega t) W(s)$ satisfies

$$\frac{\partial \tilde{w}}{\partial t} = (\tilde{L}_\varepsilon + \omega)\tilde{w} + \tilde{R} \text{ with } \tilde{R} = R e^{\omega t}/W$$

$$\tilde{w} = 0 \text{ on } \partial D, \; w(\cdot, 0) = \tilde{\psi} = \psi/W \qquad (62)$$

where the constant term of $\tilde{L}_\varepsilon + \omega$ equals $\tilde{\gamma} = W^{-1}(L_\varepsilon + \omega)W$. Now let W satisfy $(-\partial/\partial s + \alpha + \omega)W = -(1 + \omega)W$, $W(0) = 1$ with $\alpha = \max \gamma$, i.e. $W(s) = \exp((\alpha + 2\omega + 1)s)$. It is easy to check that $\tilde{\gamma} \leqslant -1 - \omega + \tilde{C}\varepsilon(1 + \alpha + 2\omega)^2$; hence for $\omega \in [0, \beta/\varepsilon]$ with β suitably chosen we have $\tilde{\gamma} \leqslant -\frac{1}{2}$. Using the maximum principle for parabolic equations (Protter and Weinberger, 1967; Friedman, 1976), we obtain

$$\tilde{w}(x,t) \mp 2\max\left\{|\psi|_0, \sup_{\tilde{t} \leqslant t} \tilde{R}(\cdot, \tilde{t})|_0\right\} \gtrless 0 \qquad (63)$$

Of course, (63) implies (62).

The regularity assumptions on ψ, R can be weakened to those in the lemma by an approximation argument: $R_n \to R$ in $L^p(Q)$, $Q = D \times (0, T)$, p sufficiently large, using L^p theory for parabolic equations (Ladyzenskaja and co-workers, 1968), and Sobolev's imbedding from $W^{1,p}(Q) \to C(\bar{Q})$ (Adams, 1975), and next $\psi_n \to \psi$ in $C_0(\bar{D})$.

□

It is well known that L_ε with Dirichlet boundary conditions generates an analytic semigroup $e^{L_\varepsilon t}$ on $C_0(\bar{D})$ (Stewart, 1980). A direct consequence of (61) is that

$$|e^{L_\varepsilon t}|_0 \leqslant C \min\left\{1, \exp\left[-\frac{\beta}{\varepsilon}(t-T)\right]\right\} \quad (64)$$

Note, that this estimate is in perfect agreement with the behaviour of the approximation for $e^{L_\varepsilon t}\chi$ given in (43). This is not completely trivial, since the free layer in (43) has a width $\sqrt{\varepsilon}$, but its erfc-structure makes it decay as in (64).

Using the characterization of analytic semigroups in terms of the resolvent of the generator (Krasnoselskii and co-workers, 1976), it is clear that

$$\sigma(L_\varepsilon) \subset \{\lambda | \mathrm{Re}\,\lambda \leqslant -\beta/\varepsilon\} \quad (65)$$

Since

$$(L_\varepsilon - \lambda)^{-1} = -\int_0^\infty e^{-\lambda t} e^{L_\varepsilon t}\,dt$$

the resolvent satisfies the estimate

$$|(L_\varepsilon - \lambda)^{-1}|_0 \leqslant C\,e^{\omega T}(\omega - \mathrm{Re}\,\lambda)^{-1} \quad (66)$$

for $\omega \in [0, \beta/\varepsilon]$, $\mathrm{Re}\,\lambda > -\omega$. This estimate in (64) is not only valid on $C_0(\bar{D})$, but on all of $C(\bar{D})$. This can be seen by using an L^p approximation argument, L^p theory for elliptic equations (Agmon, Douglis and Nirenberg, 1959), and Sobolev's imbedding from $W^{2,p}(d) \to C(\bar{D})$ for p sufficiently large. Let us now apply (64) to prove the validity of the approximations in $(22)_{\mathrm{d,b}}$. We define $C' = \hat{C}_0 + (\hat{G}_0^0 + \varepsilon\hat{G}_1^0)H(T(\phi) - s)$, $B' = B_0 + (\bar{G}_0^0 + \varepsilon\bar{G}_1^0)H(T(\phi) - s)$. Note, that compared with (22) we have included one more term in our boundary layer expansion. It is easy to verify that

$$(L_\varepsilon - \lambda)(C^\varepsilon - C') = r_d \qquad C^\varepsilon - C' = 0 \text{ on } \partial D \quad (67)_{\mathrm{d}}$$

$$(L_\varepsilon - \lambda)(B^\varepsilon - B') = r_b \qquad B^\varepsilon - B' = 0 \text{ on } \partial D \quad (67)_{\mathrm{b}}$$

with $|r_d|_0 = O(\varepsilon)$, $|r_b|_0 = O(\varepsilon)$.

If λ is fixed, then an immediate consequence of (66) is

$$|C^\varepsilon - C'|_0 = O(\varepsilon) \quad (68)_{\mathrm{d}}$$

$$|B^\varepsilon - B'|_0 = O(\varepsilon). \quad (68)_{\mathrm{b}}$$

Therefore, the order of the error specified in (22) is indeed rigorous. Of course, also the order of the differences $\Omega_{\mathrm{d}}^\varepsilon - \Omega_{\mathrm{d}}$, $\Omega_{\mathrm{b}}^\varepsilon - \Omega_{\mathrm{b}}$ is then $O(\varepsilon)$.

Hence, if $\omega_{\mathrm{d}}(\lambda) \neq 0$ $(,\omega_{\mathrm{b}}(\lambda) \neq 0)$, we can invert $1 + \Omega_{\mathrm{d}}^\varepsilon$ $(,1 - \Omega_{\mathrm{b}}^\varepsilon)$ for ε sufficiently small and the difference between u^ε and its approximation u is rigorously $O(\varepsilon)$.

If $\omega_{\mathrm{d}}(\lambda_k) = 0$ $(,\omega_{\mathrm{b}}(\lambda_k) = 0)$ it follows from Rouche's theorem (Conway, 1973), that $\det(1 + \Omega_{\mathrm{d}}^\varepsilon(\lambda))$ $(,\det(1 - \Omega_{\mathrm{b}}^\varepsilon(\lambda)))$ has $N \geqslant 1$ zeros in an $O(\varepsilon^{1/N})$ neighbourhood of λ_k with N the order of the zero of $\omega_{\mathrm{d}}(\lambda_k)$ $(,\omega_{\mathrm{b}}(\lambda_k))$. So, asymptotically close to λ_k are points of the spectrum of the controlled operator. It is also easy to check that if $\lambda(\varepsilon)$ is in the spectrum of the controlled operator and $\lim_{\varepsilon \downarrow 0} \lambda(\varepsilon) = \mu$ then μ is one of the zeros of $\omega_{\mathrm{d}}(\lambda)$ $(,\omega_{\mathrm{b}}(\lambda))$. Hence, the zeros of $\omega_{\mathrm{d}}(\lambda)$ $(,\omega_{\mathrm{b}}(\lambda))$ can be identified as the points of the spectrum of the controlled operator with a finite limit for $\varepsilon \downarrow 0$.

We can also use the above theorem to estimate rigorously the order of the error in our approximation (43) for the solution of the dynamic problem (31). We define: $v' = v + \varepsilon\{\tilde{P}_1^e H^e + \tilde{P}_1^0 H^0 + Y_1 H H_-^c H_+^c + G_1 H(1 - H_-^c)\ (1 - H^e)\}$ where $\tilde{P}_1^e$, $\tilde{P}_1^0$, Y_1, G_1 denote suitable next order terms in the various expansions. These corrections can be chosen in such a way that

$$\left(\frac{\partial}{\partial t} - L_\varepsilon\right)(v^\varepsilon - v') = O(\varepsilon^{1/5}), \quad \text{for } 0 \leqslant t \leqslant T_1$$
$$= 0, \quad \text{for } t \geqslant T_1$$
$$v^\varepsilon - v' = 0 \text{ on } \partial D$$
$$(v^\varepsilon - v')(\cdot, 0) = O(\varepsilon^{1/5}). \quad (69)$$

Furthermore, our construction took care that the remainder terms in (69) satisfy the regularity conditions as required by (61), since we avoided unbounded singularities in $L_\varepsilon v'$ at $\partial D \times \{t = 0\}$. The verification of the order functions in (66) is a long, paper-devouring business, but the calculations are rather straightforward. Now an application of (62) shows that the order of the remainder in (43) is rigorous. Consequently it is clear, that the approximation of $K_{\mathrm{d,b}}^\varepsilon$ by $K_{\mathrm{d,b}}$, see (32) and (56), takes place in the following sense:

$$|K_{\mathrm{d}}^\varepsilon(t) - K_{\mathrm{d}}(t)|_{k,i} \leqslant k_1(\omega)e^{-\omega t}\varepsilon^{1/5} + k_2\hat{I}\left(\frac{t - s_k}{\varepsilon^{2/5}}\right) \quad (70)_{\mathrm{d}}$$

$$|K_{\mathrm{b}}^\varepsilon(t) - K_{\mathrm{b}}(t)|_{k,i} \leqslant k_1\varepsilon^{1/5} + k_2\hat{I}\left(\frac{t - s_k}{\varepsilon^{2/5}}\right). \quad (70)_{\mathrm{b}}$$

Here $\hat{I}$ denotes the indicator function of the interval $[-1, 1]$. In the case of distributed control this is sufficient to show that ξ approximates ξ^ε in the following sense

$$|\xi^\varepsilon(t) - \xi(t)| \leqslant 1(v)\varepsilon^{1/5}e^{vt} \quad (71)_{\mathrm{d}}$$

where $v \in \mathbb{R}$ has to be $> v_{\mathrm{d}}^1$, with v_{d}^1 as in (28). In order to show this we observe that $\xi^\varepsilon - \xi$ satisfies the equation

$$(I - K_{\mathrm{d}}*)(\xi^\varepsilon - \xi) = \eta^\varepsilon - \eta + (K_{\mathrm{d}}^\varepsilon - K_{\mathrm{d}})*\xi + (K_{\mathrm{d}}^\varepsilon - K_{\mathrm{d}})*(\xi^\varepsilon - \xi) \quad (72)_{\mathrm{d}}$$

where * denotes the convolution operation. We consider this equation on the space $B_v = \{\xi \in$

$\{C[0,\infty)\}^p| \quad |\xi|_v = \sup|\xi(t)e^{-vt}| < \infty\}$ with $v > v_d^1$. We notice, that the equation $(I - K_d{}^*)z = y$ is for $t \geqslant T = \max(s_k)$ equivalent to an autonomous retarded differential equation

$$\frac{d}{dt}(z - y) = L(z - y) + Ly$$

with

$$(Lv)_k(t) = \sum_{i=1}^{p} \{[K_d]_{k,i}(s_k)v_i(t - t_k) - [K_d]_{k,i}(0)v_i(t) + \int_0^{s_k} (K_d)_{k,i}(\tau)v_i(t-\tau)\,d\tau\}.$$

Using the theory of autonomous retarded differential equations (Hale, 1979), it is clear that L generates a strongly continuous semigroup on $\{C[0,T]\}^p$, $T = \max s_k$ with $|e^{Lt}| \leqslant C(v')\exp(v't)$ for each $v' > v_d^1$. The solution of $(I - K_d^*)z = y$ satisfies

$$z_t = e^{Lt}z_0 + y_t + \int_0^t e^{L(t-\tau)}Ly_\tau\,d\tau$$

where $z_t \in \{C[0,T]\}^p$ is the element with $z_t(\bar{\tau}) = z(t) + \bar{\tau})$, $\bar{\tau} \in [0,T]$. It is now not difficult to show, that $I - K_d^*$ has a bounded inverse on B_v. Since $(K_d^\varepsilon - K_d)^*$ is an operator on B_v with a norm of order $\varepsilon^{1/5}$ we can solve $(72)_d$ by a Neuman series and $(72)_d$ is a consequence from the fact

$$|\eta^\varepsilon - \eta + (K_d^\varepsilon - K_d)^*\xi|_v = O(\varepsilon^{1/5}).$$

Using $(71)_d$ we find an approximation $\bar{u}$ of $\bar{u}_\varepsilon$ which satisfies

$$|\bar{u}(\cdot,t) - \bar{u}_\varepsilon(\cdot,t)|_0 \leqslant \bar{l}(v)\varepsilon^{1/5}e^{vt}, \quad v > v_d^1. \quad (73)_d$$

This shows also the spectrum of the controlled operator is in the half plane $\{\lambda | \mathrm{Re}\,\lambda \leqslant v\}$ if $v \in \mathbb{R}$, $v > v_d^1$ and ε sufficiently small and further, that the analytic semigroup generated by $L_\varepsilon + \Pi$, [van Harten, 1979, (2)], satisfies the estimate

$$|e^{(L_\varepsilon + \Pi_d)t}|_0 \leqslant C(v)e^{vt}, \quad \text{for } v \in \mathbb{R}, v > v_d^1. \quad (74)_d$$

In the case of boundary control the estimate $(70)_b$ is not sufficient to carry out an analysis leading to something analogous to $(71)_d$ and $(73)_b$, because of the derivative $\partial/\partial t$ in $(30)_b$ and $(32)_b$. Much more detailed information about the differences $K_b^\varepsilon - K_b$ and its time derivatives is necessary to do so. Our plan is to present such an analysis in a subsequent paper.

In *the case of a zero order unperturbed operator* as in (4a) we can proceed analogously as above (see also van Harten, 1979). Therefore we shall just state the results and leave the details of the derivations to the reader. For L_0 as in (4a) the solution w of (61) will be an element of $C(\bar{D} \times [0,\infty))$ satisfying an estimate

$$|w| \leqslant C(\omega)e^{-\omega t}\max\left\{|\psi|_0, \sup_{[0,t]}|R(\cdot,\tau)e^{\omega\tau}|_0\right\} (75)$$

for each $\omega \in \mathbb{R}$, $-\omega \geqslant -\hat{\gamma}$. In combination with the selfadjointness of L_ε this estimate yields: $\sigma(L_\varepsilon) \subset (-\infty, -\hat{\gamma}]$. The resolvent $(L_\varepsilon - \lambda)^{-1}$ satisfies the estimate

$$|(L_\varepsilon - \lambda)^{-1}|_0 \leqslant C(\mathrm{Re}\,\lambda + \hat{\gamma})^{-1}, \quad \text{for } \mathrm{Re}\,\lambda > -\hat{\gamma} \quad (76)$$

with a constant C independent of ε and (76) is valid on all of $C(\bar{D})$. For $\mathrm{Re}\,\lambda \leqslant -\hat{\gamma}$, $\mathrm{Im}\,\lambda \neq 0$ we can also derive an estimate for $|(L_\varepsilon - \lambda)^{-1}f|_0$ with f sufficiently regular. To do this one starts with the observation $\|(L_\varepsilon - \lambda)^{-1}\|_{L_2} \leqslant |\mathrm{Im}\,\lambda|^{-1}$. Next using repeatedly *a priori* estimates for elliptic PDEs (Agmon, Douglis and Nirenberg, 1959) and Sobolev's imbedding theorem from $W^{m,2}(D) \to C(\bar{D})$ for $m > n/2$, (Adams, 1975), we determine that

$$|(L_\varepsilon - \lambda)^{-1}f|_0 \leqslant C\frac{(1 + |\lambda|)^k}{|\mathrm{Im}\,\lambda|}\varepsilon^{-k}|f|_{C^{2k}(\bar{D})},$$

$$2k > n/2. \quad (77)$$

Including sufficiently many higher order terms in our expansions it is now easy to prove, that the orders of the remainders in $(14)_{d,b}$, $(19)_{d,b}$, (33) and $(39)_{d,b}$ are rigorously correct. Consequently if $\lambda \notin (-\infty, -\hat{\gamma}]$ and if in case of distributed control also $\lambda \notin \sigma(\Lambda)$, see $(20)_d$, the approximation u of u^ε has an error $O(\sqrt{\varepsilon})$. In the case of distributed control the points $\sigma(\Lambda)\backslash(-\infty, -\hat{\gamma}]$ are exactly the finite limits in $\Psi\backslash(-\infty, -\hat{\gamma}]$ of eigenvalues of the controlled operator. In the case of boundary control the controlled operator has no finite limits of eigenvalues in $C/(-\infty, -\hat{\gamma}]$. For the approximation of the dynamic solution in the case of distributed control we find

$$|\bar{u}(\cdot,t) - \bar{u}^\varepsilon(\cdot,t)| \leqslant l(v)\sqrt{\varepsilon}\,e^{vt} \quad (78)_d$$

for $v > v_d^0$. This shows that the spectrum of this controlled operator is for ε sufficiently small in the half plane $\mathrm{Re}\,\lambda \leqslant v$, $v > v_d^0$. The analytic semigroup generated by $L_\varepsilon + \Pi_d$ (van Harten, 1979b), satisfies the estimate

$$|e^{(L_\varepsilon + \Pi_d)t}|_0 \leqslant C(v)e^{vt}, \quad \text{for } v \in \mathbb{R}, v > v_d^0. \quad (79)_d$$

Let us conclude this section with a few remarks about stabilizability of the system. Considering the results on the asymptotic location of the spectrum of the controlled operator we cannot hope that feedback as in $(1)_{d,b}$ will improve the stability properties of the uncontrolled system, if ε is sufficiently small. From the point of view of general stabilizability results this is at a first glance somewhat puzzling. Using (Triggiani, 1975; Curtain and Pritchard, 1978) we know, that in the case of (4a) with a one-dimensional domain, when all eigenvalues of L_ε are simple, it is possible, for a fixed ε, to choose one observation point y_1^ε and one distributed control input c_1^ε, such that $\sigma(L_\varepsilon + \Pi_d) \subset (-\infty, -\alpha]$, where α can be prescribed arbitrary. But, the reason why a control (1) is not suitable to do this is rather transparent: we assumed that the input functions and observation points do not depend wildly on ε for $\varepsilon \downarrow 0$. Hence, for a fixed $\alpha > \hat{\gamma}, c_1^\varepsilon, y_1^\varepsilon$ must have a wildly fluctuating structure for $\varepsilon \downarrow 0$.

This can also be seen from the construction of the stabilizing control (see Triggiani, 1975). In the next section we shall see, that controls as in (1) can be used to optimize a different kind of performance index of the system.

5. AN EXAMPLE OF NEAR OPTIMAL FEEDBACK CONTROL

Let us consider the following controlled problem

$$\frac{\partial u}{\partial t} = L_\varepsilon u + \tilde{\Pi}_d u + h \tag{80}$$

$$u = 0 \text{ on } \partial D,\ u(\cdot, 0) = \psi$$

with $\tilde{\Pi}_d u = c_0 + c(\delta_y u - i)$ and $h = f_0 + \omega f$. Note that the control consists of a permanent part c_0 and a feedback part based on the comparison of the observation of u in the point y with the reference value i. We suppose that h is a stationary, autonomous inhomogenity, which in various situations where the system is likely to operate, has a distribution $f_0 + \omega f$, f_0, $f \in C^\infty(\bar{D})$ and ω a stochastic parameter with $E(\omega) = 0$. Now we want to determine the control parameters c_0, c, i and y in such a way, that the expected costs J are optimal, with J the following quadratic functional

$$J = E\left\{\int_D [(u_{\text{stat}} - g)^2 + \mu_0 c_0^2 + \mu c^2(\delta_y u_{\text{stat}} - i)^2]\,\mathrm{d}x\right\} \tag{81}$$

under the obvious side condition that the stationary solution u_{stat} of (80) is exponentially stable

$$|u(\cdot, t) - u_{\text{stat}}|_0 \leqslant C(v)|u_{\text{stat}} - \psi|_0\, e^{vt}, \quad \text{with } -\hat{\gamma} < v < 0. \tag{82}$$

In (81), $g \in C^\infty(\bar{D})$ has the interpretation of the ideal stationary state of the system, μ_0 and μ are constants > 0. Using the results of the previous sections we are able to solve this optimization problem in an approximate sense for $\varepsilon \downarrow 0$. For simplicity we consider the case γ constant, i.e. $\gamma \equiv \hat{\gamma}$.

In order to do so we first note, that

$$u_{\text{stat}} \simeq \gamma^{-1}\{\alpha c + c_0 + f_0 + \omega(\beta c + f)\} + \cdots \tag{83}$$

with $\alpha = [\gamma i - c_0(y) - f_0(y)]/[c(y) - \gamma]$, $\beta = -f(y)/[c(y) - \gamma]$. The remainder term in (83) only contributes as $O(\sqrt{\varepsilon})$ to the expected costs. Hence, J is approximately given by

$$J = J_0 + \int_D \{A_0 c_0^2 + 2B_0 c_0 + Ac^2 + 2Bc + 2Ecc_0\}\,\mathrm{d}x + O(\sqrt{\varepsilon}) \tag{84}$$

with J_0 the expected costs in the uncontrolled situation, $A_0 = \gamma^{-2} + \mu_0$, $B_0 = \gamma^{-1} f_0 - g$, $A = (\gamma^{-2} + \mu)(\alpha^2 + \omega_2\beta^2)$, $B = \alpha\gamma^{-1}B_0 + \omega_2\gamma^{-2}\beta f$, $E = \alpha\gamma^{-2}$, where $\omega_2 = E(\omega^2)$, the second moment of ω.

Let us first minimize the second term J_1 in (84) without consideration of the stability condition (82). Note, that

$$J_1 = \int_D \{A_0(c_0 + B_0 A_0^{-1} + EA_0^{-1}c)^2 + \hat{A}_0(c + \hat{B}_0\hat{A}_0^{-1})^2\}\,\mathrm{d}x - \int_D \{A_0^{-1}B_0^2 + \hat{A}_0^{-1}\hat{B}_0^2\}\,\mathrm{d}x$$

with

$$\begin{aligned}
\hat{A}_0 &= \alpha^2 A_1 + \beta^2 A_2,\\
A_1 &= \gamma^{-2}\{(1 + \mu\gamma^2) - (1 + \mu_0\gamma^2)^{-1}\} > 0\\
A_2 &= \gamma^{-2}(1 + \mu\gamma^2)\omega_2 > 0\\
\hat{B}_0 &= \alpha B_1 + \beta B_2,\\
B_1 &= \gamma^{-1}B_0\{1 - \gamma(1 + \mu_0\gamma^2)^{-1}\}\\
B_2 &= \gamma^{-2}\omega_2 f
\end{aligned}$$

J_1 is minimal, if

$$\begin{aligned} c_0 &= -A_0^{-1}(B_0 + Ec)\\ c &= -\hat{A}_0^{-1}\hat{B}_0 \end{aligned} \tag{85}$$

and the value of the minimum is

$$J_1^{\min} = J_1^0 - (\alpha^2 A_1 + \beta^2 A_2)^{-1}(\alpha^2 N_1 + 2\alpha\beta M + \beta^2 N_2) \tag{86}$$

with $J_1^0 = -A_0^{-1}\|B_0\|^2$, $N_1 = \|B_1\|^2$, $N_2 = \|B_2\|^2$, $M = \langle B_1, B_2\rangle$ where $\|\cdot\|$ and $\langle,\rangle$ denote the norm and inner product on $L_2(D)$. J_1^0 is independent of the choice of α, β, but the second term in (86) is still a function of $z = \beta/\alpha$. If $M \neq 0$ the best choice for z is

$$z = z_0 = (2m)^{-1}\{(a - n) + \sqrt{[(a - n)^2 + 4am^2]}\} \tag{87}$$

with $a = A_1/A_2$, $n = N_1/N_2$ and $m = M/N_2$.

Let us now think about the stability condition (82). Because of $(61)_d$ in combination with (28) this condition is satisfied if

$$c(y) - \gamma < v. \tag{88}$$

Now we are ready to construct a near optimal control. This can be done in the following way:

(i) choose the observation point y in the interior of D such that $f(y) \neq 0$ and fix the value of the control function c in y as $-k < v + \gamma$ in order to take care of (88);

(ii) calculate z_0 using (87), determine β from its definition as $\beta = f(y)/(\gamma + k)$, take $\alpha = \beta/z_0$;

(iii) then define the near optimal form of the control function c as

$$c(x) = c_{\text{unrestr}}(x) - [c_{\text{unrestr}}(y) + k]\rho\left(\frac{x-y}{\delta}\right) \quad (89)$$

with

$$c_{\text{unrestr}}(x) = -\frac{(B_1 + z_0 B_2)(x)}{\alpha(A_1 + z_0^2 A_2)}$$

according to (85). The second term sets the value $c(y)$ equal to $-k$, see (i). Here ρ is a smooth function with compact support and $\rho \equiv 1$ on a neighbourhood of $0 \in \mathbb{R}^n$ and is a suitable $O(1)$ order-function, which we shall presently relate to ε;

(iv) we then obtain c_0 as $-A_0^{-1}(B_0 + E\tilde{c})$, see (85);

(v) finally, our choice of i is determined by the consistency condition $\beta/\alpha = z_0 = -f(y)/(\gamma_i - c_0(y) - f_0(y))$, i.e.

$$i = (\gamma z_0)^{-1}\{z_0(c_0(y) + f_0(y)) - f(y)\}. \quad (90)$$

Note that our asymptotic analysis given in the previous sections remains valid for control inputs with a structure as in (89), though the order of the remainders increase to $O(\varepsilon\delta^{-2} + \sqrt{\varepsilon})$. It is easy to check that this nearoptimal control produces costs which are at most $O(\varepsilon^1 + \varepsilon\delta^{-2} + \delta^n)$ above the genuine minimum. Hence, $\delta = \varepsilon^{1/3}$ for $n = 1$, $= \varepsilon^{1/4}$ for $n \geqslant 2$ is a good choice for δ. The analysis of the non-generic case, where $M = 0$, is left to the reader as an exercise.

Let us now work out this example on a more concrete level by considering the one-dimensional problem

$$\frac{\partial u}{\partial t} = \varepsilon\frac{\partial^2 u}{\partial x^2} - [u - c_0 + c(u(y,t) - i)] + (1 + \omega)e^{-x} \quad (91)$$

on the interval $[0, 1]$. This equation models the distribution of temperature in a transparent bar, while heat is produced by absorption of light coming from the left. The stochastic parameter ω is related to the intensity of the incoming light. Let us suppose, that ω is uniformly distribution with values in $[-1, 1]$, i.e. the second moment $\omega_2 = \frac{1}{3}$. The control is enforced by regulation of the temperature of the surroundings of the bar as $u_{\text{outside}} = c_0 + c(u(y,t) - i)$. This leads to an exchange of heat with the surroundings proportional to $-[u - u_{\text{outside}}]$. Let us now determine the control parameters $c_0(x)$, $c(x)$, the observation point y and the reference value i in such a way, that the temperature keeps close to the value $\equiv 1$ on $[0, 1]$ in a stationary situation with a minimum of control by minimizing the following cost function

$$J = E\int_0^1 \{(u_{\text{stat}} - 1)^2 + c_0^2 + c^2(u_{\text{stat}}(y) - i)^2\}\,dx \quad (92)$$

while we also require that initial derivations from a stationary distribution vanish for increasing time at least as $\exp(-\frac{1}{2}t)$. For ε small, the construction of a near optimal set c_0, c, y and i now proceeds as follows:

(i) Since in this example $f_0 = f = e^{-x} \neq 0$ on $(0, 1)$ the construction of a near optimal control works for any y inside the interval $(0, 1)$. We fix the value $c(y)$ as $-k$ with the restriction $-k < \gamma + v = \frac{1}{2}$.

(ii) In this case we have $A_1 = \frac{3}{2}$, $A_2 = \frac{2}{3}$, $B_1(x) = \frac{1}{2}(e^{-x} - 1)$, $B_2(x) = \frac{1}{3}e^{-x}$, $N_1 = \frac{1}{8}[-1 + \frac{1}{4}e - e^{-2}] = 0.042$, $M = \frac{1}{12}[-1 - e^{-2} + e^{-1}] = -0.064$, $N_2 = \frac{1}{18}[1 - e^{-2}] = 0.048$, $a = 2.25$, $n = 0.875$, $m = -1.331$.

Consequently, we obtain $z_0 = -2.103$ and hence $\beta = e^{-y}/(1 + k)$, $\alpha = -0.476e^{-y}/(1 + k)$.

(iii) Next c_{unrestr} can be determined, we find

$$c_{\text{unrestr}}(x) = -0.236(1 + k)e^{y}\{0.402e^{-x} + 1\}.$$

In this case we can choose k in such a way that $c_{\text{unrestr}}(y) = -k$, namely by taking

$$k = \frac{0.402 + e^{y}}{3.829 - e^{y}}.$$

This has the obvious advantage that our near optimal choice for $c(x)$ is just $c_{\text{unrestr}}(x)$, see (89). Therefore

$$c(x) = -\frac{e^{y}}{3.829 - e^{y}}(0.402e^{-x} + 1). \quad (93)$$

(iv) Using that $A_0 = 2$, $B_0(x) = e^{-x} - 1$, $E = \alpha = -0.112\,(3.829e^{-y} - 1)$ in the expression for c_0 given in (85) we obtain

$$c_0(x) = 0.444\{1 - 1.118e^{-x}\} \quad (94)$$

which is independent of y.

(v) Finally, the reference value i is found by applying (92):

$$i = 0.444(1 + 2.147e^{-y}). \quad (95)$$

We observe, that depending on the choice of the observation point y the reference value can be larger, equal or smaller than the ideal temperature, which is equal to 1. The case $i = 1$ occurs when $y = 0.538$.

As for simplifying the choice of k we have to remark that in general this is not possible without violating (88). In that case one either has to put a restriction on the location of the observation point y

so that the equation $c_{unrestr}(y) = -k$ has a solution which satisfies (88) or one takes a suitable value for k and uses the more complex form for c given in (89).

6. CONCLUSIONS

In this paper it is shown that asymptotic analysis, especially singular perturbation theory, can be exploited to find approximations for the solutions of linear, controlled diffusion processes with a small diffusivity in a number of cases: with or without convection, with boundary or distributed feed-back control. This theory also provides us with approximate criteria for the degree of stability of the system. Furthermore it allows us to determine a near optimal feedback loop with regard to a certain cost criterion under a restriction on the degree of stability of the system.

This approach has the advantage that in many cases for the construction of these approximations one only has to deal with algebraic equations and/or ordinary differential equations, which is remarkable, because the full problem is infinite dimensional and formulated in terms of an elliptic or parabolic partial differential equation. The methods presented here are also useful for a number of analogous feedback control problems, such as diffusion processes, where the control is based on different sensors and/or the system is subject to a different kind of boundary conditions. Furthermore, generalizations to nonlinear controlled diffusion processes or to systems with more than one component are possible.

REFERENCES

Adams, R. A. (1975). *Sobolev Spaces.* Academic Press, New York.

Agmon, S., A. Douglis and N. Nirenberg (1959). Estimates near the boundary for solutions of elliptic PDE satisfying general BC. *Comm. Pure Appl. Math.*, 12.

Balas, M. J. (1982). Reduced order feedback control of distributed parameter systems via singular perturbation methods. *J. Math. Ann. Applic.*, 87.

Balas, M. J. (1979). Feedback control of linear diffusion processes. *Int. J. Control*, 29.

Besjes, J. G. (1974). Singular perturbation problems for linear parabolic differential operators of arbitrary order. *J. Math. Ann. Appl.*, 48.

Conway, J. B. (1973). *Functions of One Complex Variable.* Springer, Berlin.

Curtain, R. F. and A. J. Pritchard (1978). Infinite dimensional linear systems theory. *Lecture Notes in Control and Information Science.* Vol. 8, Springer, Berlin.

Eckhaus, W. (1979). *Asymptotic Analysis of Singular Perturbations.* North-Holland, Amsterdam.

Fife, P. C. (1974). Semi-linear elliptic boundary value problems with a small parameter. *Arch. Rat. Mech.*, 52.

Friedman, A. (1976). *Stochastic Differential Equations and Applications.* Academic Press, New York.

Groen, P. P. N. de (1976). *Singularly Perturbed Differential Operators of Second Order.* Mathematical Centre, Amsterdam.

Hale, J. (1971). *Functional Differential Equations.* Springer, Berlin.

Harten, A. van (1975). *Singularly perturbed non-linear second order elliptic boundary value problems.* Thesis, Mathematical Institute, Utrecht.

Harten, A. van (1978). Non-linear singular perturbation problems: proofs of correctness. *J. Math. Ann. Appl.*, 65.

Harten, A. van (1979a). Feedback control of singularly perturbed heating problems. *Lecture Notes in Mathematics*, p. 711. Springer, Berlin.

Harten, A. van (1979b). On the spectral properties of a class of elliptic FDE arising in feedback control theory for diffusion processes. *Preprint no.* 130, Mathematical Institute, University of Utrecht.

Harten, A. van and J. M. Schumacher (1980). Well-posedness of some evolution problems in the theory of automatic feedback control for systems with distributed parameters. *SIAM J. Contr. Opt.*, 18.

Kato, T. (1966). *Perturbation Theory for Linear Operators.* Springer, Berlin.

Krasnoselskii, M. A. and co-workers (1976). *Integral Operators in Spaces of Summable Functions.* Noordhoff, Leiden.

Ladyzenskaja, D. A., V. A. Solonnikov and N. N. Uraltseva (1968). Quasi-linear equations of parabolic type. *Am. Math. Soc. Transl.*, 23.

Lions, J. L. (1973). Perturbations singulières dans les problèmes aux limites et en contróle optimal. *Lecture Notes in Mathematics*, p. 323. Springer, Berlin.

Owens, D. H. (1980). Spatial kinetics in nuclear reactor systems. In *Modelling of Dynamical Systems.* Peregrinus, Stevenage.

Protter, M. H., Weinberger, H. F. (1967). *Maximum Principles in Differential Equations.* Prentice-Hall, London.

Schumacher, J. M. (1981). *Dynamic Feedback in Finite- and Infinite Dimensional Linear Systems.* Mathematical Centre, Amsterdam.

Schwartz, L. (1965). *Méthodes mathématiques pour les sciences physiques.* Hermann, Paris.

Stewart, H. B. (1980). Generation of semigroups by strongly elliptic operators under general boundary conditions. *Trans. AMS*, 259.

Triggiani, R. (1975). On the stabilization problem in Banach spaces. *J. Math. Ann. Appl.*, 52.

Triggiani, R. (1979). On Nambu's boundary stabilizability problem for diffusion processes. *J. Diff. Eqn*, 33.

Triggiani, R. (1980). Boundary feedback stabilizability of parabolic equations. *Appl. Math. & Opt.*, 6.

Wilkinson, J. H. (1965). *The Algebraic Eigenvalue Problem.* Oxford University Press.