Connections up to homotopy and characteristic classes *

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Introduction

The aim of this note is to clarify the relevance of “connections up to homotopy” [4, 5] to the theory of characteristic classes, and to present an application to the characteristic classes of algebroids [3, 5, 7] (and of Poisson manifolds in particular [8, 13]).

We have already remarked [4] that such connections up to homotopy can be used to compute the classical Chern characters. Here we present a slightly different argument for this, and then proceed with the discussion of the flat characteristic classes. In contrast with [4], we do not only recover the classical characteristic classes (of flat vector bundles), but we also obtain new ones. The reason for this is that (Z_2-graded) non-flat vector bundles may have flat connections up to homotopy. As we shall explain here, in this category fall e.g. the characteristic classes of Poisson manifolds [8, 13].

As already mentioned in [4], one of our motivations is to understand the intrinsic characteristic classes for Poisson manifolds (and algebroids) of [7, 8], and the connection with the characteristic classes of representations [3]. Conjecturally, Fernandes’ intrinsic characteristic classes [7] are the characteristic classes [3] of the “adjoint representation”. The problem is that the adjoint representation is a “representation up to homotopy” only. Applied to algebroids, our construction immediately solves this problem: it extends the characteristic classes of [3] from representations to representations up to homotopy, and shows that the intrinsic characteristic classes [7, 8] are indeed the ones associated to the adjoint representation [5].

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Non-linear connections

Here we recall some well-known properties of connections on vector bundles. Up to a very slight novelty (we allow non-linear connections), this section is standard [11] and serves to fix the notations.

Let \( M \) be a manifold, and let \( E = E^0 \oplus E^1 \) be a super-vector bundle over \( M \). We now consider \( \mathbb{R} \)-linear operators

\[
\mathcal{X}(M) \otimes E \to E, \quad (X, s) \mapsto \nabla_X(s)
\]  

(1)

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which satisfy
\[ \nabla_X(fs) = f\nabla_X(s) + X(f)s \]
for all \( X \in \mathcal{X}(M) \), \( s \in \mathcal{E} \), and \( f \in C^\infty(M) \), and which preserve the grading of \( E \). We say that \( \nabla \) is a non-linear connection if \( \nabla_X(V) \) is local in \( X \). This is just a relaxation of the \( C^\infty(M) \)-linearity in \( X \), when one recovers the standard notion of (linear) connection. The curvature \( k_\nabla \) of a non-linear connection \( \nabla \) is defined by the standard formula
\[ k_\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} : E \to E . \] (2)

A non-linear differential form\(^1\) on \( M \) is an antisymmetric (\( \mathbb{R} \)-multilinear) map
\[ \omega : \underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{n} \to C^\infty(M) \] (3)
which is local in the \( X_i \)'s. It is easy to see (and it has been already remarked in [4]) that many of the usual operations on differential forms do not use the \( C^\infty(M) \)-linearity, hence they apply to non-linear forms as well. In particular we obtain the algebra \( (\mathcal{A}_{nl}(M), d) \) of non-linear forms endowed with De Rham operator. (This defines a contravariant functor from manifolds to dga’s.) Considering \( E \)-valued operators instead, we obtain a version with coefficients, denoted \( \mathcal{A}_{nl}(M; E) \). Note that a non-linear connection \( \nabla \) can be viewed as an operator \( \mathcal{A}_{nl}^0(M; E) \to \mathcal{A}_{nl}^1(M; E) \) which has a unique extension to an operator
\[ d_\nabla : \mathcal{A}_{nl}^k(M; E) \to \mathcal{A}_{nl}^{k+1}(M; E) \]
satisfying the Leibniz rule. Explicitly,
\[ d_\nabla(\omega)(X_1, \ldots, X_{n+1}) = \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \check{X}_i, \ldots, \check{X}_j, \ldots X_{n+1})) + \sum_{i=1}^{n+1} (-1)^{i+1} \nabla_{X_i} \omega(X_1, \ldots, \check{X}_i, \ldots, X_{n+1}) . \] (4)

We now recall the definition of the (non-linear) connection on \( \text{End}(E) \) induced by \( \nabla \). For any \( T \in \text{End}(E) \), the operators \([\nabla_X, T] \) acting on \( (E) \) are \( C^\infty(M) \)-linear, hence define elements \( [\nabla_X, T] \in \text{End}(E) \). The desired connection is then \( \nabla_X(T) = [\nabla_X, T] \). Clearly \( k_\nabla \in \mathcal{A}_{nl}^0(M; \text{End}(E)) \), and one has Bianchi’s identity \( d_\nabla(k_\nabla) = 0 \).

We will use the algebra \( \mathcal{A}_{nl}(M; \text{End}(E)) \) and its action on \( \mathcal{A}_{nl}(M; E) \). The product structure that we consider here is the one which arises from the natural isomorphisms
\[ \mathcal{A}_{nl}(M; E) \cong \mathcal{A}_{nl}(M) \otimes_{C^\infty(M)} (E) \]
and the usual sign conventions for the tensor products (i.e. \( \omega \otimes x \cdot \eta \otimes y = (-1)^{|\eta||\omega|} \omega \otimes x \eta \)). The usual super-trace on \( \text{End}(E) \) induces a super-trace
\[ Tr_s : (\mathcal{A}_{nl}(M; \text{End}(E)), d_\nabla) \to (\mathcal{A}_{nl}(M), d) \] (5)
with the property that \( Tr_s d_\nabla = d Tr_s \). We conclude (and this is just a non-linear version of the standard construction of Chern characters [11]):

\(^1\text{as in the case of connections, the non-linearity refers to } C^\infty(M) \text{-non-linearity. As pointed out to me, the terminology might be misleading. Better names would probably be "higher order connections" and "jet-forms".} \)
Lemma 1 If $\nabla$ is a non-linear connection on $E$, then
\[
ch_p(\nabla) = Tr_s(\kappa^p) \in A^p_{\mathrm{ml}}(M)
\]
are closed non-linear forms on $M$.

Up to a boundary, these classes are independent of $\nabla$. This is an instance of the Chern-Simons construction that we now recall. Given $k+1$ non-linear connections $\nabla_i$ on $E$ ($0 \leq i \leq k$) we form their affine combination $\nabla^{\text{aff}} = (1-t_1-\ldots-t_k)\nabla_0 + t_1\nabla_1 + \ldots + t_k\nabla_k$. This is a non-linear connection on the pullback of $E$ to $\Delta^k \times M$, where $\Delta^k = \{ (t_1, \ldots, t_k) : t_i \geq 0, \sum t_i \leq 1 \}$ is the standard $k$-simplex. The classical integration along fibers has a non-linear extension
\[
\int_{\Delta^k} : A^p_{\mathrm{ml}}(M \times \Delta^k) \to A^{p-k}_{\mathrm{ml}}(M)
\]
given by the explicit formula
\[
(\int_{\Delta^k} \omega)(X_1, \ldots, X_{n-k}) = \int_{\Delta^k} \omega(\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_k}, X_1, \ldots, X_{n-k})dt_1 \ldots dt_k .
\]
We then define
\[
cs_p(\nabla_0, \ldots, \nabla_k) = \int_{\Delta^k} ch_p(\nabla^{\text{aff}}) .
\]
Using a version of Stokes' formula [2] (or integrating by parts repeatedly) we conclude

Lemma 2 The elements (8) satisfy
\[
dcs_p(\nabla_0, \ldots, \nabla_k) = \sum_{i=0}^{k} (-1)^i cs_p(\nabla_0, \ldots, \tilde{\nabla}_i, \ldots, \nabla_k) .
\]

Connections up to homotopy and Chern characters

From now on, $(E, \partial)$ is a super-complex of vector bundles over the manifold $M$,
\[
(E, \partial) : E^0 \xrightarrow{\partial} E^1 .
\]
We now consider non-linear connections $\nabla$ on $E$ such that $\nabla_X \partial = \partial \nabla_X$ for all $X \in \mathcal{A}(M)$. We say that $\nabla$ is a (linear) connection on $(E, \partial)$ if it also satisfies the identity $\nabla_{fX}(s) = f\nabla_X(s)$ for all $X \in \mathcal{A}(M)$, $f \in C^\infty(M)$, $s \in \mathcal{E}$. The notion of connection up to homotopy [4, 5] on $(E, \partial)$ is obtained by relaxing the $C^\infty(M)$-linearity on $X$ to linearity up to homotopy. In other words we require
\[
\nabla_{fX}(s) = f\nabla_X(s) + [H_X(f, X), \partial] ,
\]
where $H_X(f, X) \in \text{End}(E)$ are odd elements which are $\mathbb{R}$-linear and local in $X$ and $f$.

We say that two non-linear connections $\nabla$ and $\nabla'$ are equivalent (or homotopic) if
\[
\nabla' = \nabla + [\theta(X), \partial]
\]
for all $X \in \mathcal{A}(M)$, for some $\theta \in A^1_{\mathrm{ml}}(M; \text{End}(E))$ of even degree. We write $\nabla \sim \nabla'$. 
Lemma 3 A non-linear connection is a connection up to homotopy if and only if it is equivalent to a (linear) connection.

Proof: Assume that \( \nabla \) is a connection up to homotopy. Let \( U_a \) be the domain of local coordinates \( x^k \) for \( M \), and put

\[
\nabla^a_X = \nabla_X + [u^a(X), \partial],
\]

where \( u_a \in \mathcal{A}_m(U_a; \text{End}(E)) \) is given by

\[
u_a(\sum_k f_k \frac{\partial}{\partial x_k}) = -\sum_k H_t(f_k, \frac{\partial}{\partial x_k}),
\]

for all \( f_k \in C^\infty(U_a) \). Note that \( \nabla_X \) is linear on \( X \). Indeed, for any two smooth functions \( f, g \) and \( X = g \frac{\partial}{\partial x_k} \) we have

\[
\nabla^a_X f \nabla^a_X g - f \nabla^a_X (g \nabla^a_X g) = (\nabla f g \frac{\partial}{\partial x_k}, g \frac{\partial}{\partial x_k}) + f (\nabla g \frac{\partial}{\partial x_k} - [H_t(g \frac{\partial}{\partial x_k}, \partial)]) = \nabla g \nabla^a_X f - f g \nabla^a_X = 0.
\]

Next we take \( \{ \nu_a \} \) to be a partition of unity subordinate to an open cover \( \{ U_a \} \) by such coordinate domains and put \( \nabla^1 = \nabla + \sum_a \nu_a \nabla^a_X \). Then \( \nabla^1 = \nabla + [u, \partial] \) is a connection equivalent to \( \nabla \).

Lemma 4 If \( \nabla_0 \) and \( \nabla_1 \) are equivalent, then \( ch_p(\nabla^0) = ch_p(\nabla^1) \).

Proof: So, let us assume that \( \nabla^1 = \nabla^0 + [\theta, \partial] \). A simple computation shows that

\[
k \nabla = k \nabla_0 + [d\nabla(\theta) + R, \partial], \tag{11}
\]

where \( R(X, Y) = [\theta(X), [\theta(Y), \partial]] \). We denote by \( Z \subset \mathcal{A}_m(M; \text{End}(E)) \) the space of non-linear forms \( \omega \) with the property that \( [\omega, \partial] = 0 \), and by \( B \subset Z \) the subspace of element of type \( [\eta, \partial] \) for some non-linear form \( \eta \). The formula

\[
[\partial, \omega \eta] = [\partial, \omega] \eta + (-1)^{k-1} \omega [\partial, \eta]
\]

shows that \( ZB \subset B \), hence (11) implies that \( k \nabla = k \nabla_0 \mod B \). The desired equality follows now from the fact that \( Tr_\xi \) vanishes on \( B \).

For (linear) connections \( \nabla \) on \( (E, \partial) \), \( ch_p(\nabla) \) are clearly (linear) differential forms on \( M \) whose cohomology classes are (up to a constant) the components of the Chern character \( Ch(E) = Ch(E^0) - Ch(E^1) \). Hence an immediate consequence of the previous two lemmas is the following [4].

Theorem 1 If \( \nabla \) is a connection up to homotopy on \( (E, \partial) \), then \( ch_p(\nabla) = Tr_\xi(k \nabla) \) are closed differential forms on \( M \) whose De Rham cohomology classes are (up to a constant) the components of the Chern character \( Ch(E) \).
Flat characteristic classes

As usual, by flatness we mean the vanishing of the curvature forms. Theorem 1 immediately implies

**Corollary 1** If \((E, \partial)\) admits a connection up to homotopy which is flat, then \(Ch(E) = 0\).

As usual, such a vanishing result is at the origin of new “secondary” characteristic classes. Let \(\nabla\) be a flat connection up to homotopy. To construct the associated secondary classes we need a metric \(h\) on \(E\). We denote by \(\partial^h\) be the adjoint of \(\partial\) with respect to \(h\). Using the isomorphism \(E^* \cong E\) induced by \(h\) (which is anti-linear if \(E\) is complex), \(\nabla\) induces an adjoint connection \(\nabla^h\) on \((E, \partial^h)\). Explicitly,

\[
L_X h(s, t) = h(\nabla_X(s), t) + h(s, \nabla^h_X(t)).
\]

The following describes various possible definitions of the secondary classes, as well as their main properties (note that the role of \(i = \sqrt{-1}\) below is to ensure real classes).

**Theorem 2** Let \(\nabla\) be a flat connection up to homotopy on \((E, \partial), p \geq 1\).

(i) For any (linear) connection \(\nabla_0\) on \((E, \partial)\) and any metric \(h\),

\[
j^{p+1}(cs_p(\nabla, \nabla_0) + cs_p(\nabla_0, \nabla^h_0) + cs_p(\nabla^h_0, \nabla^h)) \in \mathcal{A}^{2p-1}(M)
\]

are differential forms on \(M\) which are real and closed. The induced cohomology classes do not depend on the choice of \(h\) or \(\nabla_0\), and are denoted \(u_{2p-1}(E, \partial, \nabla) \in H^{2p-1}(M)\).

(ii) For any connection \(\nabla_0\) equivalent to \(\nabla\), and any metric \(h\),

\[
j^{p+1}cs_p(\nabla_0, \nabla^h_0) \in \mathcal{A}^{2p-1}(M)
\]

are real and closed, and represent \(u_{2p-1}(E, \partial, \nabla)\) in cohomology.

(iii) If \(\nabla\) is equivalent to a metric connection (i.e. a connection which is compatible with a metric), then all the classes \(u_{2p-1}(E, \partial, \nabla)\) vanish.

(iv) If \(\nabla \sim \nabla'\), then \(u_{2p-1}(E, \partial, \nabla) = u_{2p-1}(E, \partial, \nabla')\).

(v) If \(\nabla\) is a flat connection up to homotopy on both super-complexes \((E, \partial)\) and \((E, \partial')\), then \(u_{2p-1}(E, \partial, \nabla) = u_{2p-1}(E, \partial', \nabla)\).

(vi) Assume that \(E\) is real. If \(p\) is even then \(u_{2p-1}(E, \partial, \nabla) = 0\). If \(p\) is odd, then for any connection \(\nabla_0\) equivalent to \(\nabla\), and any metric connection \(\nabla_m\),

\[
(-1)^{p+1}cs_p(\nabla_0, \nabla_m) \in \mathcal{A}^{2p-1}(M)
\]

are closed differential forms whose cohomology classes equal to \(\frac{1}{2}u_{2p-1}(E, \partial, \nabla)\).

Note the compatibility with the classical flat characteristic classes, which correspond to the case where \(E\) is a graded vector bundle (and \(\partial = 0\)), or, more classically, just a vector bundle over \(M\). As references for this we point out [9] (for the approach in terms of frame bundles and Lie algebra cohomology), and [1] (for an explicit approach which we follow here). For the proof of the theorem we need the following
Lemma 5  Given the non-linear connections $\nabla, \nabla_0, \nabla_1$,

(i) If $\nabla_0$ and $\nabla_1$ are connections up to homotopy then $cs_p(\nabla_0, \nabla_1)$ are differential forms;

(ii) If $\nabla_0 \sim \nabla_1$, then $cs_p(\nabla_0, \nabla_1) = 0$;

(iii) For any metric $h$, $ch_p(\nabla^h) = (-1)^p ch_p(\nabla)$ and $cs_p(\nabla^h_0, \nabla^h_1) = (-1)^p cs_p(\nabla_0, \nabla_1)$.

Proof: (i) follows from the fact that Chern characters of connections up to homotopy are differential forms. For (ii) we use Lemma 4. The affine combination $\nabla$ used in the definition of $cs_p(\nabla_0, \nabla_1)$ is equivalent to the pull-back $\nabla_0$ of $\nabla_0$ to $M \times \Delta^1$ (because $\nabla = \nabla_0 + t[\theta, \partial]$), while $ch_p(\nabla_0)$ is clearly zero. If $h$ is a metric on $E$, a simple computation shows that $k_\nabla(X, Y)$ coincides with $-k_\nabla(X, Y)^*$ where $*$ denotes the adjoint (with respect to $h$). Then (iii) follows from $Tr(A^*) = Tr(A)$ for any matrix $A$. □

Proof of Theorem 2: (i) Let us denote by $u(\nabla, \nabla_0, h)$ the forms (12). Since $(\nabla_0, \nabla_0^h)$ is a pair of connections on $E$, and $(\nabla, \nabla_0), (\nabla^h, \nabla^h_0)$ are pairs of connections up to homotopy on $(E, \partial)$ and $(E, \partial^h)$, respectively, it follows from (i) of Lemma 5 that $u(\nabla, \nabla_0, h)$ are differential forms. From Stokes formula (9) it immediately follows that they are closed. To prove that they are real we use (iii) of the previous Lemma:

$$u(\nabla, \nabla_0, h) = (-i)^{p+1}(cs_p(\nabla, \nabla_0) + cs_p(\nabla_0, \nabla) + cs_p(\nabla^h_0, \nabla^h_1)) =$$

$$+(-i)^{p+1}(-1)cs_p(\nabla^h, \nabla^h_0) + cs_p(\nabla^h_0, \nabla_0) + cs_p(\nabla, \nabla) =$$

$$= (-i)^{p+1}(-1)^p(-1)u(\nabla, \nabla_0, h) = u(\nabla, \nabla_0, h)$$

If $\nabla_1$ is another connection, using (9) again, it follows that $u(\nabla, \nabla_0, h) - u(\nabla, \nabla_1, h) = i^{p+1}dv$ where $v$ is the (linear!) differential form 

$$v = cs_p(\nabla, \nabla_0, \nabla_1) - cs_p(\nabla^h, \nabla^h_0, \nabla^h_1) + cs_p(\nabla_0, \nabla^h_0, \nabla_1) - cs_p(\nabla^h_0, \nabla^h_1)$$

(iii) clearly follows from (ii), which in turn follows from (ii) of Lemma 5 and the fact that $\nabla \sim \nabla_0$ implies $\nabla^h \sim \nabla^h_0$. To see that our classes do not depend on $h$, it suffices to show that given a linear connection $\nabla$ on a vector bundle $F$, $cs_p(\nabla, \nabla^h)$ is independent of $h$ up to the boundary of a differential form. Let $h_0$ and $h_1$ be two metrics. Although the proof below works for general $\nabla$'s, simpler formulas are possible when $\nabla$ is flat. So, let us first assume that (actually we may assume that $\nabla$ is the canonical connection on a trivial vector bundle).

From Stokes' formula (9) applied to $(\nabla, \nabla^h_0, \nabla^h_1)$, it suffices to show that $cs_p(\nabla^h_0, \nabla^h_1)$ is a closed form. We choose a family $h_t$ of metrics joining $h_0$ and $h_1$. One only has to show that $\frac{d}{dt}cs_p(\nabla^h_0, \nabla^h_1)$ are closed forms. Writing $h_t(x, y) = h_0(u_t(x), y)$, these Chern-Simons forms are, up to a constant, $Tr(u_t^{2p-1})$ where 

$$\omega_t = \nabla^{h_t} - \nabla^{h_0} = u_t^{-1}d_{\nabla_0}(u_t)$$

(here is where we use the flatness of $\nabla$). A simple computation shows that 

$$\frac{d\omega_t}{dt} = d_{\nabla_0}(v_t) + [\omega_t, v_t]$$

where $v_t = u_t^{-1} \frac{dh_t}{dt}$. Since $d_{\nabla_0}(\omega_t^2) = 0$, this implies 

$$\frac{d\omega_t}{dt} \omega_t^{2p-2} = d_{\nabla_0}(v_t \omega_t^{2p-2}) + [\omega_t, v_t \omega_t^{2p-2}]$$


Now, by the properties of the trace it follows that

$$\frac{\partial}{\partial t} Tr_s(\omega_i^{2p-1}) = dTr_s(\omega_i^{2p-2})$$

as desired. Assume now that $\nabla$ is not flat. We choose a vector bundle $F'$ together with a connection $\nabla'$ compatible with a metric $h'$, such that $\tilde{F} = F \oplus F'$ admits a flat connection $\nabla_0$. We put $\nabla = \nabla \oplus \nabla'$ and, for any metric $h$ on $F$, we consider the metric $\hat{h} = h \oplus h'$ on $\tilde{F}$. Clearly $c_{sp}(\nabla, \nabla^h) = c_{sp}(\nabla, \nabla^h)$. Using also (iii) of Lemma 5 and Stokes' formula, we have:

$$c_{sp}(\nabla, \nabla^h) = c_{sp}(\nabla_0, \nabla_0^h) - c_{sp}(\nabla_0, \nabla) + (-1)^p c_{sp}(\nabla_0, \nabla)$$

$$+ d(c_{sp}(\nabla_0, \nabla, \nabla^h) - c_{sp}(\nabla_0, \nabla_0, \nabla^h)).$$

Hence, by the flat case, $c_{sp}(\nabla, \nabla^h)$ modulo exact forms does not depend on $h$.

For (iv) one uses Stokes' formula (9) and (ii) of Lemma 5 to conclude that $c_{sp}(\nabla', \nabla_0) = c_{sp}(\nabla, \nabla_0)$ is the differential of the linear form $c_{sp}(\nabla, \nabla', \nabla_0)$. To prove (v) we only have to show (see (i)) that there exists a linear connection $\nabla^0$ on $E$ which is compatible with both $\nabla$ and $\nabla'$. For this, one defines $\nabla^0$ locally by $\nabla^0 = f\nabla_a \alpha_k^k$, and then use a partition of unity argument.

We now assume that $E$ is real. From Lemma 5,

$$c_{sp}(\nabla_m, \nabla^h_0) = (-1)^p c_{sp}(\nabla^h_m, \nabla_0) = (-1)^{p+1} c_{sp}(\nabla_0, \nabla_m).$$

Combined with Stokes' formula (9), this implies

$$dc_{sp}(\nabla_0, \nabla_m, \nabla^h_0) = (1 + (-1)^{p+1}) c_{sp}(\nabla_0, \nabla_m) - c_{sp}(\nabla_0, \nabla^h_0),$$

which proves (vi). □

Note that the construction of the flat characteristic classes presented here actually works for $\nabla$'s which are “flat up to homotopy”, i.e. whose curvatures are of type $[-, \delta]$. Moreover, this notion is stable under equivalence, and the flat characteristic classes only depend on the equivalence class of $\nabla$ (cf. (iv) of the Theorem). Note also that, as in [4] (and following [1]), there is a version of our discussion for super-connections [11] up to homotopy. Some of our constructions can then be interpreted in terms of the super-connection $\delta + \nabla$.

If $E$ is regular in the sense that $Ker(\delta)$ and $Im(\delta)$ are vector bundles, then so is the cohomology $H(E, \delta) = Ker(\delta)/Im(\delta)$, and any connection up to homotopy $\nabla$ on $(E, \delta)$ defines a linear connection $H(\nabla)$ on $H(E)$. Moreover, $H(\nabla)$ is flat if $\nabla$ is, and the characteristic classes $u_{2p-1}(E, \delta, \nabla)$ probably coincide with the classical [1, 9] characteristic classes of the flat vector bundle $H(E, \delta)$. In general, the $u_{2p-1}(E, \delta, \nabla)$'s should be viewed as invariants of $H(E, \delta)$ constructed in such a way that no regularity assumption is required. Let us discuss here an instance of this. We say that $E$ is $Z$-graded if it comes from a cochain complex

$$0 \rightarrow E(0) \rightarrow E(1) \rightarrow \ldots \rightarrow E(n) \rightarrow 0,$$

(14)

In other words, it must be of type $E = \oplus_{i=0}^n E(k)$ with the even/odd $Z$-grading, and with $\delta(E(k)) \subset E(k+1)$. As usual, we say that $E$ is acyclic if $Ker(\nabla) = Im(\nabla)$ (i.e. if (14) is exact).
Proposition 1

(i) If $(E, \partial)$ is acyclic, then any two connections up to homotopy on $(E, \partial)$ are equivalent. Moreover, if $E$ is $\mathbb{Z}$-graded, then $u_{2p-1}(E, \partial, \nabla) = 0$.

(ii) If $(E^k, \partial^k, \nabla^k)$ are $\mathbb{Z}$-graded complexes of vector bundles endowed with flat connections up to homotopy which fit into an exact sequence

$$0 \longrightarrow E^0 \xrightarrow{\delta} E^1 \xrightarrow{\delta} \cdots \xrightarrow{\delta} E^n \longrightarrow 0$$

compatible with the structures (i.e. $[\delta, \partial] = [\delta, \nabla] = [\delta, H_\nabla] = 0$), then

$$\sum_{k=0}^{n} (-1)^k u_{2p-1}(E^k, \partial^k, \nabla^k) = 0.$$  

Proof: The second part follows from (i) above and (v) of Theorem 2. To see this, we form the super-vector bundle $E = \oplus_k E^k$ (which is $\mathbb{Z}$-graded by the total degree) and the direct sum (non-linear) connection $\nabla$ acting on $E$. Then $\nabla$ is a connection up to homotopy in both $(E, \partial)$ and $(E, \partial + \delta)$. Clearly $u_{2p-1}(E, \partial, \nabla) = \sum_{k=0}^{n} (-1)^k u_{2p-1}(E^k, \partial^k, \nabla^k)$, while the exactness of (15) implies that $\partial + \delta$ is acyclic. Hence we are left with (i). For the first part we remark that the acyclicity assumption implies that $\partial^* \partial + \partial \partial^*$ is an isomorphism (“Hodge”). Then any operator $u$ which commutes with $\partial$ can be written as a commutator $[-v, \partial]$ where

$$v = ua, \quad a = -(\partial^* \partial + \partial \partial^*)^{-1} \partial^*.$$  

This applies in particular to $u = \nabla' - \nabla$ for any two connections up to homotopy on $(E, \partial)$. We now have to prove that $cs_p(\nabla, \nabla^h)$ is zero in cohomology, where $\nabla$ is a linear connection on $(E, \partial)$, and $h$ is a metric. For this we use a result of [1] (Theorem 2.16) which says that $cs_p(A, A^h)$ are closed forms provided $A = A_0 + A_1 + A_2 + \ldots$ is a flat super-connection [11] on $E$ with the properties:

(i) $A_1$ is a connection on $E$ preserving the $\mathbb{Z}$-grading,

(ii) $A_k \in \mathcal{A}^k(M; \text{Hom}(E^*, E^{*+1-k}))$ for $k \neq 1$.

Lemma 6 Given a (linear) connection $\nabla$ on the acyclic cochain complex (14), there exists a super-connection on $E$ of type

$$A = \partial + \nabla + A_2 + A_3 + \ldots : \mathcal{A}(M; E) \longrightarrow \mathcal{A}(M; E),$$

which is flat and satisfies (i) and (ii) above.

Let us show that this lemma, combined with the result of [1] mentioned above, prove the desired result. Using Stokes’ formula it follows that

$$cs_p(\nabla, \nabla^h) = cs(A, A^h) + d(cs_p(\nabla, \nabla^h, A^h)) - cs_p(\nabla, A, A^h) +$$

$$+ cs_p(\nabla, A) - cs_p(\nabla^h, A^h),$$

and we show that $cs_p(\nabla, A) = 0$ (and similarly that $cs_p(\nabla^h, A^h) = 0$). Writing $\theta = A - \nabla$ and using the definition of the Chern-Simons forms, it suffices to prove that

$$Tr_p(((1 - t^2)\nabla^2 + (t - t^2)[\nabla, \theta])^{p-1}\theta) = 0.$$
for any $t$. Since the only endomorphisms of $E$ which count are those preserving the degree, we see that the only term which can contribute is $Tr_s(\nabla^2(p-2)[\nabla, \theta]) = Tr_s(\nabla^2(p-2)[\nabla, A_2])$. But $\nabla^2(p-2)[\nabla, A_2] \partial$ commutes with $\partial$ hence its super-trace must vanish (since $Tr_s$ commutes with taking cohomology).

\[ \Box \]

**Proof of Lemma 6:** (Compare with [6]). The flatness of $A$ gives us certain equations on the $A_k$’s that we can solve inductively, using the same trick as in (16) above. For instance, the first equation is $[\partial, A_2] + \nabla^2 = 0$. Since $u_1 = \nabla^2$ commutes with $\partial$, this equation will have the solution $A_2 = u_1 a$ (with $a$ as in (16)). The next equation is $[\partial, A_3] + [A_1, A_2] = 0$. It is not difficult to see that $u_2 = [A_1, A_2]$ commutes with $\partial$, and we put $A_3 = u_2 a$. Continuing this process, at the $n$-th level we put $A_{n+1} = u_n a$ where $u_n = [\nabla, A_n] + [A_1, A_{n-1}] + \ldots$ as they arise from the corresponding equation. We leave to the reader to show that the $u_n$’s also satisfy the equations

\[ u_n = u_{n-1}[\nabla, a] + \left( \sum_{i+j=n-1} u_i u_j \right) a^2. \]

Since $[\partial, a] = -1$, $\partial$ will commute with both $[\nabla, a]$ and $a^2$, hence also with the $u_n$’s (induction on $n$). It then follows that $A_{n+1}$ satisfies the desired equation $[\partial, A_{n+1}] = -u_n$.  

\[ \Box \]

**Application to algebroids**

Recall [10] that an **algebroid** over $M$ consists of a Lie bracket $[\cdot, \cdot]$ defined on the space $\mathfrak{g}$ of sections of a vector bundle $\mathfrak{g}$ over $M$, together with a morphism of vector bundles $\rho : \mathfrak{g} \to TM$ (*the anchor of $\mathfrak{g}$*) satisfying $[X, fY] = f[X,Y] + \rho([X,Y]) f \cdot Y$ for all $X, Y \in \mathfrak{g}$ and $f \in C^\infty(M)$. Important examples are tangent bundles, Lie algebras, foliations, and algebroids associated to Poisson manifolds. It is easy to see (and has already been remarked in many other places [10], [3], [7], etc. etc.) that many of the basic constructions involving vector fields have a straightforward $\mathfrak{g}$-version (just replace $\chi(M)$ by $\mathfrak{g}$). Let us briefly point out some of them.

(a) **Cohomology:** the Lie-type formula (4) for the classical De Rham differential makes sense for $X \in \mathfrak{g}$ and defines a differential $d$ on the space $C^\ast(\mathfrak{g}) = \Lambda^\ast \mathfrak{g}^\ast$, hence a cohomology theory $H^\ast(\mathfrak{g})$. Particular cases are De Rham cohomology, Lie algebra cohomology, foliated cohomology, and Poisson cohomology.

(b) **Connections and Chern characters:** According to the general philosophy, $\mathfrak{g}$-connections on a vector bundle $E$ over $M$ are linear maps $\mathfrak{g} \times E \to E$ satisfying the usual identities. Using their curvatures, one obtains $\mathfrak{g}$-Chern classes $Ch^\theta(E) \in H^\ast(\mathfrak{g})$ independent of the connection.

(c) **Representations:** Motivated by the case of Lie algebras, and also by the relation to groupoids (see e.g. [3]), vector bundles $E$ over $M$ together with a flat $\mathfrak{g}$-connection are called representations of $\mathfrak{g}$. This time $\nabla$ should be viewed as an (infinitesimal) action of $\mathfrak{g}$ on $E$.

(d) **Flat characteristic classes:** The explicit approach to flat characteristic classes (as e.g. in [1], or as in the previous section) has an obvious $\mathfrak{g}$-version. Hence, if $E$ is a representation
of $\mathfrak{g}$, then $\text{Ch}_0(E) = 0$, and one obtains the secondary characteristic classes $u_{2p-1}(E) \in H^{2p-1}(\mathfrak{g})$. Maybe less obvious is the fact that one can also extend the Chern-Weil type approach, at the level of frame bundles (as e.g. in [9]). This has been explained in [3], and has certain advantages (e.g. for proving "Morita invariance" of the $u_{2p-1}(E)$'s and for relating them to differentiable cohomology). 

(e) Up to homotopy: All the constructions and results of the previous sections carry over to algebroids without any problem. As above, a representation up to homotopy of $\mathfrak{g}$ is a supercomplex (10) of vector bundles over $M$, together with a flat $\mathfrak{g}$-connection up to homotopy.

(f) The adjoint representation: The main reason for working "up to homotopy" is that the adjoint representation of $\mathfrak{g}$ only makes sense as a representation up to homotopy [5]. Roughly speaking, it is the formal difference $\mathfrak{g} - TM$. The precise definition is:

$$\text{Ad}(\mathfrak{g}): \quad \mathfrak{g} \xrightarrow{\rho} TM,$$

with the flat $\mathfrak{g}$-connection up to homotopy $\nabla^{ad}$ given by $\nabla^{ad}_X(Y) = [X,Y]$, $\nabla^{ad}(V) = [\rho(X), Y]$ (and the homotopies $H(f,X)(Y) = 0$, $H(f,X)(V) = V(f,X)$, for all $X,Y \in \mathfrak{g}$, $V \in \mathcal{X}(M)$).

Let us denote by $u^g_{2p-1}$ the characteristic classes $u_{2p-1}(\text{Ad}(\mathfrak{g}))$ of the adjoint representation. The most useful description from a computational (but not conceptual) point of view is given by (vi) of Theorem 2 (more precisely, its $\mathfrak{g}$-version).

1 Definition We call basic $\mathfrak{g}$-connection any $\mathfrak{g}$-connection on $\text{Ad}(\mathfrak{g})$ which is equivalent to the adjoint connection $\nabla^{ad}$.

It is not difficult to see that any such connection is also basic in sense of [7] (and the two notions are equivalent at least in the regular case). Hence we have the following possible description of the $u^g_{2p-1}$'s, which shows the compatibility with Fernandes' intrinsic characteristic classes [7, 8]:

$$u^g_{2p-1} = \begin{cases} 0 & \text{if } p = \text{even} \\ \frac{1}{2}(-1)^{\frac{p+1}{2}} c_p(\nabla_{\text{bas}}, \nabla_m) & \text{if } p = \text{odd} \end{cases}$$

where $\nabla_{\text{bas}}$ is any basic $\mathfrak{g}$-connection, and $\nabla_m$ is any metric connection on $\mathfrak{g} \oplus TM$. Hence the conclusion of our discussion is the following (which can also be taken as definition of the characteristic classes of [7, 8]).

Theorem 3 If $E$ is a representation up to homotopy then $\text{Ch}_0(E) = 0$, and the secondary characteristic classes $u_{2p-1}(E) \in H^{2p-1}(\mathfrak{g})$ of representations [4] can be extended to such representations up to homotopy. When applied to the adjoint representation $\text{Ad}(\mathfrak{g})$, the resulting classes $u^g_{2p-1}$ are (up to a constant) the intrinsic characteristic classes of $\mathfrak{g}$ [7].

More on basic connections: Let us try to shed some light on the notion of basic $\mathfrak{g}$-connection. In our context these are the linear connections which are equivalent to the adjoint connection, while in [7] they appear as a natural extension of Bott's basic connections.
Proposition 2 Let $\nabla$ be a connection on the vector bundle $\mathfrak{g}$. Then the formulas
\[
\nabla^0_X(Y) = [X,Y] + \nabla_{\rho(Y)}(X) \\
\nabla^1_X(V) = [\rho(X),V] + \rho(\nabla_V(X))
\]
$(X,Y \in \mathfrak{g}, V \in TM)$ define a basic $\mathfrak{g}$-connection $\tilde{\nabla} = (\nabla^0, \nabla^1)$.

Proof: We have $\tilde{\nabla} = \nabla^0 + [\theta, \partial]$, where $\theta$ is the (non-linear) $\text{End}(\text{Ad}(\mathfrak{g}))$-valued form on $\mathfrak{g}$ given by $\theta(X)(V) = \nabla_V(X)$, $\theta(X)(Y) = 0$. □

Depending on the special properties of $\mathfrak{g}$, there are various other useful basic connections. This happens for instance when $\mathfrak{g}$ is regular, i.e. when the rank of the anchor $\rho$ is constant. Let us argue that, in this case, the adjoint representation is (up to homotopy) the formal difference $K - \nu$, where $K$ is the kernel of $\rho$, and $\nu$ is the normal bundle $TM/F$ of the foliation $F = \rho(\mathfrak{g})$. This time, Bott's formulas [2] truly make sense on $\nu$ and $K$, making them into honest representations of $\mathfrak{g}$:
\[
\nabla_X(\hat{Y}) = [X,Y], \quad \forall X \in \mathfrak{g}, \hat{Y} \in \nu \\
\nabla_X(Y) = [X,Y], \quad \forall X \in \mathfrak{g}, Y \in K.
\]

Now, choosing splittings $\alpha : \mathcal{F} \rightarrow \mathfrak{g}$ for $\rho$, and $\beta : TM \rightarrow \mathcal{F}$ for the inclusion, we have induced decompositions
\[
\mathfrak{g} \cong K \oplus \mathcal{F}, \quad TM \cong \nu \oplus \mathcal{F}.
\]

As mentioned above, the formal difference $K - \nu$ (view it as a graded complex with $K$ in even degree, $\nu$ in odd degree, and zero differential) is a representation of $\mathfrak{g}$. On the other hand, any $\mathcal{F}$-connection $\nabla$ on $\mathcal{F}$ defines a $\mathfrak{g}$-connection on the super-complex
\[
D(\mathcal{F}) : \mathcal{F} \xrightarrow{id} \mathcal{F}
\]
(and its homotopy class does not depend on $\nabla$). Hence one has an induced $\mathfrak{g}$-connection $\nabla^{\alpha,\beta}$ on $\text{Ad}(\mathfrak{g})$, so that $(\text{Ad}(\mathfrak{g}), \nabla^{\alpha,\beta})$ is isomorphic to $(K - \nu) \oplus D(\mathcal{F})$. Explicitly,
\[
\nabla^{\alpha,\beta}_X(Y) = [X,Y - \alpha\rho(Y)] + \alpha\nabla_{\rho(Y)}(\rho X) \\
\nabla^{\alpha,\beta}_X(V) = [\rho(X),V] - \beta[\rho(X),\rho V] + \nabla_{\rho(X)}(\beta(V))
\]
for all $X, Y \in \mathfrak{g}, V \in \mathcal{X}(M)$. Note that the second part of the following proposition can also be derived from (iv) of Proposition 1.

Proposition 3 Assume that $\mathfrak{g}$ is regular. For any $\mathcal{F}$-connection $\nabla$ on $\mathcal{F}$, and any splittings $\alpha, \beta$ as above, $\nabla^{\alpha,\beta}$ is a basic $\mathfrak{g}$-connection. In particular
\[
u^{\beta}_{2p-1} = u^{\beta}_{2p-1}(K) - u^{\beta}_{2p-1}(\nu),
\]
where $K$ and $\nu$ are the representations of $\mathfrak{g}$ defined by Bott’s formulas (18), (19).
Proof: We have $\nabla^{\alpha, \beta} = \nabla^{\text{ad}} + [\theta, \partial]$, where $\theta$ is the End(Ad($g$))-valued non-linear form which is given by

$$\theta(X)(V) = \alpha[\rho(X), \beta(V)] - \alpha \beta[\rho(X), V] - [X, \alpha \beta(V)] + \alpha \nabla_{\rho(X)} \beta(V)$$

for $V \in (TM)$, while $\theta(X) = 0$ on $g$ (we leave to the reader to check that the previous formula is $C^\infty(M)$-linear on $V$). □

References


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