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# A Note on the Campbell–Hausdorff Formula

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TO NATHAN JACOBSON ON HIS 70TH BIRTHDAY

In his book ("Lie Algebras," Interscience, 1962) Jacobson proves the Campbell–Hausdorff formula for formal power series in Lie algebras. In this short note we shall prove it for finite-dimensional Lie groups making use of parts of Jacobson's proof.

1. Let $G$ be a real or complex Lie group, $\mathfrak{g}$ its Lie algebra. For $A$ and $B$ in $\mathfrak{g}$ and for sufficiently small $s$,

$$\log(\exp(sA)\exp(sB)) = \sum_{n=1}^{\infty} s^n F_n = F(s),$$

a convergent series with $F_n = F_n(A, B)$ homogeneous of degree $n$ in the coordinates of $A$ and $B$. Differentiating the relation

$$\exp(sA)\exp(sB) = \exp(F(s))$$

with respect to $s$ yields

$$\operatorname{Ad}\exp(-sB) \circ d\exp(sA)A + d\exp(sB)B = d\exp(F(s))F'(s).$$

Since $\operatorname{Ad}\exp X = \sum_{n=0}^{\infty} (n!)^{-1}(\operatorname{ad} X)^n$ and $d\exp = \sum_{n=0}^{\infty} (-1)^n((n+1)!)^{-1}$ $(\operatorname{ad} X)^n$,

one gets

$$\sum_{n=0}^{\infty} (-1)^n(n!)^{-1}s^n(\operatorname{ad} B)^n A + B$$

$$= \sum_{i=0}^{\infty} (-1)^i((i+1)!)^{-1}(\operatorname{ad} F(s))^i \sum_{j=1}^{\infty} jF_j s^{j-1}$$

$$= \sum_{n=0}^{\infty} s^n \sum_{i_1+\cdots+i_p=n+1} (-1)^{p-1}(p!)^{-1} i_p \left(\prod_{j=1}^{p-1} \operatorname{ad} F_{i_j}\right) F_{i_p}.$$



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Comparing the terms with $s^n$ on either side one finds

$$\begin{aligned} F_1 &= A + B, \\ F_{n+1} &= (-1)^n((n+1)!)^{-1}(\operatorname{ad} B)^n A \\ &\quad + \sum_{\substack{i_1+\cdots+i_p=n+1 \\ p\geqslant 2}} (-1)^p(p!(n+1))^{-1}\left(\prod_{j=1}^{p-1} \operatorname{ad} F_{i_j}\right) F_{i_p}. \end{aligned}$$

This formula shows that each $F_n$ is a homogeneous Lie polynomial of degree $n$ in $A$ and $B$ with rational coefficients which are independent of $G$ and $\mathfrak{g}$. Although convenient for computation of $F_n$ for small values of $n$, it is not suitable for deriving a general formula for $F_n$ as a Lie polynomial in $A$ and $B$.

**2.** In the free associative algebra $\mathscr{F}$ generated by two elements $X$ and $Y$ (see [1]) we have the set $\mathscr{F}_i$ of the homogeneous elements of degree $i$. Define the ideal $\mathscr{I}_m = \oplus_{i>m} \mathscr{F}_i$ and the *truncated free associative algebra* $\mathscr{F}^{(m)} = \mathscr{F}/\mathscr{I}_m$. Since the free Lie algebra $\mathscr{F}\mathscr{L}$ is spanned by its homogeneous parts $\mathscr{F}\mathscr{L} \cap \mathscr{F}_i$, we also have the *truncated free Lie algebra*

$$\mathscr{F}\mathscr{L}^{(m)} = \mathscr{F}\mathscr{L}/\mathscr{F}\mathscr{L} \cap \mathscr{I}_m \subseteq \mathscr{F}^{(m)}.$$

The projection of $\mathscr{F}$ onto $\mathscr{F}^{(m)}$ induces a linear isomorphism between $\sum_{i=0}^m \mathscr{F}_i$ and $\mathscr{F}^{(m)}$, and similarly for $\mathscr{F}\mathscr{L}^{(m)}$. It is easily seen that the Specht–Wever theorem [1, Theorem 8, p. 169] is still valid in $\mathscr{F}^{(m)}$ for homogeneous elements of degree $n \leqslant m$.

**3.** The invertible elements of $\mathscr{F}^{(m)}$ form a Lie group whose Lie algebra is $\mathscr{F}_L^{(m)}$, the Lie algebra obtained from $\mathscr{F}^{(m)}$ by taking as Lie product $[u, v] = uv - vu$. The exponential mapping and the logarithm are given by the usual power series ending with terms of degree $m$. Hence the same computations as in [1, p. 173] yield the result for $n \leqslant m$:

$$F_n(X, Y) = \sum_{\substack{p_1+q_1+\cdots+p_k+q_k=n \\ \text{all } p_i+q_i>0}} (-1)^{k-1}(p_1!\, q_1! \cdots p_k!\, q_k!\, kn)^{-1} D(X^{p_1}Y^{q_1} \cdots X^{p_k}Y^{q_k}),$$

where

$$\begin{aligned} D(Z_1 Z_2 \cdots Z_t) &= \operatorname{ad} Z_1 \circ \operatorname{ad} Z_2 \circ \cdots \circ \operatorname{ad} Z_{t-1}(Z_t) \\ &= [Z_1, [Z_2, \ldots, [Z_{t-1}, Z_t] \cdots]] \end{aligned}$$

for $Z_i = X$ or $Y$.

## Reference

1. N. Jacobson, "Le Algebras," Interscience, New York/London, 1962.