JOURNAL OF ALGEBRA 62, 393–399 (1980)

A Formula for the Characteristic Function of the Unipotent Set of a Finite Chevalley Group

T. A. SPRINGER

Mathematisch Instituut der Rijksuniversiteit, Utrecht, Netherlands

Communicated by I. N. Herstein

Received November 13, 1978

TO NATHAN JACOBSON ON HIS 70TH BIRTHDAY

1. INTRODUCTION

Let $k = \mathbb{F}_q$ be a finite field, of characteristic p. If $\mathbf{G}$ is a linear algebraic group defined over k, we denote by G its group $\mathbf{G}(k)$ of k-rational points. If $R_u\mathbf{G}$ is the unipotent radical of $\mathbf{G}$ we write $R_uG = R_u\mathbf{G}(k)$. Moreover, $q^{d(G)}$ denotes the order of a p-Sylow subgroup of G/R_uG.

Assume $\mathbf{G}$ connected and reductive. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$, and $\mathbf{T} \subset \mathbf{B}$ a maximal torus, both defined over k. If $\mathbf{N}$ is the normalizer of $\mathbf{T}$ in $\mathbf{G}$, there is a Tits system (G, B, N), with Weyl group $W = N/T$.

The parabolic subgroups $P \supset B$ (in the sense of the theory of Tits systems) are the k-rational points of the parabolic subgroups $\mathbf{P} \supset \mathbf{B}$ of $\mathbf{G}$, and P determines $\mathbf{P}$ uniquely (see [1, p. 92]). We denote by $s(P)$ the semisimple k-rank of $\mathbf{P}$.

Let $\mathbf{V}$ be the unipotent variety of $\mathbf{G}$, i.e., the set of its unipotent elements. By a result of Steinberg [9, p. 98] the number of elements of the set of its k-rational points $V = \mathbf{V}(k)$ is $q^{2d(G)}$. One also knows that now $d(G) = \dim \mathbf{G}/\mathbf{B}$.

Theorem 1 gives a formula which expresses the characteristic function ϵ_V of $V \subset G$ as a linear combination of induced characters. If H is a subgroup of G and f a class function on H, we denote by $\mathrm{Ind}_H{}^G(f)$ the induced class function on G. Also, 1_H denotes the constant function 1 on H. If S is a finite set, denote by $|S|$ its cardinal.

We can now state Theorem 1.

THEOREM 1. *We have*

$$\epsilon_V = q^{d(G)} \sum_{P \supset B} (-1)^{s(P)} |P|^{-1} \mathrm{Ind}_{R_uP}^G(1_{R_uP}). \tag{1}$$

0021-8693/80/020393-07$02.00/0

2. Proof of the Theorem

Denote by $\langle\, ,\, \rangle_H$ the standard hermitian form on the space of complex valued functions on H:

$$\langle f, g\rangle_H = |H|^{-1} \sum_{x\in H} f(x)\overline{g(x)}.$$

By Steinberg's result, quoted before, we have

$$\langle \epsilon_V, \epsilon_V\rangle_G = |G|^{-1} q^{2d(G)}.$$

Let η be the right-hand side of (1). Then

$$\begin{aligned}\langle \epsilon_V, \eta\rangle_G &= q^{d(G)} \sum_{P\supset B} (-1)^{s(P)} |P|^{-2} \langle \epsilon_V, \operatorname{Ind}_{R_uP}^G(1_{R_uP})\rangle_G \\ &= q^{d(G)} \sum_{P\supset B} (-1)^{s(P)} |P|^{-1},\end{aligned}$$

by Frobenius duality. By another result of Steinberg [9, p. 14] the last sum equals $q^{d(G)} |G|^{-1}$, whence

$$\langle \epsilon_V, \eta\rangle_G = q^{2d(G)} |G|^{-1} = \langle \eta, \eta\rangle_G .$$

The burden of the proof is to show that also

$$\langle \eta, \eta\rangle_G = q^{2d(G)} |G|^{-1}. \tag{2}$$

If (2) has been established, we have $\langle \epsilon_V - \eta, \epsilon_V - \eta\rangle_G = 0$, and (1) follows.

It follows immediately from Mackey's formula that

$$\langle \eta, \eta\rangle_G = q^{2d(G)} \sum_{P,Q\supset B} (-1)^{s(P)+s(Q)} |P|^{-1} |Q|^{-1} |R_uP\backslash G/R_uQ|. \tag{3}$$

Before dealing with the right-hand side of (3) we introduce some more notation. Let R be the (relative) root system of $\mathbf{G}$ with respect to $\mathbf{T}$, let D be the basis of R defined by $\mathbf{B}$, and denote by R^+ the corresponding set of positive roots. There is a unique bijection $S \mapsto \mathbf{P}_S$ of subsets of D onto parabolic subgroups containing $\mathbf{B}$, such that $s(\mathbf{P}_S) = |S|$. Let $\mathbf{L}_S$ denote the Levi subgroup of $\mathbf{P}_S$ containing $\mathbf{T}$.

W can be identified with the Weyl group of R. If W_S is the parabolic subgroup of W defined by S, then W_S is the Weyl group of $\mathbf{L}_S$.

If S, $S_1 \subset D$, denote by W'_{S,S_1} the "distinguished set of double coset representatives" for $W_S\backslash W/W_{S_1}$:

$$W'_{S,S_1} = \{w \in W \mid w^{-1}S \subset R^+, wS_1 \subset R^+\}$$

(see [2, p. 37]).

By a general property of Tits systems [2, p. 28] there is a bijection

$$P_S \backslash G / P_{S_1} \to W_S \backslash W / W_{S_1},$$

whence a surjective map

$$\varphi\colon R_u P_S \backslash G / R_u P_{S_1} \to W'_{S,S_1}.$$

The results of the next lemma will allow us to compute the numbers $| R_u P \backslash G / R_u Q |$ figuring in (3).

LEMMA 1. *Let* $w \in W'_{S,S_1}$.

(i) $\mathbf{P} = (\mathbf{P}_S \cap w \mathbf{P}_{S_1} w^{-1}) R_u \mathbf{P}_S$ *is a parabolic subgroup of* $\mathbf{G}$ *which contains* $\mathbf{T}$ *and lies in* $\mathbf{P}_S$. *Moreover,* $\mathbf{L}_{S \cap wS_1}$ *is the Levi subgroup of* $\mathbf{P}$ *containing* $\mathbf{S}$;

(ii) $| \varphi^{-1} w | = q^{-d(G)+d(L_{S\cap wS_1})} | P_{S \cap wS_1} |^{-1} | P_S | \, | P_T |.$

(i) is a recollection of known results. The first statement is proved in [1, p. 86]. Let $R_S \subset R$ be the root system contained in R, with basis S. The root system of the Levi subgroup L of P containing T then is $R_S \cap wR_{S_1}$, as one readily sees. Now if $\alpha \in R_S \cap wR_{S_1}$, the corresponding reflection $s_\alpha \in W$ lies in $W_S \cap wW_{S_1}w^{-1}$. By a result of Kilmoyer (for a proof see [4, p. 126]) the latter group is $W_{S\cap wS_1}$, whence $\alpha \in R_{S \cap wS_1}$. This implies that $\mathbf{L} = \mathbf{L}_{S\cap wS_1}$, as asserted.

To prove (ii), observe that each element of $\varphi^{-1}w$ is represented by an element of $L_S w L_{S_1}$. If xwx_1, $x'wx'_1$ ($x, x' \in L_S$, $x_1, x'_1 \in L_{S_1}$) lie in the same double coset, then $x^{-1}x' \in P \cap L_S$, which is the parabolic subgroup of L_S defined by $S \cap wS_1$. Moreover, if xwx_1 and xwx'_1 lie in the same double coset, then $wx_1^{-1} x'w^{-1}$ lies in $L_{wS_1} \cap wR_S P_u w^{-1}$, which is the unipotent radical of the parabolic subgroup of L_{wS_1} defined by $S \cap wS_1$. The formula of (ii) readily follows from these observations.

Using Lemma 1, we deduce from (3) that

$$\langle \eta, \eta \rangle_G = q^{d(G)} \sum_{S, S_1 \subset D} (-1)^{|S|+|S_1|} \sum_{w \in W'_{S,S_1}} | P_{S\cap wS_1} |^{0-1} q^{d(L_{S\cap wS_1})}.$$

If $w \in W$, let $D(w) = D \cap w^{-1}R^{+}$. We can rewrite this formula as

$$\langle \eta, \eta \rangle_G = q^{d(G)} \sum_{w \in W} \sum_{\substack{S, S_1 \subset D(w) \\ A \subset wS_1 \cap D}} (-1)^{|A|+|S_1|} | P_A |^{-1} q^{d(L_A)} \Big(\sum_{S \subset D(w^{-1}) - w^{-1}S_1} (-1)^{|S|} \Big).$$

The sum in brackets is 0, unless $D(w^{-1}) \subset wS_1 \subset wD(w)$, by a standard combinatorial fact.

To finish the proof of (2), we need another lemma. If $S \subset D$ we denote by $w_0(S)$ the element of maximal length in W_S. We put $w_0 = w_0(D)$.

LEMMA 2. (i) *If $w \in W$ is such that $D(w^{-1}) \subset wD$ then there is $S \subset D$ such that $w = w_0(S)w_0$.*

(ii) *If $w = w_0(S)w_0$ then $D(w^{-1}) = wD(w) = S$.*

Assume that W is as in (i) and put $S = D(w^{-1})$. We then have $w^{-1}R_S{}^+ = R^+_{w^{-1}S}$. Let $\alpha \in R^+ - R_S$. Then

$$\alpha = \sum_{\beta \in D-S} m_\beta \beta + \sum_{\gamma \in S} n_\gamma \gamma,$$

where the integers m_β, n_γ are $\geqslant 0$, and at least one of the m_β is nonzero.

If $\beta \in D - S$, then

$$w^{-1}\beta = \sum_{\delta \in D} h_\delta \, \delta,$$

where the h_δ are $\leqslant 0$, and at least one of the h_δ with $\delta \notin w^{-1}S$ is nonzero (otherwise we had $w^{-1}\beta \in -R^+_{w^{-1}S} = -w^{-1}R_S{}^+$). It follows that $w^{-1}(R^+ - R_S) \subset -R^+$, whence

$$w^{-1}w_0(S)(R^+ - R_S) = w^{-1}(R^+ - R_S) \subset -R^+.$$

Since also $w^{-1}w_0(S)\, R_S = -w^{-1}R_S \subset -R^+$, we must have $w^{-1}w_0(S) = w_0$. This proves (i). The easy proof of (ii) is omitted.

Using Lemma 2, we conclude from (4) that

$$\langle \eta, \eta \rangle_G = q^{d(G)} \sum_{S \subset D} (-1)^{|S|} \Big(\sum_{A \subset S} (-1)^{|T|} \mid P_T \mid^{-1} q^{d(L_k)} \Big).$$

The inner sum equals $| P_S |^{-1}$ [9, p. 14], whence

$$\langle \eta, \eta \rangle_G = q^{d(G)} \sum_{S \subset D} (-1)^{|S|} \mid P_S \mid^{-1} = q^{2d(G)} \mid G \mid^{-1},$$

by [loc. cit.]. This proves (2), and finishes the proof of Theorem 1.

COROLLARY 1. *Let f be a class function on G. Then*

$$\sum_{x \in V} f(x) = \sum_{P \supset B} (-1)^{s(P)} \mid G/P \mid q^{d(P)} \Big(\sum_{x \in R_u P} f(x) \Big).$$

The left-hand side equals $| G | \langle f, \epsilon_V \rangle_G$, which can be computed from (1), using Frobenius duality, giving the formula of this corollary. Clearly, the formula is equivalent to (1).

Recall that a complex valued function f on G is *parabolic* (or is a cusp form) if for any proper parabolic subgroup P of G and any $x \in G$ we have

$$\sum_{x \in R_u P} f(xy) = 0.$$

COROLLARY 2. *If f is a parabolic class function on G then*

$$\sum_{x\in V} f(x) = (-1)^{s(G)}q^{d(G)}f(e).$$

This is a direct consequence of Corollary 1. It applies, in particular, in the case that f is the character of an irreducible parabolic (or discrete series) representation of G. This confirms, in that case, a conjecture of I. G. Macdonald, according to which we should have, for any irreducible character χ of G, that

$$a(\chi) = \chi(e)^{-1} \sum_{x\in V} \chi(x)$$

is, up to sign, a power of q.

The argument used here can also be used to confirm the conjecture in the case that $\chi = \mathrm{Ind}_P{}^G\phi$, where P is a parabolic subgroup of G and ϕ a parabolic character of P/R_uP, lifted to P. See also note added in proof.

For certain other irreducible characters, the conjecture is true by a formula of Kawanaka [6, p. 541].

3. GEOMETRIC APPLICATIONS

We shall next give a more geometric formulation of the result of Theorem 1.

If $\mathbf{P}$ is a parabolic k-subgroup of $\mathbf{G}$, denote by $X(\mathbf{P})$ the k-variety $\mathbf{G}/\mathbf{P}$. For any unipotent $x \in \mathbf{G}$ put

$$X(\mathbf{P})_x = \{g\mathbf{P} \in X(\mathbf{P}) \mid g^{-1}xg \in R_u\mathbf{P}\}.$$

If $x \in G$, then $X(\mathbf{P})_x$ is defined over k. Let $X(P)_x$ be the set of its k-rational points.

THEOREM 2. *For any unipotent $x \in V$ we have*

$$\sum_{P\supset B} (-1)^{s(P)}q^{d(P)} \mid X(P)_x \mid = 1. \tag{5}$$

Let Z be the centralizer of $x \in V$ in G and C its conjugacy class. Applying Corollary 1 to the characteristic function of C, we obtain

$$\sum_{P\supset B} (-1)^{s(P)}q^{d(P)} \mid P \mid^{-1} \mid Z \mid \mid C \cap R_uP \mid = 1.$$

As is well-known, the points of $X(\mathbf{P})(k)$ are represented by the $g\mathbf{P}$ with $g \in G$. We identify $X(\mathbf{P})$ (k) with G/P, and then we have

$$X(P)_x = \{gP \in G/P \mid g^{-1}x \in R_uP\}.$$

Hence

$$|P|\ |X(P)_x| = |Z|\ |C \cap R_u(P)|.$$

Inserting this in the previous formula we obtain (5). It is readily seen that (1) is a consequence of (5) (for all $x \in V$). If $x = e$, formula (5) reduces to the formula of Steinberg [9, p. 14], used before. If $x \in G$ is a regular unipotent element, we have $|X(P)_x| = 0$ if P is not a Borel group and $|X(B)_x| = 1$, and (5) is clear. In the other cases, (5) is a nontrivial result.

The cohomological determination of the number of k-rational points of an algebraic variety over k, reveals that relations like (5), between numbers of rational points of several k-varieties, may be a reflection of geometric facts, relating these varieties (compare [3, p. 175, 1.11, 1.12]). I do not know a geometric explanation of (5), along these lines.

However, in the particular case $G = GL_n(k)$, we can draw geometric conclusions from (5). They are given in the next theorem. Assume now that $\mathbf{G} = \mathbf{GL}_n$. All unipotent classes of $\mathbf{G}$ are then represented by elements of G. We denote by $b_i(P, x)$ the ith Betti number (in the sense of l-adic cohomology, with $l \neq p$) of the variety $X(\mathbf{P})_x$.

THEOREM 3. *Let* $\mathbf{G} = \mathbf{GL}_n$, *let* $x \in V$.

(i) *We have* $b_i(P, x) = 0$ *if* i *is odd.*

(ii) *For each* $d > 0$, *we have*

$$\sum_{\substack{P \supset B \\ d(P)+i=d}} (-1)^{s(P)} b_{2i}(P, x) = 0,$$

moreover $b_0(B, x) = 1$.

(iii) *The assertions of* (i) *and* (ii), *for some* $x \in V$, *imply* (5).

It was proved by Shimomura [7] (see also [5, No. 2]) that for $\mathbf{G} = \mathbf{GL}_n$ the variety of fixed points in $X(\mathbf{P})$ of $x \in V$ can be partitioned into a finite union of locally closed subspaces, each of which is isomorphic to an affine space. It follows readily from the proof given in [5] that the subspaces of the partition are defined over k, and are k-isomorphic to an affine space. It also follows readily that our $X(\mathbf{P})_x$ have the same properties. It follows that (i) holds and that the Frobenius endomorphism acts as multiplication by q^i on the cohomology group $H^{2i}(X(\mathbf{P})_x)$. By Grothendieck's formula for numbers of k-rational points we then have

$$|X(\mathbf{P})_x| = \sum_{i \geqslant 0} q^i b_{2i}(P, x).$$

Inserting this into (5), we obtain the results stated in (ii). That (iii) holds follows by reversing the argument.

Remarks. (a) That $b_0(B, x) = 1$ simply means that the variety of fixed points of x in $\mathbf{G}/\mathbf{B}$ is connected, which is a known fact (see [8]). Theorem 3 shows that this geometric fact is hidden in (5) (if $\mathbf{G} = \mathbf{GL}_n$).

(b) From the results of [7], one could extract a combinatorial description (in terms of tableaus) of the Betti numbers $b_{2i}(P, x)$. The relations of (ii) thus can be given a combinatorial interpretation (which, however, does not appear to be very illuminating).

(c) The comparison theorem of l-adic cohomology implies that over $\mathbb{C}$ statement similar to those of Theorem 3(i) and (ii) are also true.

Note added in proof. Using Theorem 1, D. Alvis has recently obtained further results on Macdonald's conjecture, mentioned at the end of no. 2 (see D. Alvis, The duality operation in the character ring of a finite Chevalley group, *Bull. Amer. Math. Soc.*, New Series, 1 (1979), 907–911).

References

1. A. Borel and J. Tits, Groupes réductifs, *Publ. Math. IHES* **27** (1965), 55–151.
2. N. Bourbaki, "Groupes et algèbres de Lie," Chaps. IV, V, VI, Hermann, Paris, 1968.
3. P. Deligne, "Cohomologie étale," SGA 4 1/2, Lecture Notes in Mathematics 569, Springer-Verlag, Berlin/Heidelberg/New York, 1977.
4. A. van den Hombergh, About the automorphisms of the Bruhat-ordering in a Coxeter group, *Proc. Kon. Ak. v. Wet. Amsterdam (A)* **77** (1974), 125–131.
5. R. Hotta and N. Shimomura, The fixed point subvarieties of unipotent transformations on generalized flag varieties and the Green functions — combinatorial and cohomological treatments centering GL_n, to appear.
6. N. Kawanaka, Unipotent elements and characters of finite Chevalley groups, *Osaka J. Math.* **12** (1975), 523–554.
7. N. Shimomura, A theorem on the fixed point set of a unipotent transformation on the flag manifold, to appear.
8. T. A. Springer, The unipotent variety of a semisimple group, *In*: "Algebraic geometry" (papers presented at the Bombay Colloquium), pp. 373–391, Oxford Univ. Press, London, 1968.
9. R. Steinberg, Endomorphisms of linear algebraic groups, *Mem. Amer. Math. Soc.* **80** (1968).