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HIGHER ELECTRON NON-LINEARITIES IN THE DYNAMICS OF LANGMUIR COLLAPSE

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We derive a set of generalized Zakharov equations valid for both electrostatic and electromagnetic, that is, potential and non-potential, perturbations which include corrections due to higher electron non-linearities and allowing for a breakdown of slow-time scale quasi-neutrality and we show how these correction terms may halt the Langmuir collapse in two or three dimensions.

1. Introduction

An important question in contemporary plasma physics is the problem of strong plasma turbulence. It now seems to be well established both through numerical work [1, 2], and through experiments [3] that if sufficient energy is put into a plasma, cavitons will be formed, that is, localized structures which are both density depressions and electric field maxima. The numerical work is often concerned with one-dimensional models, although Nicholson and Goldman [4] deal with a two-dimensional plasma, and it is normally based on the so-called Zakharov equations which were derived by Zakharov [5] to describe the development of the modulational instability in an unmagnetized plasma. From these equations it follows that in two or three dimensions collapse will occur, and much numerical work has been devoted to a study of the existence and dynamics of the so-called Langmuir collapse (for a review of this type see, e.g., Rudakov and Tsytovich [6] or Thornhill and ter Haar [7]) it is necessary to use numerical methods as the only known analytical solution of the two- or three-dimensional Zakharov equations is the planar Langmuir soliton. Unfortunately, the numerical procedure can only be applied to the early stages of the collapse, since the Zakharov equations lose their validity when the field intensities become too large. It is the purpose of the present paper to consider a generalization of the Zakharov equations which will be valid at higher field strengths in order to be able possibly to answer the question of whether the collapse will continue until the size of the cavitons become of the order of the Debye length so that Landau damping can play a role. In this context it is interesting to note that the experiments by Antipov and collaborators [3] indicate structures which are smaller than the Langmuir solitons, but large enough for Landau damping still to be negligible.

There has been relatively little work on the limitation of the Zakharov equations. Khakimov and Tsytovich [8] (see also Tsytovich [9, a, b]) used their non-linear dielectric formalism to investigate the limit of applicability of the Zakharov equations. They claim to have taken all non-linearities up to the fifth order in the electric field into account and they derived a generalized set of equations. Using an approach similar to the original approach of Zakharov's and to the approach to be used in the present paper, Kuznetsov [10b] examined the effect of higher electrostatic non-linearities and came to the conclusion that they become important at large field amplitudes.

It is the purpose of the present study to clarify the effect of electron non-linearities upon the collapse, including both electrostatic and electromagnetic perturbations. It is important to include both potential and non-potential

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modes, as the frequently used electrostatic (potential) approximation is normally violated in the case of developed (strong) Langmuir turbulence of hot plasmas (see, e.g., Thornhill and ter Haar [7]; Nicholson and Goldman [4]). We shall follow Zakharov [5] by separating "fast" and "slow" time scales and applying a hydrodynamic approach, but we shall include higher electron non-linearities describing the scattering by stimulated fluctuations in the low-frequency electron velocity and in the $2\omega_p$-components of the electron density and velocity (ω_p: electron plasma frequency). We thus allow the possibility of the breakdown of the quasi-neutrality for slow motions (compare also [11]). We also shall apply the adiabatic scaling approach [12] to examine the collapse and the possibility of the existence of quasi-stationary structures in two or three dimensions. Although we find that in the spherically symmetric case the correction terms introduced by us halt the collapse, that is, prevent the appearance of a mathematical singularity, we have not yet investigated whether the collapse is halted before Landau damping becomes effective, that is, before the caviton reaches a size of the order of the Debye length. This means that the question whether the higher hydrodynamic non-linearities studied by us can fully stabilize the collapse still needs to be answered.

2. Derivation of the generalized Zakharov equations

We consider an isotropic, unmagnetized plasma in which the electron temperature T_e is much higher than the ion temperature T_i and we restrict ourselves to considering long-wavelength electrostatic and electromagnetic modes so that

$$(k^l r_D)^2 \ll 1, \quad (k^t c/\omega_p)^2 \ll 1, \tag{1}$$

where k^l and k^t are, respectively, the wavenumber of the electrostatic and electromagnetic waves, c is the velocity of light, and r_D is the electron Debye radius.

We consider Langmuir turbulence when the dominant plasma mode is that of the Langmuir waves with frequencies close to ω_p. We follow the original idea of well separated "slow" (ion) and "fast" electron time scales [5] and split the electron density n_e, electron velocity v_e, electric field strength E, and current density j in terms corresponding to different time scales [10b]:

$$n_e = n_0 + n_s + n_1 + n_2 + \ldots, \tag{2}$$

$$v_e = v_s + v_1 + v_2 + \ldots, \tag{3}$$

$$E = E_s + E_1 + E_2 + \ldots, \tag{4}$$

$$j = j_s + j_1 + j_2 + \ldots. \tag{5}$$

Here n_0 is the electron density corresponding to the situation where there are no waves present, the quantities with index s vary on the slow time scale, those with index 1 on the fast, ω_p time scale, those with index 2 on the fast, $2\omega_p$ time scale, and so on. It is convenient to write

$$n_1 = \tilde{n}_1 \exp(-i\omega_p t) + \tilde{n}_1^* \exp(i\omega_p t), \quad n_2 = \tilde{n}_2 \exp(-2i\omega_p t) + \tilde{n}_2^* \exp(2i\omega_p t), \tag{6}$$

$$v_1 = \tilde{v}_1 \exp(-i\omega_p t) + \tilde{v}_1^* \exp(i\omega_p t), \quad v_2 = \tilde{v}_2 \exp(-2i\omega_p t) + \tilde{v}_2^* \exp(2i\omega_p t), \tag{7}$$

and so on, where the quantities with tildes are slowly varying and where the asterisk indicates the complex conjugate quantity.

We assume that the density perturbation is not too strong, that is,

$$n_0 \gg n_s, n_1, n_2, \tag{8}$$

where, however, n_s may sometimes be taken to the comparable with, though still well below, n_0, while the basic assumption of the predominance of the Langmuir mode leads to the inequalities

$$|v_1| \gg |v_s|, |v_2|, \ldots; \quad |E_1| \gg |E_s|, |E_2|, \ldots; \ldots \tag{9}$$

Our basic equations are the Maxwell equations

$$\operatorname{div} E = 4\pi\rho, \quad \operatorname{curl} E = -\frac{1}{c}\frac{\partial H}{\partial t}, \quad \operatorname{div} H = 0, \quad \operatorname{curl} H = \frac{1}{c}\left(4\pi j + \frac{\partial E}{\partial t}\right), \tag{10}$$

where ρ is the charge density; the quantities ρ and j satisfy the relations

$$\rho = -e(n_e - n_i), \quad j = -e(n_e v_e - n_i v_i), \tag{11}$$

where n_i and v_i are the ion density and the ion velocity, and $-e$ is the electron charge.

We shall split the ion density as follows:

$$n_i = n_0 + \delta n_i, \tag{12}$$

where δn_i changes on the slow time scale and where δn_i is not necessarily equal to n_s: we are not, as in Zakharov's original treatment, assuming quasi-neutrality on the slow time scale.

The Maxwell equations (10) lead to

$$\frac{\partial^2 E}{\partial t^2} + c^2 \operatorname{curl}\operatorname{curl} E = -4\pi \frac{\partial j}{\partial t}, \tag{13}$$

which is linear in E and j and convenient for further calculations. We can split up this equation corresponding to various time scales. The equation corresponding to the ω_p-time scale is

$$\frac{\partial^2 E_1}{\partial t^2} + c^2 \operatorname{curl}\operatorname{curl} E_1 = -4\pi \frac{\partial j_1}{\partial t}. \tag{14}$$

We must draw attention to the fact that the representation (7), (8) in the case of E differs by a factor 2 from the one normally used. The quantity j_1 follows from eq. (11) and satisfies the relation

$$j_1 = -e[n_0\tilde{v}_1 + n_s\tilde{v}_1 + \tilde{n}_1 v_s + \tilde{n}_1^*\tilde{v}_2 + \tilde{n}_2\tilde{v}_1^*]. \tag{15}$$

Substituting expression (15) into eq. (14) and neglecting, as usually, $\partial^2\tilde{E}_1/\partial t^2$ as compared to $\omega_p\partial\tilde{E}_1/\partial t$ we find

$$-2i\omega_p \frac{\partial \tilde{E}_1}{\partial t} - \omega_p^2\tilde{E}_1 + c^2 \operatorname{curl}\operatorname{curl}\tilde{E}_1 = 4\pi e \frac{\partial}{\partial t}[n_0\tilde{v}_1 + n_s\tilde{v}_1 + \tilde{n}_1 v_s + \tilde{n}_1^*\tilde{v}_2 + \tilde{n}_2\tilde{v}_1^*]. \tag{16}$$

To close the equation we need expressions for n_s, v_s, $\tilde{n}_1$, $\tilde{v}_1$, $\tilde{n}_2$, and $\tilde{v}_2$. It is obvious that the dominant term on the right-hand side of eq. (16) is the one involving $n_0\tilde{v}_1$, while the other four terms are much smaller and may be regarded to be corrections.

Our basic assumptions are eq. (1), (8), and (9) and they allow us to introduce a hydrodynamic description for the electron fluid, that is, to use the equations

$$\frac{\partial n_e}{\partial t} + \operatorname{div}(n_e v_e) = 0, \tag{17}$$

$$\frac{\partial v_e}{\partial t} + (v_e \cdot \nabla) v_e = -\frac{e}{m}\left\{E + \frac{1}{c}[v_e \wedge B]\right\} - 3v_{T_e}^2 \nabla \frac{n_e}{n_0}. \tag{18}$$

Using inequalities (8) and (9) to linearize eq. (18) in n_1 and v_1 and assuming that the plasma is non-relativistic so that we can drop the term involving $\boldsymbol{B}$ we get

$$\frac{\partial \tilde{v}_1}{\partial t} = -[(\tilde{v}_1 \cdot \nabla) v_s + (\tilde{v}_1^* \cdot \nabla)\tilde{v}_2 + (v_s \cdot \nabla)\tilde{v}_1 + (\tilde{v}_2 \cdot \nabla)\tilde{v}_1^*] - \frac{e}{m}\tilde{E}_1 - 3v_{T_e}^2 \nabla \frac{\tilde{n}_1}{n_0}. \tag{19}$$

To zeroth order we get from eq. (19)

$$\tilde{v}_1^{(0)} = -\frac{\mathrm{i}e}{m\omega_p}\tilde{E}_1, \tag{20}$$

and using the Poisson equation in the form

$$\operatorname{div}\tilde{E}_1 = -4\pi e\, \tilde{n}_1, \tag{21}$$

we get to first order

$$\frac{\partial \tilde{v}_1^{(1)}}{\partial t} = \left\{\frac{\mathrm{i}e}{m\omega_p}\left[(\tilde{E}_1 \cdot \nabla) v_s + (v_s \cdot \nabla)\tilde{E}_1 - (\tilde{E}_1^* \cdot \nabla)\tilde{v}_2 - (\tilde{v}_2 \cdot \nabla)\tilde{E}_1^*\right] - \frac{e}{m}\tilde{E}_1 + \frac{3e}{m} r_D^2 \nabla(\operatorname{div}\tilde{E}_1)\right\}. \tag{22}$$

We can now use eqs. (20) and (22) to obtain from eq. (16) the relation

$$\begin{aligned} &\mathrm{i}\frac{\partial \tilde{E}_1}{\partial t} - \frac{c^2}{2\omega_p}\operatorname{curl}\operatorname{curl}\tilde{E}_1 + \frac{3}{2}\omega_p r_D^2 \operatorname{grad}\operatorname{div}\tilde{E}_1 - \frac{\omega_p}{2n_0} n_s \tilde{E}_1 \\ &= -\tfrac{1}{2}\mathrm{i}[(\tilde{E}_1 \cdot \nabla) v_s + (v_s \cdot \nabla)\tilde{E}_1 + v_s \operatorname{div}\tilde{E}_1] - \frac{\omega_p}{2n_0} n_2 \tilde{E}_1^* \\ &+ \tfrac{1}{2}\mathrm{i}[(\tilde{E}_1^* \cdot \nabla)\tilde{v}_2 + (\tilde{v}_2 \cdot \nabla)\tilde{E}_1^* - \tilde{v}_2 \operatorname{div}\tilde{E}_1^*]. \end{aligned} \tag{23}$$

If the right-hand side of eq. (23) were zero, this equation would be one of the Zakharov equations which describes both electrostatic and electromagnetic perturbations [10a, 7]. The terms on the right-hand side of eq. (23) correspond to higher electron non-linearities describing both electrostatic (Langmuir) and electromagnetic perturbations. The equation is a generalization of Kuznetsov's result [10b] in which only potential perturbations were considered. Indeed, one recovers his results by putting $\tilde{E}_1 = \nabla\psi$.

To close the set of equations we still need relations for $\tilde{n}_2$, $\tilde{v}_2$, n_s, and v_s. The electron motions at frequencies close to $2\omega_p$ can be described in the hydrodynamic framework as the phase velocity is much larger than the thermal velocity. Linearizing eqs. (17) and (18) with respect to n_2 and v_2 we find

$$\frac{\partial n_2}{\partial t} + n_0 \operatorname{div} v_2 + \operatorname{div}(n_1 v_1) = 0, \tag{24}$$

$$\frac{\partial v_2}{\partial t} + (v_1 \cdot \nabla) v_1 + \frac{e}{mc}[v_1 \wedge B_1] = -\frac{e}{m} E_2, \tag{25}$$

where E_2 which represents the electric field component due to charge separation at frequencies close to $2\omega_p$ satisfies the relation

$$\operatorname{div} E_2 = -4\pi e n_2. \tag{26}$$

Using eq. (20) for v_1 and the appropriate Maxwell equation (10) for B_1 we get from eq. (25) the following equation for v_2, where we have retained the terms with the correct frequency dependence:

$$m\frac{\partial v_2}{\partial t} = -eE_2 + F_2, \tag{27}$$

where F_2 is the potential (Miller or ponderomotive) force with frequency $2\omega_p$. If we write

$$F_2 = \tilde{F}_2 \exp(-2i\omega_p t) + \tilde{F}_2^* \exp(2i\omega_p t), \tag{28}$$

we have

$$\tilde{F}_2 = -m\left\{(\tilde{v}_1 \cdot \nabla)\tilde{v}_1 + \frac{e}{mc}[\tilde{v}_1 \wedge \tilde{B}_1]\right\} = \frac{-1}{8\pi n_0}\quad(\tilde{E}_1 \cdot \tilde{E}_1). \tag{29}$$

We note from eq. (27) that at the frequency $2\omega_p$ the driving force derives both from charge separation and from the ponderomotive force.

To find n_2 we first combine eqs. (24), (26), (20), and (21) to find div $\tilde{v}_2$ in the form

$$\operatorname{div} \tilde{v}_2 = \frac{i}{12\pi n_0 m\omega_p} \operatorname{div}\{\tilde{E}_1 \operatorname{div} \tilde{E}_1 + \nabla(\tilde{E}_1 \cdot \tilde{E}_1)\}, \tag{30}$$

and hence

$$\tilde{n}_2 = \frac{1}{6\pi m\omega_p^2} \operatorname{div}\{\tilde{E}_1 \operatorname{div} \tilde{E}_1 + \tfrac{1}{4}\nabla(\tilde{E}_1 \cdot \tilde{E}_1)\}. \tag{31}$$

Eqs. (30) and (31) are identical with Kuznetsov's equations [10b] which were derived assuming that there were only electrostatic perturbations.

Next we must derive the "slow-timescale" electron equations. As the slow-timescale phase velocity v_{ph}^{s} will be of the order of the ion-sound speed which is small compared to the electron thermal velocity, we may assume that the averaged electron density, defined by the relation

$$\langle n_e \rangle \equiv \langle n_0 + n_s + n_1 + n_2 \rangle = n_0 + n_s, \tag{32}$$

will be given by a Boltzmann distribution in the field of an effective potential U_{eff} [5, 7]. This effective potential consists of a slow-time scale charge separation potential $-e\phi_s$ and a ponderomotive potential U_{pond} obtained, like $\langle n_e \rangle$, by averaging over the fast-time scale motion:

$$U_{eff} = -e\phi_s + U_{pond} \tag{33}$$

with

$$U_{pond} = \frac{e^2}{m\omega_p^2} (\tilde{E}_1 \cdot \tilde{E}_1^*). \tag{34}$$

For $\langle n_e \rangle$ we have thus

$$\langle n_e \rangle = n_0 \exp \frac{e\phi_s - U_{pond}}{T_e}, \tag{35}$$

whence

$$n_s = n_0 [\exp \{(e\phi_s - U_{pond})/T_e\} - 1]. \tag{36}$$

If the electron distribution is stationary, we get by linearizing the equation of continuity (17) with respect to n_s and v_s and retaining only the slow-time scale terms

$$\operatorname{div} v_s = -\operatorname{div} [\tilde{n}_1 \tilde{v}_1^* + \tilde{n}_1^* \tilde{v}_1]/n_0, \tag{37}$$

and hence, using eqs. (20) and (21)

$$\operatorname{div} v_s = \frac{i}{4\pi m n_0 \omega_p} \operatorname{div} [\tilde{E}_1^* \operatorname{div} \tilde{E}_1 - \tilde{E}_1 \operatorname{div} \tilde{E}_1^*]. \tag{38}$$

To eliminate $e\phi_s$ we need an equation for the ion motions. As the ion temperature is lower than T_e, v_{ph}^{s} will be higher than the ion thermal velocity v_{Ti} and we can apply a hydrodynamic description for the ion motion. (If $v_{ph}^{s} \ll v_{Ti}$, one can use the so-called "static" approximation [5].)

For the moment we shall assume that we may linearize the ion equations (see the discussion at the end of this paper). We then have

$$\frac{\partial \delta n_i}{\partial t} + n_0 \operatorname{div} v_i = 0, \tag{39}$$

$$\frac{\partial v_i}{\partial t} = -\frac{e}{m} \nabla \phi_s, \tag{40}$$

where we have in the equation of motion omitted the pressure term, since $T_e \gg T_i$ by assumption, and neglected the direct ion ponderomotive force in comparison to the ambipolar "electron" force.

If we restrict ourselves to density perturbations for which $n_s/n_0 \ll 1$ – which means that we can, for instance, not apply our results directly for a comparison with the latest experimental results of Antipov et al. [3] as they have values n_s/n_0 up to 0.3 – we can linearize eq. (36) and we then have

$$\frac{n_s T_e}{n_0} + U_{\text{pond}} = e\phi_s, \tag{41}$$

while Poisson's equation becomes

$$\nabla^2 \phi_s = 4\pi e(n_s - \delta n_i). \tag{42}$$

Combining eqs. (41) and (42) to eliminate ϕ_s we have

$$n_s = \delta n_i + \nabla^2 U_{\text{pond}}/4\pi e^2 + r_D^2 \nabla^2 n_s. \tag{43}$$

If we assume that the spatial variations of n_s are on length scales much larger than the Debye radius, we can drop the last term on the right-hand side of eq. (43) to get

$$n_s = \delta n_i + \nabla^2 U_{\text{pond}}/4\pi e^2. \tag{44}$$

We can now eliminate n_s and v_i from eqs. (39), (40), and (44) and using eq. (34) we find the familiar driven ion-sound equation

$$\frac{\partial^2 \delta n_i}{\partial t^2} - c_s^2 \nabla^2 \delta n_i = \frac{1}{4\pi M} \nabla^2 (\tilde{E}_1 \cdot \tilde{E}_1^*). \tag{45}$$

Let us briefly remind ourselves of a fundamental requirement for the validity of our results [13]. This is the requirement that the distance travelled by an electron during a time ω_p^{-1} is small compared to the characteristic fast-timescale length scale k^{-1}, that is,

$$v_s \omega_p^{-1} \ll k^{-1} \quad \text{and} \quad u_1 \omega_p^{-1} \ll k^{-1}; \tag{46}$$

we note that we shall allow for v_1 to be more important than v_s, however, Using eq. (20) and (38) this leads to the condition

$$\frac{W}{n_0 T_e} \ll (k r_D)^{-2}, \tag{47}$$

where W is the energy density in the Langmuir wave.

The condition (47) also ensures that one can neglect the effect of electrons trapped in the finite-amplitude, fast-time scale electric field of the wave [14]. The trapping is a special case of a strong Landau-like resonant interaction involving strong non-linear electron-orbit modifications.

Trapping in spatially localized wavepackets is also negligible for sufficiently broad wavepacket spectra when condition (47) is satisfied. We must add that, anyway, trapping is inherently a kinetic effect so that we cannot consider it in the hydrodynamic description used here.

3. Adiabatic scaling and spherical collapse

In this section we investigate the effect of the non-linear correction terms on the stability of spherical or circular configurations in the three- and two-dimensional cases, respectively. At the end of the present paper we shall briefly discuss less symmetrical structures, but eq. (23) is too complicated for a quantitative discussion of the general case.

The spherical collapse was already discussed in Zakharov's original paper [5] but the solutions he considered were criticized by Litvak et al. [15] and by Degtyarev et al. [16]. For the spherically symmetric case eq. (23) becomes

$$iE_t + \frac{\partial}{\partial r}\left[\frac{1}{r^{d-1}}\frac{\partial}{\partial r}(r^{d-1}E)\right] - nE - \frac{\beta|E|^2E}{r^2} = 0, \tag{48}$$

where we have introduced dimensionless variables by the substitutions

$$t \to \left(\frac{3}{2\mu\omega_p}\right)^{-1} t,\quad r \to \left(\frac{3}{2}r_D\mu^{-1/2}\right)^{-1} r,\quad \delta n_i \to \left(\frac{4\mu n_0}{3}\right)^{-1} n,\quad E_1 \to (16\pi\mu n_0 T_e/3)^{-1/2}E, \tag{49}$$

with

$$\mu = \frac{m}{M}. \tag{50}$$

The quantity d is the dimensionality ($d = 1, 2$, or 3) of the system, β is a function of d and given by the equations

$$\beta(d=1) = 0,\quad \beta(d=2,3) = \frac{8}{27}\mu. \tag{51}$$

The term with β comes from the electron non-linearities which thus do not contribute in the one-dimensional case, as was independently confirmed by Kuznetsov [10b] and by Khakimov and Tsytovich [8] (see also [17]).

In the derivation of eq. (48) we have eliminated $\tilde{n}_2$, $\tilde{v}_2$, n_s, and v_s from eq. (23) by using eq. (30), after integrating over r, (31), (37), and (38).

We shall follow the approach used in an earlier paper [12] and for the moment for stationary states represent the density perturbation n as an as yet unspecified function $-Q(|E|^2)$ of the plasmon density $|E|^2$:

$$n = -Q(|E|^2). \tag{52}$$

Of course, n satisfies the equation

$$n_{tt} - \nabla^2 n = \nabla^2|E|^2. \tag{53}$$

In the static case, where the time-derivative can be neglected we have clearly

$$Q(x) = x. \tag{54}$$

Using eq. (52) we get eq. (48) in the form

$$iE_t + \frac{\partial}{\partial r}\left\{\frac{1}{r^{d-1}}\frac{\partial}{\partial r}(r^{d-1}E)\right\} + Q(|E|^2)E - \frac{\beta|E|^2E}{r^2} = 0. \tag{55}$$

Eq. (55) has the constants of motion

$$N = \int |E|^2 \mathrm{d}^d r, \tag{56}$$

the plasmon number, and

$$H = \int \left\{ |(\nabla \cdot E)|^2 - R(|E|^2) + \frac{\beta |E|^4}{2r^2} \right\} \mathrm{d}^d r, \tag{57}$$

the Hamiltonian. In eq. (57) we have $R(|E|^2)$ which is defined by the equation

$$Q(x) = \frac{\mathrm{d}R(x)}{\mathrm{d}x}. \tag{58}$$

We now consider a scaling factor $\lambda(t)$ such that

$$r \to r/\lambda, \quad E \to \lambda^{d/2} E; \tag{59}$$

this scaling leaves the plasmon number (action) invariant and may thus be called to be an adiabatic scaling. The Hamiltonian scales as follows:

$$H \to C \int r^{d-1} \mathrm{d}r \left\{ \lambda^2 \left| \frac{1}{r^{d-1}} \frac{\partial}{\partial r} (r^{d-1} E) \right|^2 - \lambda^{-d} R(r^d |E|^2) + \lambda^{d+2} \beta \frac{|E|^4}{2r^2} \right\}, \tag{60}$$

where C is a numerical factor depending on the value of d:

$$C(d=1) = 1, \quad C(d=2) = 2\pi, \quad C(d=3) = 4\pi. \tag{61}$$

If now

$$\lim_{x \to \infty} Q(x) \propto x^{\nu}, \tag{62}$$

stability against collapse, that is, as $\lambda \to \infty$, is guaranteed, if

$$\nu < \sigma + \frac{2}{d} = \nu_{\mathrm{crit}}, \tag{63}$$

where σ is a constant depending on d:

$$\sigma(d=1) = 0, \quad \sigma(d=2,3) = 1. \tag{64}$$

Condition (63) changes into condition $\nu < 2/d$ for the Zakharov equations (see, e.g. [12]) when $\beta = 0$. We also see that in the case (54), when $\nu = 1$, the one-dimensional case leads to stationary solutions – which are, of course, the Langmuir solitons – even without the electron non-linearities. Finally, we note that, if the electron non-linearities are taken into account, condition (63) is satisfied even for two- and three-dimensional plasmas in the case of the ponderomotive force (54). In the spherical approximation the corrections due to the higher-order electron non-linearities thus lead to the possibility of quasi-stationary solutions also in the two- or three-dimensional case. We still need to consider, however, whether the absence of a mathematical singularity also implies the absence of a physical collapse to dimensions of the size of the Debye length, as seems to be the case judging from the experimental results of Antipov et al. [3] who formed apparently stable structures of the size of several Debye lengths.

In order to study this question we shall estimate the magnitude of the extra correction term in eq. (48). As long as $W/n_0T_e \ll \mu$ we can neglect the n_{tt} term in eq. (53) (see [7]) so that eq. (54) holds and the self-focusing term in eq. (48) is simply $|E|^2E$. If we require, say, that the correction term is an order of magnitude smaller than the self-focusing term, we have

$$\frac{\beta|E|^2E}{r^2} \sim \frac{1}{10}|E|^2E, \tag{65}$$

or

$$r^2 \sim 10\beta, \tag{66}$$

whence, restoring dimensions, we get

$$r \sim 3r_D. \tag{67}$$

This means that the stabilizing action of the higher-order electron non-linearities does not start to be fully effective until the collapse has already proceeded quite far.

4. Qualitative discussion of the collapse

Before discussing the collapse of a caviton structure, that is, a localized field accompanied by a density depression, centred at the origin, we must draw attention that the second term in eq. (48) can be written in the form $\partial^2E/\partial r^2 + (d-1)r^{-1}\partial E/\partial r - (d-1)E/r^2$ so that the origin is a singular point of that equation, even if $\beta = 0$. One should therefore for the spherical case impose the boundary condition $E(0, t) = 0$. This may have been the reason why Degtyarev et al. [16] considered the spherical collapse of a spherical layer of radius R and thickness δ, with $\delta \ll R$, with a soliton-like field structure in the r-direction. As $\delta \gg r_D$, so that a fortiori $R \gg r_D$, the evolution of this model will not be affected by our correction term. By the same token, however, it is only possible in this model to study the very first stages of the collapse, as the condition on δ and R will soon be violated.

In the more general case (see also [15] and [18]), we expect that there will be two regions with different dynamical properties. There will be a "core" region with a radius of the order of a few times r_D which will be stabilized through the effect of the higher order electron non-linearities, and there will be a "shell" region which collapses towards the centre. As the collapse proceeds, the field amplitude increases and the condition necessary for the static approximation to hold will be violated. This means that one should consider the hydrodynamic approximation [7], but that would mean using the full equation (53) when W/n_0T_e approaches the value μ, but is still below it so that the collapse is subsonic. The self-focusing term will become increasingly important and it will drive the collapse towards the sonic, or even supersonic ($W/n_0T_e > \mu$) regime. The transition to the supersonic regime is subject to a certain amount of controversy (see [19]) as eq. (53) is no longer valid for near-sonic motion. However, numerical experiments [20] seem to indicate, at least for non-dissipative cases, the existence of a supersonic regime. This part of the collapse certainly needs a further study, especially the question whether such a supersonic stage will actually occur in real plasmas.

If eq. (48) remains valid during the collapse we may perhaps expect in the later stages of the collapse a hydrodynamic stabilization so that stable cavitons with spatial dimensions of the order of the "core" radius may be formed and these may be the structure seen by Antipov and coworkers.

Strictly speaking, as soon as $(W/n_0 T_e)(kr_D)^2\mu^{-1}$ becomes of the order of unity, corrections to the ion motion will have to be considered [21] and eq. (39) and (40) must be replaced by the proper non-linear equations [11b].

Let us finally briefly consider the general, non-spherically-symmetric case. As compared to the original Zakharov equations, we note that in eq. (16) we have three extra terms. These terms are of the relative order $(kr_D)^2$ in the static regime, but they may become important when $W/n_0 T_e$ becomes larger. For example, in the simple model of a self-similar supersonic collapse [22] we have $n_s/n_0 = \chi(W/n_0 T_e)^{2/3}$, where $\chi \sim 0.1$ to 1.0 and for the relative magnitude of the correction terms we get $(W/n_0 T_e)^{1/3}(kr_D)^2\chi^{-1}$ which shows the possible importance of the correction terms in the hydrodynamic regime.

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