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LAGRANGE'S EARLY CONTRIBUTIONS TO THE THEORY OF FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

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SUMMARIES

In 1776, J. L. Lagrange gave a definition of the concept of a "complete solution" of a first-order partial differential equation. This definition was entirely different from the one given earlier by Euler. One of the sources for Lagrange's reformulation of this concept can be found in his attempt to explain the occurrence of singular solutions of ordinary differential equations. Another source of the new definition is contained in an earlier treatise of Lagrange [1774] in which he elaborated an approach to first-order partial differential equations briefly indicated by Euler. The method of "variation of constants," which was fundamental to his argument, suggested to Lagrange the reformulation of the concept of a "complete solution." In the present paper I shall discuss both sources of the new definition of "completeness."

Im Jahre 1776 gab J. L. Lagrange eine neue und von der Eulerschen grundverschiedene Definition des Begriffes "vollständige Lösung" einer partiellen Differentialgleichung erster Ordnung. Einer der Ursprünge dieser neuen Begriffsbildung liegt in Lagrange's Versuch das Auftreten singulärer Lösungen gewöhnlicher Differentialgleichungen zu klären. In einer früheren Arbeit von Lagrange aus dem Jahre 1774 findet man ein weiteres Motiv für die neue Definition. In diesem Artikel arbeitet Lagrange einen von Euler angedeuteten Weg zur Lösung partieller Differentialgleichungen erster Ordnung aus. Die dabei von ihm entwickelte Methode der "Variation der Konstanten" führt ihn unmittelbar auf den späteren Begriff einer "vollständigen Lösung." Beide Ursprünge der neuen Definition werden in diesem Aufsatz diskutiert.

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1. INTRODUCTION

This new Analysis deals essentially with the need and the way, which he [d'Alembert] *has explained, of introducing arbitrary functions into the integrations of formulas involving partial differences* [1].

This was how the Abbé Charles Bossut characterized the newly developed calculus of partial differential equations in his "Discours Préliminaire" [Bossut 1784, cxii] of the mathematical volumes of the *Encyclopédie Méthodique*. Bossut stressed particularly the importance of the intimate relation between arbitrary functions and partial differential equations. The new Analysis referred to by Bossut had its origin in the work of d'Alembert, who in 1747 had studied the general cause of winds [d'Alembert 1747, art. 87ff.] and the motion of the vibrating string [d'Alembert 1749]. In these studies d'Alembert came across systems of differential expressions; when he integrated these systems, which are equivalent to second-order partial differential equations, he had to introduce two arbitrary functions. (A thorough discussion of d'Alembert's contribution to the theory of partial differential equations can be found in Demidov [1974].)

The notion of partial differential equations and an adequate notation for them were introduced by Leonhard Euler in 1753 [cf. Truesdell 1960, 260; Demidov 1975, 219]. In the 1760s the first surveys of results pertaining to partial differential equations appeared. Euler summarized his results as early as 1764 in a paper presented to the Petersburg Academy; in the third volume of *Institutiones Calculi Integralis* [Euler 1770], Euler gave his results in the context of an introductory textbook. In a very systematic way and using numerous examples by way of illustration, Euler presented a uniform approach to partial differential equations. This approach is based on a few simple techniques which render differential expressions integrable. At some points Euler's approach breaks down and appears to be fundamentally inadequate. D'Alembert [1768] devoted one of the articles of *Opuscules Mathématiques* to partial differential equations. Thus by 1770 the new analysis of partial differential equations had established itself as a separate branch of mathematics.

Euler's textbook gives a clear criterion for assessing completeness of the integration of a partial differential equation. According to Euler the integration of an equation of order n is *complete* if the integral contains n arbitrary functions. In Euler's view partial differential equations are just like ordinary differential equations, in that the role of the arbitrary constant is taken over by an arbitrary function. This analogy between arbitrary constants and arbitrary functions becomes clear, for instance, when Euler explains how the arbitrary constants or functions are determined:

> *... in the case of ordinary integrations the arbitrary constant introduced is always determined by the nature of the problem, the solution of which required these integrations; here too the nature of the problem solved by this type of integration* [i.e., integration *of partial differential equations*] *will always determine the arbitrary function introduced by the integration.* [Euler 1770, 38].

In his famous article on singular solutions of differential equations, presented to the Berlin Academy in 1775, Lagrange [1776] gave an entirely different interpretation of completeness of a solution of a partial differential equation. Let

$$Z(x,\ y,\ z,\ \frac{dz}{dx},\ \frac{dz}{dy}) = 0 \qquad (1)$$

be a first-order partial differential equation. Lagrange calls a solution of (1) *complete* if it contains two arbitrary constants. Thus a complete solution z is of the form

$$z = \phi\ (x, y, a, b),\quad a, b \text{ constant},$$

or, as is usual in 18th-century analysis, the complete solution z is defined implicitly by

$$V\ (x, y, z, a, b) = 0. \qquad (2)$$

For Euler, the complete solution produces all particular solutions of the differential equation by specification of the arbitrary elements. This can be seen, for instance, in the quotation given above: a problem leads to a differential equation, and the nature of the problem determines the arbitrary constants or functions. So at least all relevant solutions of the differential equation are supposed to be contained in the complete solution. Lagrange's concept of completeness differs considerably from Euler's, in that Lagrange's complete solution does not yield all particular solutions of the differential equation through a mere specification of the arbitrary elements. In this article I shall trace the origins of Lagrange's revision of the concept of completeness, restricting myself to first-order partial differential equations.

2. EULER'S *INSTITUTIONES CALCULI INTEGRALIS* [1770] AND LAGRANGE'S ARTICLE *"SUR L'INTÉGRATION DES ÉQUATIONS A DIFFÉRENCES PARTIELLES DU PREMIER ORDRE"* [1774]

Lagrange's first contribution to the theory of first-order partial differential equations appeared in 1774 in the Berlin *Mémoires* for the year 1772 [cf. Taton 1974]. In this article Lagrange adopted an approach to partial differential equations which Euler had indicated in the third volume of his *Institu-*

tiones [Euler 1770, Sect. 128] at a point where his usual methods seemed to fail. In the *Institutiones* Euler treats partial differential equations formally, without relating them to practical examples. He develops his theory in a series of problems of increasing complexity and generality; his main ordering principle is the number of variables appearing in the differential equation. Euler's integrations always start with the total differential

$$dz = pdx + qdy, \tag{3}$$

and they are based on the *a-priori* condition that $pdx + qdy$ is integrable whatever the partial differential equation under consideration may be [2]. Thus this condition guarantees the existence of solutions. As relation (3) is equivalent to

$$d(z - px) = -xdp + qdy, \tag{4}$$

$$d(z - qy) = pdx - ydq, \tag{5}$$

and

$$d(z - px - qy) = -xdp - ydq, \tag{6}$$

the integrability of $pdx + qdy$ implies the integrability of the right-hand sides of (4) to (6); replacement of (3) by (4), (5), or (6) amounts to the choice of another pair of independent variables. For a partial differential equation

$$F(x,y,p,q) = 0 \tag{7}$$

in which z does not occur explicitly, the integrability condition for $pdx + qdy$ gives rise to two methods of integration, which shall be designated here as (a) the method of total differentials and (b) the method of direct integration.

(a) Method of Total Differentials

If $pdx + qdy$ can be transformed by means of (7) into the form hdt, then the integral of (7) can be represented by the equations $z = f(t)$ and $h = f'(t)$, where f is an arbitrary function. This method is founded on the fundamental principle that, for variables $s,h,$ and t the relation $ds = hdt$ implies $h = f'(t)$ and $s = f(t)$, where f is an arbitrary function. Euler had already stated this principle in [1740, Sect 7]. I will illustrate this method with an example from the *Institutiones*, Section 138. Let the partial differential equation have the form

$$px + qy = 0. \tag{8}$$

Then by elimination of q, (3) takes the form

$$dz = pdx - \frac{px}{y}dy = py\left(\frac{dx}{y} - \frac{xdy}{yy}\right) = pyd\left(\frac{x}{y}\right),$$

and thus the solution is given by $z = f(x/y)$, where f is arbitrary.

(b) Method of Direct Integration

If by means of (7) one of the differential expressions on the right-hand sides of (3) to (6) can be transformed in such a way that one of the differentials has a coefficient involving only the two independent variables, then the solution can be given by integration of this differential. Here is an example by Euler [1770, Sect. 110]: let

$$x = X(p, q). \tag{9}$$

Elimination of x from the right-hand side of (6) yields $d(z - px - qy) = -X(p, q)\,dp - y\,dq$. Then $z - px - qy = -\int X(p, q)\,dp + f(q)$ and $y = -\int X_q(p, q)\,dp + f'(q)$ represent together with (9), a solution to (9), parametrized by p and q.

In these methods it is crucial that the given partial differential equation (7) does not contain z, so that the right-hand sides of (3) to (6) must remain free of z after one of the variables is eliminated by means of the given equation. When he came to treat equations of the type $V(p, q, z) = 0$, Euler mentioned this limitation of his methods and indicated a more general approach:

> *Thus the case remains where an equation between* p, q, *and* z *is given; it is immediately clear that in this case the quantities* p *and* q *in the equation* dz = pdx + qdy *cannot be regarded as functions of* x *and* y, *as they are also dependent on* z; *so their form cannot be determined from the condition that the formula* pdx + qdy *be integrable.*

Euler then indicates another approach which might lead to success:

> *but--without any difference--the condition should hold that the differential equation* dz - pdx - qdy = 0 *be possible* [see [2]]; *for that it is required by the principles given above, that ...*
>
> $$p\left(\frac{dq}{dz}\right) - q\left(\frac{dp}{dz}\right) + \left(\frac{dq}{dx}\right) - \left(\frac{dp}{dy}\right) = 0. \tag{10}$$
>
> *Therefore, given any equation between* p, q, *and* z, *one should try to find, in general, those conditions for which this requirement be satisfied.* [Euler 1770, Sect. 128].

But instead of pursuing this approach Euler develops more elaborate applications of his methods. Perhaps the most impressive result he achieves in this way is his integration of the linear equation $Z(z) = pP(x, y) + qQ(x, y)$. It was left to Lagrange to pick up the line of thought Euler had indicated.

In "Sur l'intégration des équations a différences partielles du premier ordre," Lagrange [1774] pursues the approach which Euler had indicated. With Eq. (10) as the starting point, Lagrange investigates the general first-order partial differential equation in three variables, (1) $Z(x, y, z, p, q) = 0$. He observes that Euler's Eq. (10) is not only of importance as a criterion to establish integrability of a differential expression $dz - pdx - qdy$; Eq. (10) can also be used to reduce the integration of a given first-order partial differential equation (1) to the integration of a linear first-order partial differential equation. This advantage is clearly pointed out by Lagrange, although at that time he had not worked out a general method for integrating linear equations [3]. But the linear equation (10) has another property, which at this point is even more important:

> *Moreover we would point out that it is not necessary to solve this equation in a complete way, but it is sufficient to find some value for* p *that satisfies this equation, provided this value contains an arbitrary constant; for we shall see in a moment how the general and complete solution of the given equation can be found with such a value for* p [Lagrange 1774, Sect. 3].

Euler's Eq. (10) represents the condition of integrability of

$$dz - pdx - qdx = 0. \qquad (11)$$

Lagrange assumes that the given partial differential equation (1) can be written explicitly as

$$q = f(x, y, z, p) \text{ [4]}. \qquad (12)$$

Combined with (12), Eq. (10) becomes a linear first-order equation for p with independent variables $x, y,$ and z. Lagrange does not write down this equation, which in our notation is

$$\frac{\partial p}{\partial y} + (f - pf_p)\frac{\partial p}{\partial z} - f_p\frac{\partial p}{\partial x} - f_x - pf_p = 0. \qquad (13)$$

(f_p denoting $\partial f/\partial p$, etc.). Now if $p = P(x, y, z)$ satisfies Eq. (13), then

$$p = P(x, y, z) \quad \text{and} \quad q = Q(x, y, z) = f(x, y, z, P(x, y, z))$$

satisfy both (10) and (1), and the equation of differentials,

$$dz - P(x, y, z)dx - Q(x, y, z)\, dy = 0, \qquad (14)$$

is integrable. The integral $N(x, y, z) = 0$, for which

$$dN(x, y, z) = I(x, y, z)(dz - P(x, y, z)dx - Q(x, y, z)dy) \qquad (15)$$

holds ($I(x, y, z)$ is an integrating factor), defines implicitly a solution $z = F(x, y)$ of the given first-order partial differential equation (1). Lagrange develops a "variation of constants" procedure to find the complete solution of (1)--in the Eulerian sense--from a one-parameter family $p = P(x, y, z, a)$ of solutions of Eq. (10). $P(x, y, z, a)$ gives rise to the expressions $q = Q(x, y, z, a)$, $I(x, y, z, a)$, and $N(x, y, z, a)$ so that the following equation holds

$$dN(x, y, z, a) = I(x, y, z, a)(dz - P(x, y, z, a)\,dx - Q(x, y, z, a)dy) \qquad (16)$$

$N(x, y, z, a) = 0$ defines a one-parameter family of solutions of (1).

Now Lagrange varies the constant--i.e., he assumes that a is a function $A(x, y, z)$--and asks under which conditions $p = P(x, y, z, a)$ will also yield a solution of (1) with variable a. Because of the variability of a, relation (16) is transformed into

$$dN(x, y, z, a) = I(x, y, z, a)(dz - P(x, y, z, a)dx - Q(x, y, z, a)dy) + \frac{dN(x, y, z, a)}{da}\,da, \qquad a = A(x, y, z) \qquad (17)$$

or equivalently

$$dN(x, y, z, a) - \frac{dN(x, y, z, a)}{da}da = I(x, y, z, a)(dz - P(x, y, z, a)dx - Q(x, y, z, a)dy). \qquad (18)$$

As Eq. (1) holds for $p = P(x, y, z, a)$ and $q = Q(x, y, z, a)$ irrespective of the variability of a, one obtains a solution of (1) if the right-hand side of (18) is integrable. Integrability of the right-hand side of (18) is equivalent to integrability of $(dN(x, y, z, a)/da)\,da$, which in turn is equivalent to the condition that $dN(x, y, z, a)/da$ is a function of a alone. This is the condition on $a = A(x, y, z)$ that Lagrange was looking for.

Summarizing, if $a = A(x, y, z)$ is such that $(dN(x, y, a)/da)da$ is a function $f'(A(x, y, z))$ of $A(x, y, z)$ alone, then

$$N(x, y, z, A(x, y, z)) - f(A(x, y, z)) = 0$$

also defines a solution z of Eq. (1).

This result can be stated otherwise: for any arbitrary function f, the system

$$N(x, y, z, a) - f(a) = 0, \qquad \frac{dN(x, y, z, a)}{da} - f'(a) = 0 \qquad (19)$$

yields a solution of the given partial differential equation (1) by elimination of a.

In Lagrange's own words:

> *It is clear that this value of* z *will be complete as it will contain one arbitrary function*. [Lagrange 1774, Sect. 6].

In one of the more philosophical paragraphs at the end of this article, Lagrange asks whether one could not obtain the complete solution--in Euler's terminology--of (1) from a one-parameter family of solutions $N(x, y, z, a) = 0$ of (11) itself, instead of from a one-parameter family of solutions $p = P(x, y, z, a)$ of (13). According to Lagrange this is not in general possible, as the parameter in $N(x, y, z, a)$ may vanish in the course of differentiation. But two parameters in the solution of (11) would suffice. Differentiation would at least leave one parameter in dN, as is required for the "variation of constants" procedure:

> *But it will not be surprising at all that a particular solution which contains two arbitrary constants is sufficient to permit the derivation of the complete solution; close examination tells us that this solution almost entirely meets the conditions of the differential equation, as one cannot let the two arbitrary constants vanish without arriving at an equation which simultaneously involves the partial derivatives* dz/dx *and* dz/dy; *in fact it is necessary to have three equations, since two quantities are to be eliminated; so there have to be two equations in addition to the given equation; these two can only be obtained by two different differentiations, one with respect to variable* x *and one with respect to variable* y. [Lagrange 1774, Sect. 11].

Lagrange's terminology is still entirely Eulerian here, "complete solution" denoting a solution containing one arbitrary function. But Lagrange has already clearly noticed that a solution containing two arbitrary constants has a form of completeness as well, in that it yields the differential equation by elimination of the constants. In his article of 1776 on singular solutions Lagrange takes the latter property as the definition of a complete solution.

3. LAGRANGE'S NEW CONCEPT OF COMPLETENESS

Lagrange's exposé of the new concept of completeness of a partial differential equation is contained in "Sur les intégrales particulières des equations différentielles" [1776]. "Intégrales particulières," in Lagrange's terminology, are what we nowadays call singular solutions. The greater part of Lagrange's article is devoted to the occurrence of singular solutions to ordinary differential equations. I shall deal briefly with Lagrange's treatment of ordinary differential equations and their singular solutions. Here Lagrange's new view of completeness produced its first spectacular results.

Lagrange explains that a singular solution is a solution which is not contained in the complete integral of the ordinary differential equation; that is, it cannot be found by giving specific values to the constants which occur in the complete solution. Singular solutions can be found by differentiation instead of by integration, of the differential equation. According to Lagrange,

> *the particular integrals escape from the ordinary method of integration* [Lagrange 1776, Sect. 1].

As early as 1715 Brook Taylor had come across such a solution:

> ... *this is an extraordinary solution to the problem* [Taylor 1715, p. 27].

In 1758 Euler devoted an entire article to singular solutions. Even then--40 years after Taylor--he considered their occurrence to be a "Paradoxe dans le Calcul Intégrale":

> *For if the integral equation, which is found after all prescribed precautions, does not exhaust the extent of the differential equation, the problem permits solutions which the integration will not yield, and thus one will arrive at a defective solution. Doubtless this seems to run counter to the ordinary principles of integral calculus* [Euler 1758, Sect. 32].

In 1775, when Lagrange read the article on "intégrales particulières" to the Berlin Academy, the occurrence of singular solutions was, on the whole, still incomprehensible to mathematicians. It is to Lagrange's credit that he gave a thorough explanation, both geometrically and algebraically, of this phenomenon, thus fully justifying his statement:

> *So one has to regard the theory which we have just given more as a necessary supplement than as an exception to the general rule of integral calculus* [Lagrange 1776, Sect. 7].

Lagrange's treatment of singular solutions of a first-order ordinary differential equation may be summarized as follows: Let

$$Z(x, y, \frac{dy}{dx}) = 0 \qquad (20)$$

be a first-order ordinary differential equation, with the solution

$$V(x, y, a) = 0. \qquad (21)$$

To both Euler and Lagrange (21) represents the complete solution of Eq. (20) as it contains an arbitrary constant. But they stress different properties of the complete solution (21). To

Euler the most important property is that (21) contains the constant a, which is introduced by integration of the differential equation (20), while the prescribed precautions are observed. Lagrange stresses the inverse relationship between (20) and (21), emphasizing that solution (21) is equivalent to (20) in the sense that elimination of a from (21) necessarily leads to (20). This may be illustrated by the following scheme:

$$\text{Euler:} \quad Z(x, y, \frac{dy}{dx}) = 0 \quad \xrightarrow{\text{integration}} \quad V(x, y, a) = 0.$$

$$\text{Lagrange:} \quad Z(x, y, \frac{dy}{dx}) = 0 \quad \xleftarrow{\text{elimination of } a} \quad V(x, y, a) = 0.$$

Lagrange's interpretation of this elimination procedure has important consequences for his treatment of both ordinary and partial differential equations. For each a, Eq. (21) defines a solution of the differential equation (20). By differentiation of (21) with a constant, one arrives at

$$V_x(x, y, a)dx + V_y(x, y, a)dy = 0. \tag{22}$$

Hence

$$\frac{dy}{dx} = P(x, y, a), \qquad P(x, y, a) = \frac{-V_x(x, y, a)}{V_y(x, y, a)}. \tag{23}$$

Elimination of the constant a from Eqs. (21) and (23), which both apply to a solution of (20), leads back to the differential equation (20).

Having explained completeness in this way, Lagrange asks the crucial question: does the elimination procedure, (21), (23) → (20), work only if a is constant, or might a be variable as well, depending on x and y? In the case of constant a the differentials of x and y are related by

$$dy = P(x, y, a)dx. \tag{24}$$

With a variable a, differentiation of (21) yields

$$V_x(x, y, a)dx + V_y(x, y, a)dy + V_a(x, y, a)da = 0, \tag{25}$$

and so (24) becomes

$$dy = P(x, y, a)dx + Q(x, y, a)da; \qquad Q(x, y, a) = -\frac{V_a(x, y, a)}{V_y(x, y, a)}. \tag{26}$$

Now if the condition

$$Q(x, y, a)da = 0 \tag{27}$$

holds, then (26) will reduce to (24), and the elimination scheme (21), (23) → (20) will remain unchanged. There are two ways of satisfying (27):

$$da = 0 \tag{27a}$$

$$Q(x, y, a) = 0 \tag{27b}$$

Equation (27a) states that a is constant; this is the case already dealt with. Equation (27b) in general defines a in terms of x and y, and thus leads to variable a. Substitution of this variable a in the complete solution (21), or equivalently, the elimination scheme

$$(21),\ (27b) \xrightarrow[\text{of } a]{\text{elimination}} S(x, y) = 0 \qquad (28)$$

yields a finite equation between x and y. This equation also represents a solution of (20); but as a is variable now (defined by (27b)), the solution thus arrived at is, according to Lagrange, not contained in the complete integral (21) of (20). Thus grange has arrived at a singular solution of Eq. (20) [5].

Having explained singular solutions algebraically as the result of a "variation of constants" procedure, Lagrange goes on to give a geometrical explanation. In geometrical terms Eq. (21) represents an infinity of curves in the plane; for each of those curves the tangent at a point (x, y) is determined by the differential equation (20). If there is an envelope of this family (a curve which at each of its points is tangent to a curve from the family (21) [6]) then the equation of the envelope also satisfies the differential equation (20). As Lagrange shows, the equation of the envelope can be derived from the equation of the family by exactly the same elimination procedure, (21), (27b) → (28), that yielded the singular solution. Thus Lagrange has explained the singular solution as the envelope of the curves contained in the complete solution.

Let me summarize the two main features of Lagrange's successful approach to singular solutions of ordinary differential equations:

(a) A marked change of view on completeness enables Lagrange to explain the occurrence of singular solutions algebraically with a "variation of constants" procedure.

(b) Singular solutions are explained geometrically as envelopes of the curves in the complete integral.

In my view Lagrange was motivated by this successful approach to ordinary differential equations to try out a uniform approach to all types of differential equations. This uniform approach consists of (A) an appropriate concept of a complete solution, and (B) application of the "variation of constants" procedure to find all solutions not contained in the complete solution as defined under (A).

The appropriate definition of completeness in the case of a first-order partial differential equation is definition (2), which I have given in Section 1. The complete solution of a first-order partial differential equation $Z(x, y, z, dz/dx, dz/dy) = 0$ is a solution of the form (2) $V(x, y, z, a, b) = 0$, containing two arbitrary constants a and b. Constants a and b play the same role as the arbitrary constant a in the complete solution (21) $V(x, y, a) = 0$ of the ordinary differential equation (20) $Z(x, y, dy/dx) = 0$.

By partial differentiation of (2) one can find expressions $P(x, y, z, a, b)$ and $Q(x, y, z, a, b)$ for which the following equations hold.

$$\frac{dz}{dx} = P(x, y, z, a, b), \tag{29}$$

$$\frac{dz}{dy} = Q(x, y, z, a, b). \tag{30}$$

Elimination of a and b from the system of equations (2), (29), (30) leads to the given differential equation (1). This is the very property of (2) which Lagrange had indicated at the end of his 1774 article. Thus one can understand why Lagrange revised Euler's definition of a complete solution.

Lagrange then applies his "variation of constants" procedure to find all solutions which are not contained in the complete solution. For constant a and b, Eq. (2) yields the following relation between the differentials of z, x, and y:

$$dz = pdx + qdy, \quad p = P(x, y, z, a, b), \quad q = Q(x, y, z, a, b). \tag{31}$$

For variable a and b, we have instead

$$dz = pdx + qdy + rda + sdb, \tag{32}$$

where $r = R(x, y, z, a, b)$ and $s = S(x, y, z, a, b)$. Equation (32) reduces to (31) when the following condition holds.

$$rda + sdb = 0. \tag{33}$$

If $a = A(x, y, z)$ and $b = B(x, y, z)$ are so that (33) is satisfied, then Eqs. (29) and (30) determining dz/dx and dz/dy still hold, and elimination of $A(x, y, z)$ and $B(x, y, z)$ from the system (2), (29), (30) still yields the given partial differential equation (1). So for such variable a and b equation

$$V(x, y, z, A(x, y, z), B(x, y, z)) = 0 \tag{34}$$

also represents a solution of Eq. (1).

Lagrange shows two ways of obtaining variable a and b which satisfy (33):

(I) Letting ϕ be an arbitrary function, put $b = \phi(a)$. Equation (33) then takes the forms $(r + \phi'(a)s)da = 0$. Now $r + \phi'(a)s = 0$, with r and s as in (32), in general defines a variable $a = A(x, y, z)$. From the value of a one finds $b = \phi(A(x, y, z)) = B(x, y, z)$.

(II) If a and b are such that $r = R(x, y, z, a, b) = 0$ and $s = S(x, y, z, a, b) = 0$, then these two equations determine variable $a = A(x, y, z)$ and $b = B(x, y, z)$ as well.

In both case (I) and case (II) an elimination procedure is involved. Explicitly they are:

(I') For arbitrarily chosen ϕ eliminate a from

$$V(x, y, z, a, \phi(a)) = 0, \quad V_a(x, y, z, a, \phi(a)) + \phi'(a)V_b(x, y, z, a, \phi(a)) = 0.$$

(II) Eliminate a and b from

$$V(x, y, z, a, b) = 0, \quad V_a(x, y, z, a, b) = 0, \quad V_b(x, y, z, a, b) = 0.$$

Lagrange calls the solution of the given partial differential equation (1), obtained by (I), the "intégrale générale" of (1). It is Euler's complete solution, ϕ being the arbitrary function. Geometrically he explains the general solution as the envelope of an arbitrarily chosen one-parameter subfamily of the surfaces contained in the complete solution (2). The solution arising from (II) Lagrange calls "intégrale particulière;" Lagrange explains this singular solution as the envelope of the entire two-parameter family of surfaces contained in the complete solution.

4. SOME FINAL REMARKS

Euler and Lagrange both defined concepts of complete solutions to first-order partial differential equations, Euler in his textbook *Institutiones Calculi Integralis,* Volume 3, and Lagrange in his research article "Sur les intégrales particulières ..." of 1776. Their definitions differ considerably and reveal two very different views on partial differential equations and their complete integration.

Euler's complete solution is characterized by an arbitrary function. Almost everywhere in the *Institutiones* it is Euler's ultimate aim to arrive at the complete solution of a given partial differential equation. Euler pursues this aim mainly by integrating the differential expressions related to the differential equation. Lagrange's complete solution, on the other hand, is characterized by the occurrence of only two arbitrary constants. It is also found by integration. But far from being the final result itself, it is only an intermediate means for arriving at it. Lagrange subdivides the process of solving a partial differential equation. Integration is needed only to find a certain subset of the entire set of solutions; once this subset is obtained, the "variation of constants" method, consisting of differentiation and elimination, is used to construct the entire set of solutions from the subset.

The development of the theory of first-order partial differential equations can be called a success story in 18th-century analysis. From d'Alembert's first exploration of partial differential equations, it took only 40 years for a uniform theory of first-order partial differential equations to be formed. But this successful development did not follow a direct path of gradual discovery, each step supporting and leading on to the next. Lagrange's first contribution [1774] can be regarded a "Gestaltswitch" in the development of the theory of first-order partial differential equations. The important feature of this contribution lies more in the new interpretation of known facts than in the presentation of new facts. Almost all the examples treated by Lagrange [1774] were solved by Euler as well; even the algebraic forms in which both men give their solutions are very much alike. The new feature of Lagrange's article is the idea that an incomplete solution in Eulerian terminology would still suffice to produce all solutions of a first-order partial differential equation. This idea became central to Lagrange's reformulation in 1776 of the concept of a complete solution.

Lagrange's new concept of a complete solution and the associated "variation of constants" method provided a structure for the set of all solutions of a first-order partial differential equation. Lagrange paid some attention to the geometrical interpretation of this structure, in that he interpreted the general and singular solutions as envelopes of certain subfamilies of the entire family of surfaces represented by the complete solution. But Lagrange did not elaborate these geometrical aspects in great detail. As a purely analytical mathematician, he may well have indicated them mainly for didactic purposes. In fact, it was not until the contributions of Monge that these geometrical features were explored fully, thus reinstating geometry once more as a field of research in its own right.

NOTES

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All translations from French and Latin are mine.

1. "Partial differences;" (Bossut: "différences partielles") are the summands pdx and qdy of the total (18th century: "complete") differential $dz = pdx + qdy$ of a variable z. There was no commonly accepted notation for the partial derivatives p and q of z. In the articles I am dealing with here, Lagrange wrote simply $p = dz/dx$ and $q = dz/dy$. Euler consistently used the notation $p = (dz/dx)$ and $q = (dz/dy)$. In this article I will retain original notations as much as possible.

2. A differential expression

$$Pdx + Qdy + Rdz \quad (P,Q,R \text{ are expressions in } x, y, z) \qquad (*)$$

can be (a) the complete differential of some finite expression in the three variables x, y, z, or (b) a complete differential as under (a) divided by a finite factor (which is called an "integrating factor"), or (c) neither (a) nor (b).

In case (a) Euler calls (*) *integrable*; if (*) is either of type (a) or of type (b), he calls (*) *possible* or *real*. A differential expression (*) which is not *possible* Euler calls *absurd*.

Lagrange uses a different terminology: In both case (a) and case (b), (*) is called *integrable;* so Lagrange's *integrable* is Euler's *possible*. Euler's *integrable* Lagrange calls *"intégrable d'elle-même"* (case (a)). In the articles we are dealing with Lagrange has no special name for a differential expression in case (c).

The terminology used by Euler and Lagrange also applies to equations of differentials

$$Pdx + Qdy + Rdz = 0. \qquad (**)$$

In his *Institutiones*, Section 1, Euler asserted that the differential expression (*) $Pdx + Qdy + Rdz$ is possible whenever P, Q, and R satisfy the *character*:

$$P((\frac{dQ}{dz}) - (\frac{dR}{dy})) + Q((\frac{dR}{dx}) - (\frac{dP}{dz})) + R((\frac{dP}{dy}) - (\frac{dQ}{dx})) = 0 \qquad (***)$$

He in fact only proved that the possibility of (*) implies (***).

3. Lagrange later found such a method, which he published in the 1779 *Mémoires* of the Berlin Academy. It consists of the reduction of a linear first-order partial differential equation to a system of simultaneous ordinary differential equations. Both methods, the 1774 linearization and the 1779 reduction, were combined by Charpit in the 1780s to yield the well-known Lagrange-Charpit method for the integration of a first-order partial differential equation in three variables. For further information the reader is referred to Kline [1972, 533-535]. However, Kline seems to be interested only in that part of Lagrange's [1774] article which had immediate consequences for the Lagrange-Charpit method. Kline's presentation of Lagrange's argument is correct only as far as the reduction to a linear first-order partial differential equation is concerned.

4. The notation of the formulas given here differs from Lagrange's original notation. In his 1774 article Lagrange does not write downs Eqs. (1) and (12) explicitly; he gives only a verbal circumscription. In the article of 1776 on singular solutions, Lagrange writes (1) simply as $Z = 0$ and specifies verbally the quantities that appear; (12) does not occur in [Lagrange 1776]. Lagrange would write Eq. (16) in the following form

$$dN = I(dz - pdx - qdy).$$

The distinction between upper and lowercase is Lagrange's. Uppercase characters represent expressions; lowercase characters represent variables. Lagrange's notation does not account for dependencies between variables or for the variables involved in an expression. These dependencies are stated verbally. For the sake of distinctness I have taken the liberty of specifying the quantities involved in an expression. So, for instance, I write $N(x, y, z, a)$ instead of N for an expression containing the variables x, y, and z and the constant a.

When it was clear from the context that Lagrange considered that a variable was defined by an expression, I have introduced a new symbol for this expression. For instance, in Eq. (16) Lagrange considered p to be a solution of Eq. (10) containing an arbitrary constant; therefore instead of p, I have used the notation $P(x, y, z, a)$.

5. Lagrange calls every solution of the differential equation (20) arrived at by means of a variable a defined by (27b), a singular solution [Lagrange 1776, Sect. 4]. To him such a solution is necessarily different from the solutions which one finds by giving the constant a in the complete solution a specific value. Lagrange seems not to be aware of the fact that in some cases both a variable a and a constant a produce the same solution.

6. Lagrange defines an envelope of a one-parameter family of curves as a curve which is tangent to all curves from the family [Lagrange 1776, Sect. 21]. He in fact considers the envelope to be formed by intersection of successive curves from the family [Sect. 22].

REFERENCES

Where reference is made to an article in one of the 18th-century Academy *Mémoires*, the year after the author's name indicates the year of publication of the volume in which the article appeared; the year after the name of the journal and the number of the volume indicates the Academy year covered by this volume

d'Alembert, J. 1747. *Réfléxions sur la cause générale des vents* Paris.

——— 1749. Recherches sur la courbe que forme une corde tenduë mise en vibration. *Histoire de l'Académie Royale des Sciences et Belles Lettres de Berlin* 3 (1747), pp. 214-219.

——— 1768. Recherches de calcul intégral. In *Opuscules mathématiques*, Vol. 4, pp. 225-282 Paris.

Bossut, Ch. 1784. Discours préliminaire. In *Encyclopédie méthodique, mathématiques* Vol. 1. Paris.

Demidov, S. S. 1974. Differentsialnye uravneniya s chastnymi proizvodnymi v rabotakh zh. dalambera. *Istoriko-matematicheskie Issledovaniya* 19, 94-124.

——— 1975. Vozniknovenie teorii differentsialnykh uravnenii s chastnymi proizvodnymi. *Istoriko-Matematicheskie*

Issledovaniya 20, 204-220.

Euler, L. 1740. Additamentum ad dissertationem de infinitis curvis eiusdem generis. *Commentarii Academiae Scientiarum Imperialis Petropolitanae* 7 (1734-1735), 184-200; also in *Leonhardi Euleri opera omnia*, Ser. 1, Vol. 22, pp. 57-75 Basel, 1936.

——— 1758. Exposition de quelques paradoxes dans le calcul intégral. *Mémoires de l'Académie des Sciences de Berlin* 12 (1756), 300-321; also in *Leonhardi Euleri opera omnia* Ser. 1, Vol. 22, pp. 214-236 Basel, 1936.

——— 1764. Investiqatio functionum ex data differentialium conditione. *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres, Berlin* 9 (1762-1763), 170-212; also in *Leonhardi Euleri opera omnia* Ser. 1, Vol 23, pp. 1-41 Basel, 1938.

——— 1770. *Institutionum calculi integralis volumen tertium...* Petropolis: Acad. Imp. Sci. also in *Leonhardi Euleri opera omnia*, ser. 1, Vol 13. Leipzig/Berlin, 1914

Kline, M. 1972. *Mathematical thought from ancient to modern times*. New York: Oxford Univ. Press.

Lagrange, J. L. 1774. Sur l'intégration des équations a différences partielles du premier ordre. *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres, Berlin* (1772), 353-372; also in *Oeuvres de J. L. Langrange*, Vol. e pp. 549-575. Paris, 1869.

——— 1776. Sur les integrales particulières des équations différentielles. *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres, Berlin* (1774), 197-275; also in *Oeuvres de J. L. Lagrange*, Vol. 4, pp. 5-108. Paris, 1869.

Taton, R. 1974. Inventaire chronologique de l'oeuvre de Lagrange *Revue d'histoire des sciences* 27, 3-36.

Taylor, B. 1715. *Methodus incrementorum directa et inversa*. London.

Truesdell, C. 1960. The rational mechanics of flexible or elastic bodies 1638-1788. In *Leonhardi Euleri opera omnia*, Ser. 2, Vol. 11. Zürich.