

ON THE APPROXIMATION BY LÜROTH SERIES

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Abstract

Let $x \in (0, 1]$ and p_n/q_n , $n \geq 1$ be its sequence of Lüroth Series convergents. Define the approximation coefficients $\theta_n = \theta_n(x)$ by $\theta_n = q_n x - p_n$, $n \geq 1$. In [BBDK] the limiting distribution of the sequence $(\theta_n)_{n \geq 1}$ was obtained for a.e. x using the natural extension of the ergodic system underlying the Lüroth Series expansion. Here we show that this can be done without the natural extension. We also will get a bound on the speed of convergence. Using the natural extension we will study the distribution for a.e. x of the sequence $(\theta_n, \theta_{n+1})_{n \geq 1}$ and related sequences like $(\theta_n + \theta_{n+1})_{n \geq 1}$. It turns out that for a.e. x the sequence $(\theta_n, \theta_{n+1})_{n \geq 1}$ is distributed according to a continuous singular distribution function G . Furthermore we will see that two consecutive θ 's are positively correlated.

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1 Introduction

Let $x \in (0, 1]$, then

$$x = \frac{1}{a_1} + \frac{1}{a_1(a_1 \Leftrightarrow 1)a_2} + \cdots + \frac{1}{a_1(a_1 \Leftrightarrow 1) \cdots a_{n-1}(a_{n-1} \Leftrightarrow 1)a_n} + \cdots, \quad (1)$$

where $a_n \geq 2$, $n \geq 1$. J. Lüroth, who introduced the series expansion (1) in 1883, showed (among other things) that every irrational number x has a unique infinite expansion (1) and that each rational either has a finite or an infinite periodic expansion, see also [L] and [Pe]. The series expansion (1) of x is called the *Lüroth Series* of x .

Dynamically the Lüroth series expansion (1) of x is generated by the operator $T : [0, 1] \rightarrow [0, 1]$, defined by

$$Tx := \lfloor \frac{1}{x} \rfloor \left(\lfloor \frac{1}{x} \rfloor + 1 \right) x \Leftrightarrow \lfloor \frac{1}{x} \rfloor, \quad x \neq 0; \quad T0 := 0, \quad (2)$$

(see also figure 1), where $\lfloor \xi \rfloor$ denotes the greatest integer not exceeding ξ . For $x \in [0, 1]$ we define $a(x) := \lfloor \frac{1}{x} \rfloor + 1$, $x \neq 0$; $a(0) := \infty$ and $a_n(x) = a(T^{n-1}x)$ for $n \geq 1$. From (2) it follows that $Tx = a_1(a_1 \Leftrightarrow 1)x \Leftrightarrow (a_1 \Leftrightarrow 1)$, and therefore

$$x = \frac{1}{a_1} + \frac{1}{a_1(a_1 \Leftrightarrow 1)}Tx = \frac{1}{a_1} + \frac{1}{a_1(a_1 \Leftrightarrow 1)a_2} + \cdots + \frac{T^n x}{a_1(a_1 \Leftrightarrow 1) \cdots a_n(a_n \Leftrightarrow 1)}.$$

Putting

$$\frac{p_n}{q_n} = \frac{1}{a_1} + \sum_{k=1}^{n-1} \frac{1}{a_1(a_1 \Leftrightarrow 1) \cdots a_k(a_k \Leftrightarrow 1)a_{k+1}}, \quad n \geq 1, \quad (3)$$

where $q_1 := a_1$; $q_n = a_1(a_1 \Leftrightarrow 1) \cdots a_{n-1}(a_{n-1} \Leftrightarrow 1)a_n$, $n \geq 2$, it follows from (3) that

$$x \Leftrightarrow \frac{p_n}{q_n} = \frac{T^n x}{q_n(a_n \Leftrightarrow 1)}, \quad n \geq 1. \quad (4)$$

From $a_n \geq 2$ and $0 \leq T^n x \leq 1$ it follows that the series from (1) converges to x . We will write

$$x = \langle a_1, a_2, \dots, a_n, \dots \rangle \quad \text{and} \quad \frac{p_n}{q_n} = \langle a_1, a_2, \dots, a_n \rangle. \quad (5)$$

In [JdV], H. Jager and C. de Vroedt showed that the stochastic variables $a_1(x), \dots, a_n(x), \dots$ are independent¹ with $\lambda_1(a_n = k) = \frac{1}{k(k-1)}$ for $k \geq 2$, and that T is measure preserving and ergodic with respect to Lebesgue measure. From the ergodicity of T and Birkhoff's Individual Ergodic Theorem a number of results were obtained, analogous to classical results on continued fractions, e.g.

$$\lim_{n \rightarrow \infty} (a_1 a_2 \cdots a_n)^{1/n} = e^c, \quad \text{a.e. where } c \approx 1.25,$$

$$\lim_{n \rightarrow \infty} \log(x \Leftrightarrow \frac{p_n}{q_n}) = \Leftrightarrow d, \quad \text{a.e., where } d \approx 2.03.$$

Here and in the following a.e. will be with respect to Lebesgue measure.

Figure 1

In view of (4) it is natural to define and study the so-called *approximation coefficients* $\theta_n = \theta_n(x)$, $n \geq 1$, defined by

$$\theta_n = \theta_n(x) := q_n \left| x \Leftrightarrow \frac{p_n}{q_n} \right|, \quad n \geq 1.$$

As in the case of the regular continued fraction these θ 's give an indication of "the quality of approximation of x by its n -th convergent² p_n/q_n ", see also [JK]. Note that the absolute value signs are in fact superfluous here. In view of (4) one has

$$\theta_n = \frac{T^n x}{a_n \Leftrightarrow 1}, \quad n \geq 1. \quad (6)$$

¹Here and in the following λ_n will denote Lebesgue measure on \mathbf{R}^n .

²In case of the regular continued fraction one defines $\Theta_n := q_n |q_n x - p_n|$, $n \geq 1$, where p_n/q_n is the n -th regular convergent of x .

Putting $T_n := T^n x$ it follows from (2) and (5) that

$$T_n = \langle a_{n+1}, a_{n+2}, \dots \rangle .$$

We say that T_n is *the future of x at time n* . Similarly is

$$V_n = \langle a_n, a_{n-1}, \dots, a_1 \rangle = \frac{1}{a_n} + \frac{1}{a_n(a_n \Leftrightarrow 1)a_{n-1}} + \dots + \frac{1}{a_n(a_n \Leftrightarrow 1) \dots a_2(a_2 \Leftrightarrow 1)a_1}$$

the past of x at time n . Putting $V_0 := 0$, from (6) one sees that θ_n is expressed in terms of both the past (viz. a_n) and the future. Therefore, in order to obtain the distribution of the sequence $(\theta_n)_{n \geq 1}$ for a.e. x the natural extension of the ergodic system $((0,1], \mathcal{B}_1, \lambda_1, T)$ (here \mathcal{B}_1 is the collection of Borel sets of $(0,1]$) was constructed in [BBDK].

Theorem 1 ([BBDK]) *Let $\Omega := [0,1] \times [0,1]$ and \mathcal{B}_2 be the collection of Borel sets of Ω . Let $\mathcal{T} : \Omega \rightarrow \Omega$ be defined by*

$$\mathcal{T}(x, y) := \left(Tx, \frac{1}{a(x)} + \frac{y}{a(x)(a(x) \Leftrightarrow 1)} \right), \quad (x, y) \in \Omega,$$

then the system

$$([0,1] \times [0,1], \mathcal{B}_2, \lambda_2, \mathcal{T}_\varepsilon)$$

is the natural extension of $([0,1], \mathcal{B}_1, \lambda_1, T_\varepsilon)$. Moreover, $([0,1] \times [0,1], \mathcal{B}_2, \lambda_2, \mathcal{T}_\varepsilon)$ is Bernoulli.

From this theorem we have the following lemma.

Lemma 1 ([BBDK]) *For almost all x the two-dimensional sequence*

$$\mathcal{T}^n(x, 0) = (T_n, V_n), \quad n \geq 1,$$

is uniformly distributed over $\Omega = [0,1] \times [0,1]$.

The distribution of the sequence $(\theta_n)_{n \geq 1}$ now follows from lemma 1.

Theorem 2 ([BBDK]) *For almost all x and for every $z \in (0,1]$ the limit*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \# \{ 1 \leq j \leq N : \theta_j(x) < z \}$$

exists and equals $F(z)$, where

$$F(x) = \sum_{k=2}^{\lfloor \frac{1}{z} \rfloor + 1} \frac{z}{k} + \frac{1}{\lfloor \frac{1}{z} \rfloor + 1}, \quad 0 < z \leq 1. \quad (7)$$

Taking the first moment, theorem 2 yields that for a.e. x

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \theta_n = \frac{\zeta(2) \Leftrightarrow 1}{2} = 0.322467 \dots, \quad (8)$$

where $\zeta(s)$ is the zeta-function.

In fact one needs not use the natural extension to study the distribution of the sequence $(\theta_n)_{n \geq 1}$. Since

$$T_n = \frac{1}{a_{n+1}} + \frac{T_{n+1}}{a_{n+1}(a_{n+1} \Leftrightarrow 1)},$$

see also (2), it follows that

$$a_{n+1}T_n = 1 + \frac{T_{n+1}}{a_{n+1} \Leftrightarrow 1}$$

and therefore (6) yields that

$$\theta_{n+1} = a_{n+1}T_n \Leftrightarrow 1, \quad n \geq 1, \quad (9)$$

i.e. the distribution of the sequence $(\theta_n)_{n \geq 1}$ can be obtained from $([0, 1], \mathcal{B}, \lambda, T_\varepsilon)$.

In general the ergodic theorem does not yield any information on the speed of convergence, see also [P], section 3.2, where examples are given to show that convergence can be arbitrarily slow. Here we are however in the situation that we can apply theorem 5.8 from [JdV], which yields the following much stronger result.

Theorem 3 *For almost all x and for every $z \in (0, 1]$ one has for every $\varepsilon > 0$*

$$\frac{1}{N} \# \{ 1 \leq j \leq N : \theta_j(x) < z \} \Leftrightarrow F(z) = o(N^{-\frac{1}{2}} \log^{\frac{3+\varepsilon}{2}} N), \quad n \rightarrow \infty.$$

In this paper we will study the distribution for a.e. x of the sequence $(\theta_n, \theta_{n+1})_{n \geq 1}$ and related sequences like $(\theta_n + \theta_{n+1})_{n \geq 1}$. We will show that two consecutive θ 's are positively correlated.

2 On the relation between θ_n and θ_{n+1}

From (7) and (9) it is natural to define the map $\Psi : \Omega \rightarrow \Omega$, given by

$$\Psi(x, y) := \left(\frac{x}{a(y) \Leftrightarrow 1}, a(x)x \Leftrightarrow 1 \right), \quad (x, y) \in \Omega.$$

Obviously one has

$$\Psi(T_n, V_n) = (\theta_n, \theta_{n+1}), \quad n \geq 1. \quad (10)$$

Putting

$$V_{A,B} := \{(x, y) \in \Omega : a(x) = A, a(y) = B\}, \quad A, B \geq 2,$$

one finds

$$V_{A,B} = \left(\frac{1}{A}, \frac{1}{A \Leftrightarrow 1} \right] \times \left(\frac{1}{B}, \frac{1}{B \Leftrightarrow 1} \right].$$

For $(x, y) \in V_{A,B}$ one has $\Psi(x, y) = (\frac{x}{B-1}, Ax \Leftrightarrow 1)$ (where $1/A < x \leq 1/(A \Leftrightarrow 1)$). Hence putting

$$\begin{cases} \alpha := \frac{x}{B-1} \Leftrightarrow x = (B \Leftrightarrow 1)\alpha \\ \beta := Ax \Leftrightarrow 1 \end{cases}$$

yields

$$\beta = A(B \Leftrightarrow 1)\alpha \Leftrightarrow 1, \quad \alpha \in \left(\frac{1}{A(B \Leftrightarrow 1)}, \frac{1}{(A \Leftrightarrow 1)(B \Leftrightarrow 1)} \right]. \quad (11)$$

Thus we see that Ψ maps the rectangle $V_{A,B}$ onto the line segment $L_{A,B}$, which has endpoints $(\frac{1}{A(B-1)}, 0)$ and $(\frac{1}{(A-1)(B-1)}, \frac{1}{A-1})$. Notice that from (10) and (11) one has

$$\theta_{n+1} = a_{n+1}(a_n \Leftrightarrow 1)\theta_n \Leftrightarrow 1, \quad n \geq 1, \quad (12)$$

and $(\theta_n, \theta_{n+1}) \in \Xi$, where

$$\Xi := \bigcup_{A,B \geq 2} L_{A,B},$$

see also figure 2.

Figure 2

Notice, that from (7) it follows that always

$$0 \leq \theta_n < 1, \quad n \geq 1.$$

Note that figure 2 shows that a Vahlen-type theorem as one has for the continued fraction (see [JK]) is not possible for Lüroth Series. That is, there does not exist a constant $c < 1$, such that for every x one has

$$\min(\theta_n(x), \theta_{n+1}(x)) < c$$

(recall that for continued fractions always $0 \leq \Theta_n(x) < 1$ and $\min(\Theta_n(x), \Theta_{n+1}(x)) < 1/2$). However, it is also clear from figure 2, that

$$(T_n, V_n) \notin V_{2,2} \Leftrightarrow \theta_n < \frac{1}{2}$$

and

$$(T_n, V_n) \in (\frac{1}{2}, \frac{3}{4}) \times (\frac{1}{2}, 1] \Rightarrow \theta_{n+1} < \frac{1}{2}.$$

We have the following proposition, which follows directly from lemma 1 (see also theorem 2 with $z = 1/2$).

Proposition 1 *For almost all x one has with probability $3/4$ that $\theta_n < \frac{1}{2}$ and with probability $7/8$ that*

$$\min(\theta_n(x), \theta_{n+1}(x)) < \frac{1}{2}.$$

Furthermore, given that $\theta_n < 1/2$ one has with probability $5/6$ that $\theta_{n+1} < 1/2$. The same holds when θ_n and θ_{n+1} are interchanged.

Remarks In view of (12) it is obvious that for a.e. x two consecutive θ 's are NOT independent. In fact proposition 1 suggests that two consecutive θ 's are positively correlated. That this is the case almost surely is shown in section 3.2. The situation here is similar to that for the regular continued fraction; there Vahlen's theorem suggests that two consecutive Θ 's are negatively correlated. This is indeed the case as was shown by Vincent Nolte in an unpublished document, see also [N].

3 On the distribution of $(\theta_n, \theta_{n+1})_{n \geq 1}$

In this section we will show in 3.1 that for almost all x the sequence $(\theta_n, \theta_{n+1})_{n \geq 1}$ is distributed according to a continuous singular distribution function G . Before stating the result we first recall the definition of a continuous distribution function, see also [T], p. 20. In 3.2 we will study for a.e. x the distribution of the sequence $(\theta_n + \theta_{n+1})_{n \geq 1}$, which will then be used to show that two consecutive θ 's are positively correlated.

3.1 A continuous singular distribution function

Definition 1 A distribution function G is said to be continuous singular if it is continuous and if there exists a Borel set S with Lebesgue measure zero such that $\mu_G(S) = 1$. Here μ_G denotes the Lebesgue-Stieltjes measure determined by G .

We have the following theorem.

Theorem 4 For almost all x and for all $(z_1, z_2) \in [0, 1] \times [0, 1]$ the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \# \{ 1 \leq j \leq N : \theta_j(x) < z_1, \theta_{j+1} < z_2 \}$$

exists and equals $G(z_1, z_2)$, where G is given by

$$G(\xi, \eta) := \sum_{A, B \geq 2} \lambda_2(V_{A, B}^*(\xi, \eta)), \quad (\xi, \eta) \in \Omega, \quad (13)$$

where

$$V_{A, B}^*(\xi, \eta) := \{(\alpha, \beta) \in V_{A, B} : \alpha < \min((B \Leftrightarrow 1)\xi, \frac{1+\eta}{A})\}. \quad (14)$$

Finally, G is a continuous singular distribution function with support Ξ .

Proof The first assertion follows from (7), (9) and lemma 1. In order to show that G is a continuous distribution function we have to show, see also [T], section 2.2 :

- (i) $G(x_1, x_2) \rightarrow 1$ as $\min(x_1, x_2) \rightarrow \infty$.
- (ii) For each $i \in \{1, 2\}$, $G(x_1, x_2) \rightarrow 0$ as $x_i \rightarrow \Leftrightarrow \infty$.
- (iii) $G(x_1, x_2)$ is continuous.
- (iv) Let $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$, where $a_i < b_i$, $i \in \{1, 2\}$ and put

$$[\mathbf{a}, \mathbf{b}] := \{\mathbf{x} = (x_1, x_2) \in \mathbf{R}^2 : a_i < x_i \leq b_i, i \in \{1, 2\}\}.$$

Then for each cel $[\mathbf{a}, \mathbf{b}] \subset \mathbf{R}^2$ we must have

$$\Delta_{\mathbf{a}}^{\mathbf{b}} G \geq 0,$$

where

$$\Delta_{\mathbf{a}}^{\mathbf{b}} G = G(b_1, b_2) \Leftrightarrow G(a_1, b_2) \Leftrightarrow G(b_1, a_2) + G(a_1, a_2).$$

Notice that (i) and (ii) follow from the definition of G ; clearly G is monotone in each of its coordinates, and in case $x_i < 0$ (for $i \in \{1, 2\}$) one has that $G(x_1, x_2) = 0$. In case $\min(x_1, x_2) \geq 1$ it follows that $G(x_1, x_2) = 1$. That G is continuous clearly follows from (13). In order to prove (iv) we introduce for $A, B \geq 2$ a function $G_{A,B} : \Omega \rightarrow \mathbf{R}$, given by

$$G_{A,B}(\xi, \eta) := \lambda_2(V_{A,B}^*(\xi, \eta)), \quad (\xi, \eta) \in \Omega,$$

where $V_{A,B}^*(\xi, \eta)$ is as in (14). Notice that

$$G(\xi, \eta) := \sum_{A,B \geq 2} G_{A,B}(\xi, \eta), \quad (\xi, \eta) \in \Omega.$$

It is now sufficient to show that for all $A, B \geq 2$ and each $\text{cel } (\mathbf{a}, \mathbf{b}] \subset \Omega$ one has

$$\Delta_{\mathbf{a}}^{\mathbf{b}} G_{A,B} \geq 0.$$

Fix $A, B \geq 2$ and let

$$m(\xi, \eta) = m_{A,B}(\xi, \eta) := \min \left((B \Leftrightarrow 1)\xi, \frac{1+\eta}{A} \right)$$

and $\pi_1(\xi, \eta) = \pi_{(A,B),1} := (B \Leftrightarrow 1)\xi$, $\pi_2(\xi, \eta) = \pi_{(A,B),2} := \frac{1+\eta}{A}$, one has the following, possibly overlapping, cases.

(I) $m(a_1, b_2) < m(b_1, b_2)$ and

(Ia) $m(a_1, a_2) < m(b_1, a_2)$.

Notice that the monotonicity of π_2 as a function of its first coordinate yields that

$$\pi_1(a_1, a_2) < \pi_2(a_1, a_2)$$

and therefore $m(a_1, b_2) = m(a_1, a_2)$, from which it follows, by definition of $G_{A,B}$:

$$\Delta_{\mathbf{a}}^{\mathbf{b}} G_{A,B} = G_{A,B}(b_1, b_2) \Leftrightarrow G_{A,B}(b_1, a_2) \geq 0.$$

(Ib) $m(a_1, a_2) = m(b_1, a_2)$.

In this case one has

$$\Delta_{\mathbf{a}}^{\mathbf{b}} G_{A,B} = G_{A,B}(b_1, b_2) \Leftrightarrow G_{A,B}(a_1, b_2) > 0.$$

(II) $m(a_1, b_2) = m(b_1, b_2)$, which implies that $\pi_2(a_1, b_2) \leq \pi_1(a_1, b_2)$, which in turn yields that

$$\pi_2(a_1, a_2) \leq \pi_1(a_1, a_2).$$

But then we only can have that

$$m(a_1, a_2) = m(b_1, a_2),$$

from which it at once follows that

$$\Delta_{\mathbf{a}}^{\mathbf{b}} G_{A,B} = 0.$$

(III) $m(b_1, a_2) < m(b_1, b_2)$: see case **(I)**.

(IV) $m(b_1, a_2) = m(b_1, b_2)$: see case (II).

In order to show that $\mu_G(\Xi) = 1$, or equivalently that $\mu_G(\Xi^c) = 0$, it is sufficient to show that for each cel $(\mathbf{a}, \mathbf{b}] \subset \Omega$, for which

$$\text{card}((\mathbf{a}, \mathbf{b}] \cap \Xi) \leq 2,$$

one has that $\mu_G((\mathbf{a}, \mathbf{b}]) = 0$, which is equivalent with

$$\Delta_{\mathbf{a}}^{\mathbf{b}} G = 0.$$

Notice that we may assume that $(\mathbf{a}, \mathbf{b}]$ is contained in \mathcal{S}_k for some $k \geq 2$, where

$$\mathcal{S}_k := \left(\frac{1}{k}, \frac{1}{k \Leftrightarrow 1}\right] \times [0, 1], \quad \text{for } k \geq 2.$$

Obviously there are only finitely many values of A and B such that $L_{A,B} \cap \mathcal{S}_k \neq \emptyset$. Let A and B two such values, then $(\mathbf{a}, \mathbf{b}]$ either "lies above" $L_{A,B}$ or "below" $L_{A,B}$. Let $\mathcal{U} = \mathcal{U}(\mathbf{a}, \mathbf{b})$ be the collection of all pairs (A, B) for which $(\mathbf{a}, \mathbf{b}]$ "lies above" $L_{A,B}$.

Clearly one has

$$\mu_G((\mathbf{a}, \mathbf{b}]) = \mu_G((\mathbf{a}^*, \mathbf{b}]) \Leftrightarrow \mu_G((\mathbf{a}^*, \mathbf{b}^*]),$$

where $\mathbf{a}^* := (a_1, 0)$ and $\mathbf{b}^* := (b_1, a_2)$. For $(A, B) \in \mathcal{U}$ we now define $L_{A,B}^*$ by

$$L_{A,B}^* := \{(x, y) \in L_{A,B} : a_1 \leq x \leq b_1\},$$

then

$$\begin{aligned} \mu_G((\mathbf{a}^*, \mathbf{b}]) &= G(b_1, b_2) \Leftrightarrow G(a_1, b_2) = \lambda_2\left(\bigcup_{(A,B) \in \mathcal{U}} \Psi^{-1} L_{A,B}^*\right) \\ &= G(b_1, a_2) \Leftrightarrow G(a_1, a_2) = \mu_G((\mathbf{a}^*, \mathbf{b}^*]), \end{aligned}$$

from which the theorem follows. \square

3.2 On the correlation between θ_n and θ_{n+1}

In section 2 we saw that it is likely that θ_n and θ_{n+1} are positively correlated. In order to show this, we first give some definitions.³

Definition 2 *The correlation-coefficient $\rho(\theta_n, \theta_{n+1})$ of θ_n and θ_{n+1} is defined by*

$$\rho(\theta_n, \theta_{n+1}) := \frac{E(\theta_n \theta_{n+1}) \Leftrightarrow E(\theta_n)E(\theta_{n+1})}{\sqrt{V(\theta_n)}\sqrt{V(\theta_{n+1})}},$$

where $E(\theta_n)$ is the expectation of θ_n , as given in (8) and $V(\theta_n)$ is the variance of θ_n , defined by

$$V(\theta_n) := E(\theta_n^2) \Leftrightarrow (E(\theta_n))^2.$$

The nominator of $\rho(\theta_n, \theta_{n+1})$ equals the covariance $C(\theta_n, \theta_{n+1})$ of θ_n and θ_{n+1} .

³In section 3.1 we assume that the stochastic variables θ_n for $n \geq 1$ are all identically distributed with distribution function F as given in (7).

Definition 3 Let X and Y be two stochastic variables. Then the covariance $C(X, Y)$ of X and Y is given by

$$C(X, Y) := E((X \Leftrightarrow E(X))(Y \Leftrightarrow E(Y))) .$$

Notice that $C(X, Y) > 0$ indicates that whenever X (resp. Y) is bigger/smaller than its mean $E(X)$ (resp. $E(Y)$), the same is likely to hold for Y (resp. X), i.e. X and Y are positively correlated. Similarly $C(X, Y) < 0$ tells us that X and Y are negatively correlated. In case $C(X, Y) = 0$ we say that X and Y are uncorrelated. One has that

$$X \text{ and } Y \text{ are independent} \Rightarrow X \text{ and } Y \text{ are uncorrelated,}$$

but the converse does not hold in general.

Theorem 5 For almost all x one has that

$$\rho(\theta_n, \theta_{n+1}) = \frac{(\zeta(2) \Leftrightarrow 1)((1 \Leftrightarrow 3\zeta(2) + 4\zeta(3))}{4\zeta(3) \Leftrightarrow (\zeta(2) \Leftrightarrow 1)(1 + 3\zeta(2))} = 0.5744202 \dots$$

and therefore θ_n and θ_{n+1} are positively correlated a.s.

Since $E(\theta_n) = E(\theta_{n+1})$ and $E(\theta_n^2) = E(\theta_{n+1}^2)$ one has

$$\rho(\theta_n, \theta_{n+1}) = \frac{E(\theta_n \theta_{n+1}) \Leftrightarrow E(\theta_n)^2}{V(\theta_n)} .$$

Notice, that from theorem 2 one has that F has density f , where

$$f(x) = \sum_{\ell=2}^k \frac{1}{\ell}, \quad \text{for } x \in \left(\frac{1}{k}, \frac{1}{k \Leftrightarrow 1}\right], \quad k \geq 2.$$

Taking second moments thus yields

$$E(\theta_n^2) = \sum_{k=2}^{\infty} \int_{\frac{1}{k}}^{\frac{1}{k-1}} z^2 f(z) dz = \sum_{k=2}^{\infty} \frac{1}{3k} \frac{1}{(k \Leftrightarrow 1)^3} = \frac{1 \Leftrightarrow \zeta(2) + \zeta(3)}{3} = 0.185708 \dots$$

But then it follows from (8) that

$$V(\theta_n) = \frac{4\zeta(3) \Leftrightarrow (\zeta(2) \Leftrightarrow 1)(1 + 3\zeta(2))}{12} = 0.0817226 \dots$$

In order to find $E(\theta_n \theta_{n+1})$ we will determine $E((\theta_n + \theta_{n+1})^2)$, since

$$E((\theta_n + \theta_{n+1})^2) = 2E(\theta_n^2) + 2E(\theta_n \theta_{n+1}) .$$

Hence we find for Lüroth series that

$$E(\theta_n \theta_{n+1}) = \frac{1}{2} E((\theta_n + \theta_{n+1})^2) \Leftrightarrow E(\theta_n^2) .$$

From (7) and (9) one has

$$\theta_n + \theta_{n+1} = (a_{n+1} + \frac{1}{a_n \Leftrightarrow 1}) T_n \Leftrightarrow 1, \quad n \geq 1, \quad (15)$$

and from the definition of $L_{A,B}$ it follows that

$$\frac{1}{a_{n+1}(a_n \Leftrightarrow 1)} < \theta_n + \theta_{n+1} \leq \frac{a_n}{(a_{n+1} \Leftrightarrow 1)(a_n \Leftrightarrow 1)}, \quad n \geq 1. \quad (16)$$

From lemma 1, (16) and (17) we have the following theorem.

Theorem 6 For almost all x and for every $z \in (0, 2]$ the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \# \{ 1 \leq j \leq N : \theta_j(x) + \theta_{j+1} < z \}$$

exists and equals $S(z)$, where S is a continuous distribution function with density s , given by

$$s(z) = \sum_{A, B \geq 2} \frac{B \Leftrightarrow 1}{A(B \Leftrightarrow 1) + 1} \mathbf{1}_{(\frac{1}{A(B-1)}, \frac{B}{(A-1)(B-1)}]}(z).$$

Here $\mathbf{1}_{(\alpha, \beta]}(z)$ is the indicator fuction of the interval $(\alpha, \beta]$, i.e.

$$\mathbf{1}_{(\alpha, \beta]}(z) := \begin{cases} 1 & , z \in (\alpha, \beta], \\ 0 & , z \notin (\alpha, \beta]. \end{cases}$$

Theorem 6 now at once yields that

$$\begin{aligned} E((\theta_n + \theta_{n+1})^2) &= \sum_{B=2}^{\infty} \sum_{A=2}^{\infty} \frac{1}{B} \int_{\frac{1}{A(B-1)}}^{\frac{B}{(A-1)(B-1)}} \frac{1}{A(B \Leftrightarrow 1) + 1} z^2 dz \\ &= \sum_{A, B \geq 2} \frac{A^2 B^2 + AB(A \Leftrightarrow 1) + (A \Leftrightarrow 1)^2}{3BA^3(A \Leftrightarrow 1)^3(B \Leftrightarrow 1)^3} = \frac{3 \Leftrightarrow 3\zeta(2) + 2\zeta(2)\zeta(3)}{3} = 0.6737 \dots \end{aligned}$$

But then

$$E(\theta_n \theta_{n+1}) \Leftrightarrow E(\theta_n)E(\theta_{n+1}) = \frac{(\zeta(2) \Leftrightarrow 1)((1 \Leftrightarrow 3\zeta(2) + 4\zeta(3))}{12},$$

and therefore the first assertion of theorem 5 is immediate. It follows that θ_n and θ_{n+1} are indeed positively correlated. \square

Notice that the proof of theorem 6 can easily be adapted to derive the distribution for a.e. x of the sequence $(\theta_n \Leftrightarrow \theta_{n+1})_{n \geq 1}$. We leave this to the reader, but mention one - surprising - case : one has that the probability $\bar{P}(\theta_n < \theta_{n+1})$ that θ_n is smaller than θ_{n+1} is $0.391 \dots$, i.e. θ_{n+1} has the tendency to be smaller than its predecessor. To be more precise, we have the following proposition.

Proposition 2 For almost all x one has that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \# \{ 1 \leq j \leq N : \theta_j(x) < \theta_{j+1} \} = 0.391 \dots$$

Proof From (6) and (9) it follows that $\theta_n < \theta_{n+1}$ is equivalent with

$$T_n > \frac{a_n \Leftrightarrow 1}{a_{n+1}(a_n \Leftrightarrow 1) \Leftrightarrow 1}.$$

But then lemma 1 yields that $\lim_{N \rightarrow \infty} \frac{1}{N} \# \{ 1 \leq j \leq N : \theta_j(x) < \theta_{j+1} \}$ exists for a.e. x , and equals $\lambda(\mathcal{D})$, where \mathcal{D} is given by

$$\mathcal{D} = \bigcup_{A, B \geq 2} \left[\frac{B}{AB \Leftrightarrow 1}, \frac{B}{A(B \Leftrightarrow 1) \Leftrightarrow 1} \right) \times \left[0, \frac{1}{B} \right],$$

see also figure 3. The proposition now follows from

$$\lambda(\mathcal{D}) = \sum_{A=2}^{\infty} \sum_{B=2}^{\infty} \frac{1}{(AB \Leftrightarrow A \Leftrightarrow 1)(AB \Leftrightarrow 1)B} = 0.391 \dots \square$$

Figure 3

Final remarks Recently Jose Barrionuevo, Bob Burton and the present authors generalized the whole concept of Lüroth Series, see also [BBDK]. A new class of series expansions, the so-called *Generalized Lüroth Series* (or GLS), was introduced and their ergodic properties were studied. Examples of these GLS are the recent *alternating Lüroth Series*, as introduced by S. Kalpazidou and A. and J. Knopfmacher, but also familiar expansions like r -adic expansions (for $r \in \mathbf{Z}$, $r \geq 2$). Although β -expansions are not in this class, it turned out that many important ergodic properties of these expansions can be obtained using the appropriate GLS-expansion, see also [DKS]. Here we only want to mention that the entire approach of this paper can be carried over to GLS-expansions. In order to keep the exposition clear and easy we only dealt with the "classical" Lüroth expansion. Details are left to the reader, see also section 3.1 of [BBDK].

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