THE EXPANSION OF THE FOKKER–PLANCK EQUATION INCLUDING A CRITICAL POINT

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The known expansion of the master equation for weak diffusion in an external potential applies to both the *monostable* and the *bistable* case, but fails at the critical point. This can be remedied by taking as zeroth order approximation a suitably defined set of eigenfunctions. The resulting expansion is uniformly valid in a range of the pump parameter *including the critical point.*

1. Introduction. We shall consider the Fokker–Planck equation [1–3]

$$\frac{\partial P(z,t)}{\partial t} = -\frac{\partial}{\partial z}[\alpha_1(z)P] + \tfrac{1}{2}\vartheta\frac{\partial^2}{\partial z^2}[\alpha_2(z)P]\,, \quad (1.1)$$

in the weak diffusion limit $\vartheta \downarrow 0$. For $\vartheta = 0$ it is equivalent to the deterministic equation

$$\dot{z} = \alpha_1(z)\,. \quad (1.2)$$

Let $\varphi(t)$ be a solution of eq. (1.2) with a certain initial value. Suppose it is asymptotically stable in the linear approximation, i.e. $\alpha_1'(\varphi) < 0$. This will be called the normal case. Then one obtains approximate solutions of eq. (1.1) setting $z = \varphi(t) + \vartheta^{1/2}\xi$. In terms of ξ eq. (1.1) separates into a linear Fokker–Planck equation [4,5]

$$\frac{\partial P(\xi,t)}{\partial t} = -\alpha_1'(\varphi)\frac{\partial}{\partial \xi}\xi P + \tfrac{1}{2}\alpha_2(\varphi)\frac{\partial^2 P}{\partial \xi^2} \quad (1.3)$$

and a corrective remainder. This remainder is small and can be incorporated into the solution systematically in successive powers of $\vartheta^{1/2}$. Clearly, eq. (1.3) ceases to be meaningful as a zeroth order approximation at the critical point where $\alpha_1'(\varphi) = 0$. In that case initial fluctuations would grow beyond bound, reflecting the absence of linear stability in eq. (1.2). In this note we outline the expansion of eq. (1.1) in terms of ϑ including such marginal or critical cases.

We take the diffusion function $\alpha_2(z)$ to be even and positive,

$$\alpha_2(z) = \alpha_2^{(0)} + \alpha_2^{(2)}z^2 + \alpha_2^{(4)}z^4 + \dots\,, \quad (1.4)$$

and consider the odd drift function

$$\alpha_1(z) = \alpha_1^{(1)}z + \alpha_1^{(3)}z^3 + \alpha_1^{(5)}z^5 + \dots\,. \quad (1.5)$$

Obviously, eq. (1.2) allows for one steady state $\varphi_0 = 0$, which is stable if $\alpha_1^{(1)} < 0$. When $\alpha_1^{(1)} > 0$, it becomes unstable [6,7], Assuming $\alpha_1^{(3)} < 0$, eq. (1.2) then yields two non-zero stable steady states φ_+ and $\varphi_- = -\varphi_+$. Throughout it is understood that eq. (1.2) has no other real steady state solutions.

We now distinguish between three regimes of the pump parameter: (A) normal region well *above* the critical point, $\alpha_1^{(1)} < 0$ and of order ϑ^0 (monostable case); (B) normal region well *below* the critical point, $\alpha_1^{(1)} > 0$ and of order ϑ^0 (bistable case); (C) *critical* region, $\alpha_1^{(1)}$ of order $\vartheta^{1/2}$.

2. The critical region. This is region C, where $\alpha_1^{(1)}$ is of order $\vartheta^{1/2}$ including zero. Putting

$$z = \vartheta^{1/4}\eta\,, \quad t = \vartheta^{-1/2}\tau\,, \quad \alpha_1^{(1)}/\alpha_1^{(3)} = \vartheta^{1/2}\Delta\,, \quad (2.1)$$

into eq. (1.1) with eqs. (1.4) and (1.5) one obtains

$$\frac{\partial P(\eta,\tau)}{\partial\tau} = -\alpha_1^{(3)}\frac{\partial}{\partial\eta}\eta(\Delta+\eta^2)P + \tfrac{1}{2}\alpha_2^{(0)}\frac{\partial^2 P}{\partial\eta^2}$$
$$-\vartheta^{1/2}\left[\alpha_1^{(5)}\frac{\partial}{\partial\eta}\eta^5 P - \tfrac{1}{2}\alpha_2^{(2)}\frac{\partial^2}{\partial\eta^2}\eta^2 P\right]$$
$$-\vartheta\left[\alpha_1^{(7)}\frac{\partial}{\partial\eta}\eta^7 P - \tfrac{1}{2}\alpha_2^{(4)}\frac{\partial^2}{\partial\eta^2}\eta^4 P\right] + \dots . \qquad (2.2)$$

Considering formally η, τ and Δ as quantities of order unity, and disregarding terms in eq. (2.2) that vanish as $\vartheta \downarrow 0$, one is left with the irreducible zeroth order approximation. By means of a trivial transformation it is written in the standard form [6,8]

$$\frac{\partial P(x,s)}{\partial s} = \frac{\partial}{\partial x}U'(x)P + \frac{\partial^2 P}{\partial x^2}, \qquad U(x) = \tfrac{1}{2}\kappa x^2 + \tfrac{1}{4}x^4 . \qquad (2.3)$$

This represents a simple diffusion process in the quartic potential $U(x)$. See fig. 1. If the solution of eqs. (2.3) is known, one can systematically calculate the higher order corrections in eq. (2.2). Of course it suffices to know the Green function solution, which is determined by the complete set of eigenfunctions (see e.g. refs. [9,10]).

3. Well above the critical point. By definition the perturbational series shown in eq. (2.2) is valid in region C (critical region). It is not difficult to see that it retains its validity in region A. Put $\eta = \vartheta^{1/4}\xi$ and return to the original coefficients and time scale. The irreducible part leads to the proper linear noise approximation about $\varphi_0 = 0$. Further, the correction term in eq. (2.2) of order $\vartheta^{p/2}$ ($p = 1, 2, \dots$) simply becomes of order $\vartheta^{1/2}\vartheta^{p/2}\vartheta^{(p-1)/2} = \vartheta^p$ (the first $\vartheta^{1/2}$ arises from the time scaling).

4. Well below the critical point. The connection with this region (B) is complicated by the existence of the *two* stable steady states $\varphi_\pm$. In order to obtain the correct normal fluctuations, we first rewrite the diffusion and drift function as

$$\alpha_2(z) = a_2^{(0)} + a_2^{(2)}(z^2-\varphi_\pm^2) + a_2^{(4)}(z^2-\varphi_\pm^2)^2 + \dots , \qquad (4.1)$$
$$\alpha_1(z) = z[a_1^{(1)} + a_1^{(3)}(z^2-\varphi_\pm^2) + a_1^{(5)}(z^2-\varphi_\pm^2)^2 + \dots] . \qquad (4.2)$$

Comparing eq. (4.1) with eq. (1.4), and eq. (4.2) with eq. (1.5), one readily obtains the simple relations

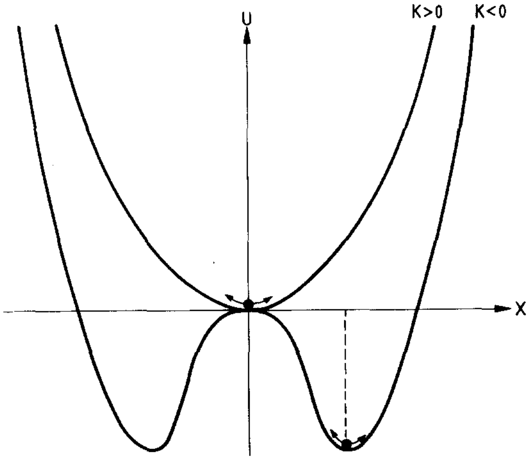

Fig. 1. The irreducible potential $U(x)$ following eqs. (2.3).

between the original and the new coefficients. In particular, $a_1^{(1)} = \alpha_1(\varphi_\pm) = 0$, $a_1^{(3)} = \alpha_1'(\varphi_\pm)/2\varphi_\pm^2$ and $a_2^{(0)} = \alpha_2(\varphi_\pm)$. In region C $a_n^{(m)} = \alpha_n^{(m)}$ + terms of order $\vartheta^{1/2}$ that vanish at the critical point, so that $a_n^{(m)}$ can be considered as a renormalized version of $\alpha_n^{(m)}$. Now putting

$$z = \vartheta^{1/4}\eta, \quad t = \vartheta^{-1/2}\tau, \quad \varphi_\pm^2 = -\vartheta^{1/2}\nabla, \qquad (4.3)$$

into eq. (1.1) with eqs. (4.1) and (4.2), one finds

$$\frac{\partial P(\eta,\tau)}{\partial\tau} = -a_1^{(3)}\frac{\partial}{\partial\eta}\eta(\nabla+\eta^2)P + \tfrac{1}{2}a_2^{(0)}\frac{\partial^2 P}{\partial\eta^2}$$
$$-\vartheta^{1/2}\left[a_1^{(5)}\frac{\partial}{\partial\eta}\eta(\nabla+\eta^2)^2 P - \tfrac{1}{2}a_2^{(2)}\frac{\partial^2}{\partial\eta^2}(\nabla+\eta^2)P\right]$$
$$-\vartheta\left[a_1^{(7)}\frac{\partial}{\partial\eta}\eta(\nabla+\eta^2)^3 P - \tfrac{1}{2}a_2^{(4)}\frac{\partial^2}{\partial\eta^2}(\nabla+\eta^2)^2 P\right] + \dots . \qquad (4.4)$$

Considering formally η, τ and ∇ as quantities of order unity, and disregarding terms in eq. (4.4) that vanish as $\vartheta \downarrow 0$, one obtains the below-critical irreducible zeroth order. Once again, by a trivial transformation it is cast into the standard form (2.3) with $\kappa \leqslant 0$. Note that eq. (2.2) can be obtained formally from eq. (4.4) by replacing all $a_n^{(m)}$ by $\alpha_n^{(m)}$, changing ∇ into Δ in the irreducible part and setting $\nabla \equiv 0$ in the correction terms.

The expansion shown in eq. (4.4) is obviously correct at the critical point $\nabla = \Delta = 0$, and below it in region C. One checks without effort that it is also valid in region B. Put $\eta = \vartheta^{-1/4}\varphi_\pm + \vartheta^{1/4}\xi$ and return to the original time scale. In view of the mentioned results for $a_1^{(3)}$

and $a_2^{(0)}$, the irreducible representation leads to the correct linear noise approximation at $\varphi_\pm$. Further, each correction factor within square brackets in eq. (4.4) scales up to order $\vartheta^{-1/2}$, which is precisely compensated by the time scaling.

5. Conclusions. We have shown how to expand the Fokker–Planck equation for a diffusion process with small diffusion coefficient ϑ and pump parameter $\alpha_1^{(1)}$, including the critical point $\alpha_1^{(1)} = 0$. The expansion proceeds in successive powers of $\vartheta^{1/2}$. It is, however, not a genuine power series as each term also contains higher orders as well. In the normal regimes well above and well below the critical point the evaluation yields the equilibrium fluctuations.

The essence of the procedure is the proper recognition of the critical region, which allows the separation of the original Fokker–Planck equation into its irreducible part and a corrective remainder. Further, a certain renormalization of coefficients in the drift function and diffusion function guarantees the correction terms to remain small also well below the critical point (in the bistable regime). More details and a similar treatment of the general markovian master equation will be reported elsewhere [11].

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