ON THE STRUCTURE OF NONLEADING LOGARITHMS

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A recursive procedure for expressing all nonleading logarithms in terms of the coefficients of the β-function and the anomalous dimensions is derived. A part of the nonleading logarithms, which dominates not far from the Landau singularity, is summed to all orders. The behaviour of the running coupling constant in this region is different from that predicted by the simple Landau pole.

The problem of the nonleading corrections to the leading-logarithm (LL) predictions of gauge theories attracts a lot of attention. Their understanding from the point of view of diagrams [1] and their extension to other hard processes [2] is intensively studied and is being adapted to practical calculations. The only complete calculation up to now provides the first nonleading corrections for deep inelastic scattering (DIS) [3].

In this letter we would like to discuss the structure of all nonleading logarithms as it emerges from the solution of the renormalization group (RG) equation. A recursive procedure for computing all nonleading logarithms will be derived. As an example the first few nonleading terms for the proton/gluon propagator will be calculated explicitly. It turns out that there also exists a region of q^2 (which we propose to label as "not far from the Landau ghost pole") where explicit summation of an important part of the nonleading contributions can be carried out. Hence an analytic expression for the photon propagator will be obtained and the approach to the Landau singularity will be discussed.

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The following discussion, if not stated otherwise, will apply to both QCD and QED. For definiteness let us consider the inverse of the "invariant charge" $[\alpha d(q^2, \alpha)]^{-1}$ of QED. It satisfies the RG equation [4]

$$[\partial/\partial t - \beta(\alpha)\alpha\, \partial/\partial\alpha]\, (\alpha d)^{-1} = 0\ , \qquad (1)$$

where

$$t = \log(-q^2/m^2), \qquad \alpha = \alpha_m = e^2/4\pi\ ,$$

$$\beta(\bar{\alpha}) = \mathrm{d}\log\bar{\alpha}(\mu)/\mathrm{d}\log\mu^2\ .$$

It is convenient to introduce the following parametrization:

$$(\alpha d)^{-1} = \frac{1}{\alpha}\sum_{N=0}^{\infty} \alpha^N d_N(\alpha t)\ . \qquad (2)$$

Here $\alpha^N d_N(x)$ corresponds to the nonleading logarithms of the Nth order ($x = \alpha t$). d_0 is just the LL result. Substituting (2) in (1) and comparing the same powers of α one obtains

$$(1-\beta_1 x)d_N' - \beta_1(N-1)d_N = \sum_{K=2}^{N+1} \beta_K\,[(N-K)d_{N+1-K} + x d_{N+1-K}'],\ \ N \geqslant 1, \qquad (3)$$

with

$$\beta(\alpha)=\sum_{N=1}^{\infty}\alpha^N\beta_N\,,\quad d_N'=\mathrm{d}d_N(x)/\mathrm{d}x\,.$$

Eq. (3) can be simplified using the standard method of variation of constants. Introducing the new variable $\xi=1-\beta_1 x$ one obtains

$$d_N\equiv C_N(\xi)/\xi^{N-1}\,, \tag{4}$$

and

$$\dot{C}_N=-\sum_{K=2}^{N+1}\frac{\beta_K}{\beta_2}\,\xi^{K-3}$$

$$\times\,[(N-K)C_{N+1-K}+\xi(\xi-1)\dot{C}_{N+1-K}]\,, \tag{5}$$

where $\dot{C}_N=\mathrm{d}C_N(\xi)/\mathrm{d}\xi$. With the boundary conditions $C_0(\xi)=1$, and $C_N(1)=0$ for $N\geqslant 1$, the set of equations (5) determines $C_N(\xi)$ once $C_0,\ldots,C_{N-1}$ are known. The recursive procedure suggested by eqs. (5) is a simple alternative to the usual algebraic determination of the nonleading corrections. In table 1 we quote the first few nonleading contributions to $(\alpha d)^{-1}$ [5]. We emphasize that we concentrate here on the structure of the relation between $\beta(\alpha)$ $(\beta_1,\beta_2,\ldots)$ and $d^{-1}(\alpha,t)$ [‡1]. The problem of calculating higher coefficients of $\beta(\alpha)$ is outside the scope of this letter.

Combining eq. (5) with the present knowledge of $\beta(\alpha)$ gives some nontrivial information about the region $\xi\approx\alpha$ [‡2]. To this end substitute eq. (4) in (2) to obtain

$$(\alpha d)^{-1}=\sum_{N=0}^{\infty}(\alpha/\xi)^{N-1}C_N(\xi)\,. \tag{6}$$

It follows from eq. (5) that the C_N's have the structure

$$C_N(\xi)=\sum_{K,L=0}^{N-1}W_{KL}^{N}\,\xi^L\log^K\xi\,,\quad N\geqslant 2\,. \tag{7}$$

Needless to say that any conclusion about the nature of the singularity at $\xi=0$ (Landau ghost) must be based on the full series (6) (cf. ref. [5]). Summation of (6)

Table 1

N	$C_N(\xi)$ a)
0	1
1	$-\bar\beta_2\log\xi$
2	$\bar\beta_2^2\log\xi-(\bar\beta_2^2+\bar\beta_3)(\xi-1)$
3	$\frac{1}{2}\bar\beta_2^3\log^2\xi+\bar\beta_2\bar\beta_3\log\xi+\frac{1}{2}(\bar\beta_2^3+\bar\beta_2\bar\beta_3-\bar\beta_4)(\xi^2-1)$ $+\bar\beta_2(\bar\beta_2^2+\bar\beta_3)(\xi-1)$

a) $\bar\beta_K=-\beta_K/\beta_1$, $K=1,2,\ldots$.

with the exact C_N's cannot be done. However, in the region

$$|\log\xi|>1\,,\quad \alpha/\xi<1\,, \tag{8a,b}$$

$$\xi<|\log\xi|\,,\quad (\alpha/\xi)\log\xi\sim\text{const.}\,, \tag{8c,d}$$

the main contribution to eq. (7) is given by the highest power of $\log\xi$ and lowest power of ξ. Closer inspec-

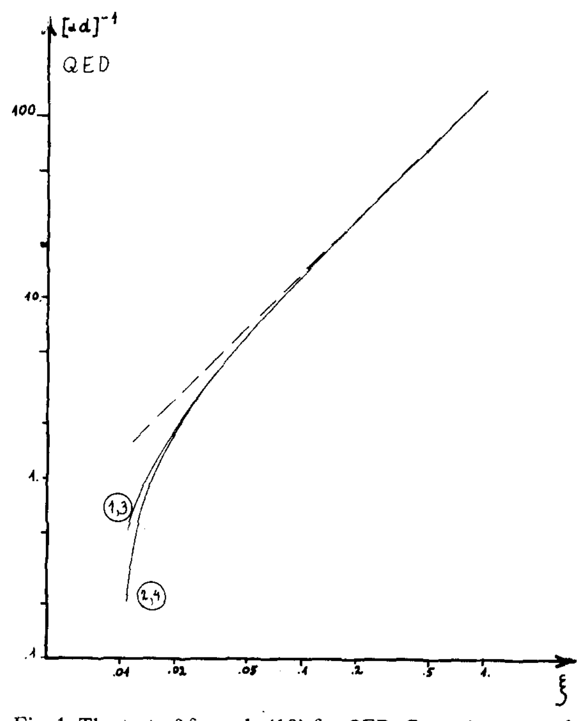

Fig. 1. The test of formula (10) for QED. Curve 1: exact solution of eq. (1) with $\beta(\alpha)$ approximated by the first two terms of the power series. Curve 2: same as curve 1 but $\beta(\alpha)$ approximated by the first three terms of the series expansion. Curve 3: Leading logarithms of the ξ approximation [eq. (10)]. Curve 4: Leading logarithms and first nonleading correction. Dashed line: leading-logarithm approximation.

‡1 Generalization to quantities with nonzero anomalous dimension $\gamma(\alpha)$ is straightforward.

‡2 For QCD this corresponds to $Q\approx\Lambda$.

tion of eq. (5) shows that

$$W^N_{N-1,0} = (N-1)^{-1}(-\beta_2/\beta_1)^N, \quad N \geqslant 2. \tag{9}$$

Neglecting other contributions gives the "leading log ξ" (LLξ) approximation for the invariant charge:

$$\alpha d = \frac{\alpha}{\xi + \alpha(\beta_2/\beta_1)\log[\xi + \alpha(\beta_2/\beta_1)\log\xi]}. \tag{10}$$

Formula (10) contains the nonleading logarithms to any order calculated in the leading log ξ approximation. It is expected to work in the region (8). Note that the highest power of log ξ in eq. (7) is generated only by the β_1 and β_2 coefficients. Hence the result (10) is independent of the renormalization prescription.

Fig. 1 compares the "exact" solutions of eq. (1) with the LL formula (dashed line), LL plus first nonleading (1NL) correction (curve 4) and the LLξ result (curve 3) given by eq. (10). By "exact" we mean the solutions of eq. (5) obtained by the standard method of characteristics with $\beta(\alpha)$ approximated, for QED, by

(1) the first two terms (curve 1) and

(2) the first three terms (curve 2) [6].

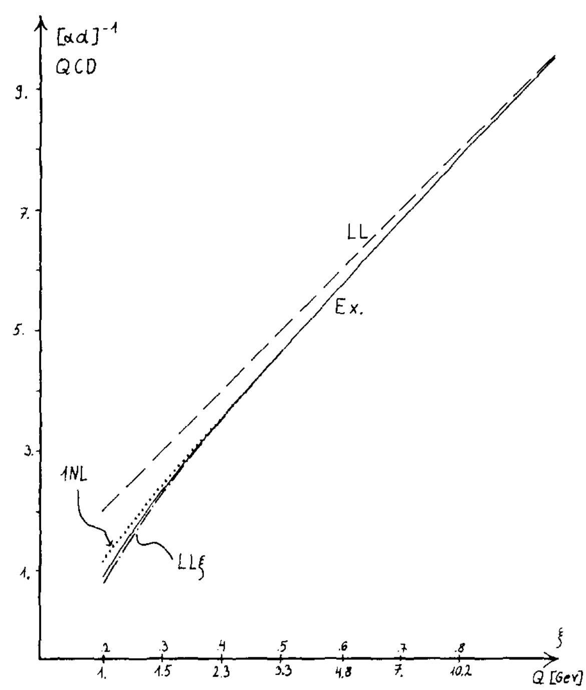

Fig. 2. Same as fig. 1 but for QCD. No β_3 coefficient is yet available for comparison.

Accidentally curves 1 and 2 coincide, on the graph, with curves 3 and 4, respectively. It is apparent from fig. 1, that below $\xi \approx 0.02$ higher terms in the expansion of $\beta(\alpha)$ will be important, hence our test is trustworthy only for $\xi > o(0.02)$. It turns out that for $\xi > 0.02$ the main correction is due to the first nonleading term. All nonleading logarithms of higher order change the 1NL result by at most 10%. They get more important for $\xi < 0.02$, but the exact result, to compare with, is not known there. At the estimated lower limit of validity of the LLξ approximation the 1NL result is changed by 100% ‡3.

In conclusion, we have proposed a recursive procedure which allows easy generation of the nonleading logarithms provided $\beta(\alpha)$ and $\gamma(\alpha)$ are known to the required accuracy. The analytic summation of the important pieces of all nonleading logarithms was performed for the example of the running coupling constant of QED (for QCD the procedure is the same, c.f. fig. 2). The region of validity, of the approximation used, ends not far from the Landau singularity $1 - \beta_1\alpha\ln(-q^2/m^2) = O/\alpha$). The running coupling constant rises faster, with $-q^2$, than both the leading-logarithm formula and the LL with the first nonleading correction. For QCD, a similar method allows one to explore a region of smaller Q^2 than considered up to now but larger than the confinement scale.

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‡3 In fact the bound (8b) has to be restricted to $\xi > \xi^*$ where ξ^* is the pole of eq. (10). At $\xi = \xi^*$ the two leading terms cancel, hence the nonleading contributions become important (for QED $\xi^* \approx 0.011$).

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