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THE COMPLEXITY OF FINDING UNIFORM EMULATIONS ON FIXED GRAPHS *

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Uniform emulations are a method to obtain efficient and structure-preserving simulations of large networks on smaller networks. We show that for every fixed graph H, which is connected (or strongly connected in the case of directed graphs) but not complete, the problem to decide whether another (strongly) connected graph G can be uniformly emulated on H is NP-complete.

Keywords: Emulation, parallel computing, NP-completeness

1. Introduction

Parallel algorithms are normally designed for execution on a suitable network of N processors with N depending on the size of the problem to be solved. In practice, N will be large and varying, whereas processor networks will be small and fixed. The resulting disparity between algorithm design and implementation must be resolved by simulating a network of some size N on a fixed and smaller size network of a similar or different kind, in a structure-preserved manner. For this purpose, a notion of simulation, termed emulation, was first proposed by Fishburn and Finkel [6]. Independently, Berman [1] proposed a similar notion. A detailed study was presented by Bodlaender and Van Leeuwen [4].

Definition. Let $G=(V_G, E_G)$ and $H=(V_H, E_H)$ be networks of processors (graphs). We say that G can be *emulated on H* if there exists a function $f: V_G \rightarrow V_H$ such that, for every edge $(g, g') \in E_G$, $f(g)=f(g')$ or $(f(g), f(g')) \in E_H$. The function f is called an *emulation function* or, in short, an emulation of G on H. We call G the *guest graph* and H the *host graph*.

Let f be an emulation of G on H. Any processor $h \in V_H$ must actively emulate the processors belonging to $f^{-1}(h)$ in G. When $g \in f^{-1}(h)$ communicates information to a neighbouring processor g', then h must communicate the corresponding information either 'internally', when it emulates g' itself, or to a neighbouring processor $h=f(g')$ otherwise. If all processors act synchronously in G, then the emulation will be slowed by a factor proportional to

$$\max_{h \in V_H} |f^{-1}(h)|.$$

Definition. Let G and H be as above. The emulation f is said to be (computationally) *uniform* iff, for all $h, h' \in V_H$, $|f^{-1}(h)| = |f^{-1}(h')|$.

Every uniform emulation has associated with it a fixed constant c, called the computation factor, such that, for all $h \in V_H$, $|f^{-1}(h)| = c$. It means that every processor of H emulates the same number of processors of G.

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Graphs and directed graphs representing the interconnection structure of a processor network will be connected and strongly connected, respectively. We shall therefore mainly consider uniform emulations of (strongly) connected graphs on (strongly) connected graphs. We assume the reader to be familiar with the theory of NP-completeness (see [7]).

In [2,5], the following problem was considered:

UNIFORM EMULATION

Instance: Connected graphs $G=(V_G, E_G)$ and $H=(V_H, E_H)$.

Question: Is there a uniform emulation of G on H?

The problem was proved to be NP-complete, even if various additional constraints are imposed on G, H and the computation factor $|V_G|/|V_H|$. In [3], the existence of polynomial-time approximation algorithms that approximate

$$\min\left\{\max_{h\in V_H} |f^{-1}(h)| \mid f \text{ emulates } G \text{ on } H\right\}$$

was discussed.

In [5] it was proved that UNIFORM EMULATION stays NP-complete if H is fixed to any graph, obtained by removing one edge from an undirected, complete graph with at least three nodes. (For instance, H can be fixed to the connected graph with three nodes and two edges.) In this paper we generalize this result to all (strongly) connected graphs that are not complete.

2. Preliminary definitions and results

Definition. Let $G=(V, E)$ be an undirected, bipartite graph. We say that G contains a *balanced complete bipartite subgraph* (abbreviated as BCBS) of $2K$ nodes, if there are two disjoint subsets V_1, $V_2 \subseteq V$ such that $|V_1|=|V_2|=K$ and $u\in V_1$, $v\in V_2$ implies $\{u, v\}\in E$.

Given a bipartite graph G and a $K\in N^+$, the problem to decide whether G contains a BCBS with $2K$ nodes is NP-complete [7]. In [5], the following variant of BALANCED COMPLETE BIPARTITE SUBGRAPH was proved to be NP-complete.

2.1. Lemma ([3]). *Let $n\in N^+$, $n\geqslant 3$. The following problem is NP-complete:*

Instance: *Bipartite graph $G=(V, E)$, with $n \mid |V|$.*

Question: *Does G contain a BCBS with $2|V|/n$ nodes?*

Proof. The proof uses a (simple) transformation from BALANCED COMPLETE BIPARTITE SUBGRAPH. □

Definition. Let

$$P_3=(\{1, 2, 3\}, \{(1, 2), (2, 3)\})$$

be the undirected graph with three nodes and two edges (= a path with three nodes)—see Fig. 1.

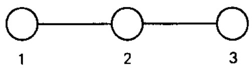

Fig. 1.

2.2. Lemma. *Let V_1, V_2 be disjoint finite sets, and let $G=(V_1\cup V_2, E_G)$ be an undirected bipartite graph with edges between nodes of V_1 and V_2 only, i.e.,*

$$(v, w)\in E_G \Rightarrow (v\in V_1 \Leftrightarrow w\in V_2).$$

Let $\tilde{G}=(V_1\cup V_2, \tilde{E}_G)$ be the undirected, bipartite graph with

$$\tilde{E}_G=\{(v, w) \mid v\in V_1 \wedge w\in V_2 \wedge (v, w)\notin E_G\}.$$

Then, there is a uniform emulation of $\tilde{G}$ on P_3 with $f(V_1)\subseteq\{1, 2\}$ and $f(V_2)\subseteq\{2, 3\}$ if and only if G contains a BCBS with $2|V|/c$ nodes.

Proof. First suppose f is a uniform emulation of $\tilde{G}$ on H. Choose $W_1=f^{-1}(1)$ and $W_2=f^{-1}(3)$. Now, $W_1\subseteq V_1$, $W_2\subseteq V_2$, and $v\in W_1$, $w\in W_2 \Rightarrow (v, w)\notin \tilde{E}_G$, hence $(v, w)\in E_G$. So, G contains a BCBS with $2|V|/c$ nodes.

Now suppose G contains a BCBS with $2|V|/c$ nodes, i.e., there are sets $W_1\subseteq V_1$, $W_2\subseteq V_2$ with $|W_1|=|W_2|=|V|/c$ and $v\in W_1$, $w\in W_2 \Rightarrow (v, w)\in E_G \Rightarrow (v, w)\notin\tilde{E}_G$. Let $f(W_1)=1$, $f(W_2)$

$= 3$, and $f(V_1 \setminus W_1) = f(V_2 \setminus W_2) = 2$. It is easy to check that f is a uniform emulation of $\tilde{G}$ on P_3 and $f(V_1) \subseteq \{1, 2\}$, $f(V_2) \subseteq \{2, 3\}$. □

2.3. Lemma. *The following problem is NP-complete:*

Instance: *Disjoint finite sets V_1, V_2, undirected graph $G = (V_1 \cup V_2, E_G)$.*

Question: *Is there a uniform emulation f of G on P_3 with $f(V_1) \subseteq \{1, 2\}$ and $f(V_2) \subseteq \{2, 3\}$?*

Proof. This lemma can be proven using Lemmas 2.1 and 2.2. □

2.4. Lemma. *Let $H = (V_H, E_H)$ be a directed graph, with $V = \{1, 2, 3\}$ and $(1, 2) \in E_H$, $(2, 3) \in E_H$, $(1, 3) \notin E_H$. (H is one of the eight graphs, shown in Fig. 2.) The following problem is NP-complete:*

Instance: *Disjoint finite sets V_1, V_2, directed graph $G = (V_1 \cup V_2, E_G)$.*

Question: *Is there a uniform emulation f of G on H, with $f(V_1) \subseteq \{1, 2\}$, $f(V_2) \subseteq \{2, 3\}$?*

Proof. The proof is similar to the one for the undirected case. □

3. Main results

3.1. Theorem. *Let $H = (V_H, E_H)$ be a connected, undirected graph that is not complete. The following problem is NP-complete:*

Instance: *A connected undirected graph $G = (V_G, E_G)$.*

Question: *Is there a uniform emulation of G on H?*

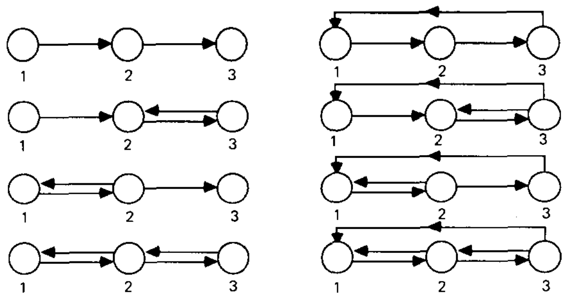

Fig. 2. Eight possible choices for H in Lemma 2.4.

Proof. Let $H = (V_H, E_H)$ be a connected, undirected graph that is not complete. Clearly, the problem is in NP. To prove NP-completeness, we transform the problem from Lemma 2.3 to it.

Let disjoint finite sets V_1, V_2 and an undirected graph $G = (V_1 \cup V_2, E_G)$ be given. We shall construct a connected graph $G' = (V', E')$ such that there is a uniform emulation of G' on H iff there is a uniform emulation f of G on P_3 with $f(V_1) \subseteq \{1, 2\}$ and $f(V_2) \subseteq \{2, 3\}$. Let $c = \frac{1}{3} | V_G |$ (c is the computation factor of the emulation of G on P_3).

From the fact that H is not complete it follows that there exist nodes $v_1, v_2, v_3 \in V_H$ with $(v_1, v_2) \in E_H$, $(v_2, v_3) \in E_H$, and $(v_1, v_3) \notin E_H$. (The subgraph of H induced by $\{v_1, v_2, v_3\}$ is isomorphic to P_3.) We let G' consist of the following parts:

- $3c + 1$ copies of H; from the first c copies of H we omit the nodes v_1, v_2, v_3; we connect copies of the same and adjacent nodes;
- a copy of G.

Each copy of v_1 and v_2 is connected to each node in V_1; each copy of v_2 and v_3 is connected to each node in V_2.

Definition. Let $G' = (V', E')$ be the undirected graph, with

$$V' = V_G \cup \{ v_{x,i} \mid x \in V_H,\ 1 \leqslant i \leqslant 3c + 1,\ x \notin \{ v_1, v_2, v_3 \} \vee i \geqslant c + 1 \}$$

and

$$E' = E_G \cup \{ (v_{x,i}, v_{y,j}) \mid v_{x,i}, v_{y,j} \in V' \text{ and } (x, y) \in E_H \} \cup \{ (v_{w,i}, y) \mid y \in V_1,\ w \in \{ v_1, v_2 \} \} \cup \{ (v_{w,i}, y) \mid y \in V_2,\ w \in \{ v_2, v_3 \} \}.$$

3.1.1. Lemma. *There is a uniform emulation of G' on H if and only if there is a uniform emulation f of G on P_3 with $f(V_1) \subseteq \{1, 2\}$ and $f(V_2) \subseteq \{2, 3\}$.*

Proof. First suppose there is a uniform emulation g of G on P_3 with $f(V_1) \subseteq \{1, 2\}$ and $f(V_2) \subseteq \{2, 3\}$. Define $f: V' \to V_H$ as follows: $f(v_{x,i}) = x$ and, for $y \in V_G$, $f(y) = v_{g(y)}$. It is easy to check that f is a uniform emulation of G' on H.

Now suppose there is a uniform emulation f of G' on H. For every $x \in V_H$, let $N(x)$ be the set consisting of x and its neighbours, i.e.,

$$N(x) = \{x\} \cup \{y \mid (x, y) \in E_H\}.$$

Number the nodes in H with $w_1, \ldots, w_{|V_H|}$, in order of nonincreasing degree, i.e., if degree(w_i) > degree(w_j), then $i < j$.

3.1.1.1. Claim. *$\forall i$, $0 \leqslant i \leqslant |V_H|$, there exists a uniform emulation f^i of G' on H such that, $\forall j, k, l$ with $v_{w_j,k}, v_{w_j,l} \in V' \wedge 0 \leqslant j \leqslant i$, $f^i(v_{w_j,k}) = f^i(v_{w_j,l})$.*

Proof. We use induction on i. For $i = 0$, the claim immediately follows.

Now, let f^i be given. Notice that for w_j, $1 \leqslant j \leqslant i$, and $v_{w_{i+1},l} \in V'$,

$$f^i(v_{w_j,1}) = f^i(v_{w_{i+1},l}) \Rightarrow w_j \in \{v_1, v_2, v_3\}.$$

Therefore, if $w_{i+1} \notin \{v_1, v_2, v_3\}$, then there are at most $3c$ nodes $v \in \{v_{w_{i+1},l} \mid 1 \leqslant l \leqslant 3c+1\}$ with $\exists j$, $1 \leqslant j \leqslant i$,

$$f^i(v) = f^i\left(\{v_{w_j,k} \mid c+1 \leqslant k \leqslant 3c+1\}\right).$$

If $w \in \{v_1, v_2, v_3\}$, then there are at most $2c$ such nodes in $\{v_{w_{i+1},l} \mid c+1 \leqslant l \leqslant 3c+1\}$. Hence, there exists a $v_{w_{i+1},l} \in V_G$ such that there is no j, $1 \leqslant j \leqslant i$, with

$$f^i\left(\{v_{w_j,k} \mid c+1 \leqslant k \leqslant 3c+1\}\right) = f^i(v_{w_{i+1},l}).$$

Let such an l be given and write $v = v_{w_{i+1},l}$. Look at the degree of $f(v)$. Uniformity prevents that degree($f(v_{x,j})$) < degree(x), for all $v_{x,j} \in V' \setminus V_G$. Hence, degree($f(v)$) = degree($w_{i+1}$). (We use the fact that the degrees of $w_1, \ldots, w_{|V_H|}$ are nonincreasing.)

Further notice that for every $y \in N(f^i(v))$ there is a $w \in V_G$ with $f^i(w) = y$ and $(v, w) \in E'$. Now, let $f^i(v_{w_{i+1},m}) \neq f^i(v)$. Every neighbour of v is a neighbour of $v_{w_{i+1},m}$. Hence, $N(f^i(v_{w_{i+1},m})) \supseteq N(f^i(v))$. Choose a node y with $f^i(y) = f^i(v)$ and y not of the form $v_{w_{i+1},m}$ $(1 \leqslant m \leqslant 3c+1$; if $w_{i+1} \in \{v_1, v_2, v_3\}$, then $c+1 \leqslant m \leqslant 3c+1)$. For all $z \in V_G$, $(y, z) \in E_G$ implies $f^i(z) \in N(f^i(v))$. We obtain a new uniform emulation $\tilde{f}^i$ by 'exchanging' the images of $v_{w_{i+1},m}$ and y:

$$z \notin \{y, v_{w_{i+1},m}\} \Rightarrow \tilde{f}^i(z) = f^i(z),$$

$\tilde{f}^i(y) = f^i(v_{w_{i+1},m})$, and $\tilde{f}^i(v_{w_{i+1},m}) = f^i(y)$.

$\tilde{f}^i$ maps every neighbour of y upon a node in $N(f^i(v)) \subseteq N(\tilde{f}^i(y))$, and every neighbour of $v_{w_{i+1},m}$ is v or a neighbour of v, so is mapped upon a node in $N(f^i(v)) = N(\tilde{f}^i(v_{w_{i+1},m}))$. So, $\tilde{f}^i$ is again a uniform emulation of G' on H, but now $\tilde{f}^i(v_{w_{i+1},m}) = \tilde{f}^i(v)$.

By repeated use of this 'image-exchanging' process one can obtain a uniform emulation f^{i+1} with $f^{i+1}(v_{w_{i+1},m}) = f^{i+1}(v)$ for all $v_{w_{i+1},m} \in V' \setminus V_G$. With the induction hypothesis one proves that $\forall j, k, l$ with $v_{w_j,k}, v_{w_j,l} \in V'$, and $0 \leqslant j \leqslant i+1$,

$$f^{i+1}(v_{w_j,k}) = f^{i+1}(v_{w_j,l}). \quad \square$$

Now, let $g = f_{|V_H|}$. g, restricted to the set of nodes $\{v_{3c+1} \mid x \in V_H\}$ can be seen as a graph isomorphism of H. Hence, we have the following lemma.

3.1.1.2. Lemma. *There is a uniform emulation $\tilde{g}$ of G' on H with $f(v_{x,i}) = x$ for all $v_{x,i} \in V' \setminus V_G$.*

Notice that if $w \in V_G$, then $\tilde{g}(w) \in \{v_1, v_2, v_3\}$, and $\tilde{g}$ maps c nodes of V_G on each of the nodes v_1, v_2, v_3. Further, $w \in V_1$ implies that $\tilde{g}(w)$ must be adjacent to

$$\tilde{g}\left(\{v_{v_1,k} \mid c+1 \leqslant k \leqslant 3c+1\}\right) = v_1$$

and to

$$\tilde{g}\left(\{v_{v_2,k} \mid c+1 \leqslant k \leqslant 3c+1\}\right) = v_2.$$

Hence, $\tilde{g}(w) \in \{v_1, v_2\}$. Likewise, $w \in V_2$ implies $\tilde{g}(w) \in \{v_2, v_3\}$. So, the mapping $h: V_G \to \{1, 2, 3\}$, given by $h(w) = i \Leftrightarrow g(w) = v_i$, is a uniform emulation of G on P_3 with $h(V_1) \subseteq \{1, 2\}$ and $h(V_2) \subseteq \{2, 3\}$, therewith proving Lemma 3.1.1. $\square$

Finally, notice that the construction of G' can be carried out in polynomial time in $|V_G|$. Hence, the problem stated in Theorem 3.1 is NP-complete. $\square$

With the following simple observation we have a complete classification of the complexity of finding uniform emulations on fixed, connected, undirected graphs.

3.2. Proposition. *Let* $G = (V_G, E_G)$ *be an undirected graph and let* K_n *be the complete graph with n nodes. There is a uniform emulation of G on* K_n *iff* $n \mid |V_G|$.

For the general case of graphs that are not necessarily connected, we mention the following results.

3.3. Corollary. *Let H be an undirected graph such that at least one connected component of H is not complete. The following problem is NP-complete*:

Instance: *An undirected graph* $G = (V_G, E_G)$.

Question: *Is there a uniform emulation of G on H?*

Proof. The proof is done by transformation from the problem of Theorem 3.1. □

3.4. Proposition. *Let H be an undirected graph such that each connected component of H is complete. Then there exists a polynomial-time algorithm that decides whether a graph G can be uniformly emulated on H.*

Proof. This problem becomes the question whether we can allocate the connected components of G to the connected components of H such that the numbers of nodes that are allocated to components of H are proportional to its size. By exhaustive search over all allocations this is solvable in polynomial time. (We use the fact that a graph can be separated in its connected components in linear time.) □

For directed graphs, a result similar to that of Theorem 3.1 can be proved.

3.5. Theorem. *Let* $H = (V_H, E_H)$ *be a strongly connected, directed graph that is not complete. The following problem is NP-complete*:

Instance: *Strongly connected directed graph* $G = (V_G, E_G)$.

Question: *Is there a uniform emulation of G on H?*

The proof is similar to the one of the undirected case, and uses Lemma 2.4.

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