On Response Time and Cycle Time Distributions in a Two-Stage Cyclic Queue

O.J. Boxma and P. Donk
Mathematical Institute, University of Utrecht, 3508 TA Utrecht, The Netherlands

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We consider a two-stage closed cyclic queueing model. For the case of an exponential server at each queue we derive the joint distribution of the successive response times of a customer at both queues, using a reversibility argument. This joint distribution turns out to have a product form. The correlation coefficient is calculated and shown to be non-positive.

For the case of one general server and one exponential server we derive two approximations for the joint distribution of the response times. The numerical results based on these approximations are compared with simulation results. The first approximation which heavily relies on results for the M/G/1 queue with finite capacity, is somewhat complicated, but it yields exact marginal distributions and appears to be very accurate. The second one is less accurate, but very easily applicable.

Keywords. Two-stage Cyclic Queue, Multi-programmed Computer System, Cycle Time, Joint Distribution of Response Times, Reversibility.

Onno J. Boxma was born in The Hague, The Netherlands, in 1952. He received his master's degree in Delft Technological University in 1974 and his Ph.D. in the University of Utrecht in 1977. During 1978–1979 he was an IBM Postdoctoral Fellow in Yorktown Heights, New York. His main research interests are in the area of queueing theory and computer networks.

Peter Donk was born in Giessenburg, The Netherlands, in 195[illegible]. He received his master's degree in the University of Utrecht in 1981. Presently he is involved in a governmental project on Motorway Control and Signaling Systems.

1. Introduction

We consider the two-stage cyclic queueing network shown in Fig. 1. Q_1 and Q_2 are single server queues with an FCFS discipline. The system contains N customers, who cyclically visit Q_1 and Q_2.

Service times in Q_1 are independent, identically distributed stochastic variables (s.v.) τ_i, $i = 1, 2, \ldots,$ with

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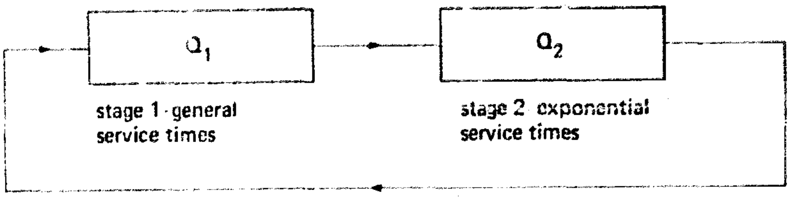

Fig. 1.

distribution function

$$B(t) \overset{\text{def}}{=} \Pr\{\tau_1 < t\}, \quad t > 0,$$

and

$$\beta \overset{\text{def}}{=} \mathrm{E}\{\tau_1\} < \infty,$$
$$\beta_n \overset{\text{def}}{=} \mathrm{E}\{\tau_1^n\}, \qquad n = 2, 3, \ldots, \tag{1.1}$$
$$\beta(\rho) \overset{\text{def}}{=} \mathrm{E}\{e^{-\rho\tau_1}\}, \qquad \mathrm{Re}\, \rho \geq 0.$$

Service times in Q_2 are independent, negative exponentially distributed stochastic variables with mean α. The service processes in Q_1 and Q_2 are also independent.

The model described above will be denoted as $\mathrm{G/M}/N^{(\mathrm{C})}$. It is a simple model of a multiprogrammed computer system with customers representing programs, stage 2 representing the central processing unit and stage 1 representing a data storage and transfer facility which has access to and can transfer information for only one program at a time (cf. Lavenberg [8]). It has been observed by several authors [7,8] that Q_1 is equivalent with an M/G/1 finite capacity queue with capacity N (i.e., $(N-1)$ waiting places). The queue length and waiting time distributions in such an $\mathrm{M/G}/1-N$ queue (and hence in Q_1) have been obtained by Cohen [5] (see also [8,10] for algorithms which efficiently determine queue length and waiting time distributions).

In the present study we are interested in the joint distribution of the sojourn or response times (waiting plus service times) at the two queues. In the case of exponentially distributed service times at both queues ($\mathrm{M/M}/N^{(\mathrm{C})}$) we obtain an expression for the Laplace–Stieltjes Transform (LST) of the joint distribution of the response times at both queues (Section 2). Thus we generalize a result of Chow [3] who derived the LST of the cycle time (= sum of two successive response times) distribution for the $\mathrm{M/M}/N^{(\mathrm{C})}$ model.

As a by-product we obtain the covariance and correlation coefficient of the sojourn times at both queues; they are shown to be non-positive.

For the general $\mathrm{G/M}/N^{(\mathrm{C})}$ case we did not succeed in obtaining an exact expression for the LST of the joint response time distribution. However, in Section 3 we derive two approximations for this LST. The first one, (3.7), is still rather complicated, but very accurate; in particular, it yields exact marginal response time distributions.

In this approximation we use an exact expression, obtained in [2] for the $\mathrm{M/G}/1-N$ queue, for the joint distribution of the number of customers present immediately before an (admitted) arrival and the response time of this arriving customer. The second approximation can be used more easily to obtain numerical results, but it is slightly less accurate. Extensive numerical comparisons are displayed in Appendix A.

2. Exponential cyclic queues

Consider the $\mathrm{G/M}/N^{(\mathrm{C})}$ cyclic model described above. Denote by z_n the number of customers left behind in Q_1 at r_n, the epoch of the nth departure from Q_1 after $t = 0$. As is observed by Cohen [5] in his discussion of the $M/G/1-N$ queue (which – as remarked above – is equivalent with Q_1 in the $G/M/N^{(C)}$ model), $\{z_n, n=1,2,\dots\}$ is an aperiodic Markov chain with stationary transition probabilities and with a finite and discrete state space. Hence all its states are positive recurrent, and the chain has a stationary distribution. Defining

$$z_i \stackrel{\text{def}}{=} \lim_{n\to\infty} \Pr\{z_n = i \mid z_1 = z\}, \quad i=0,1,\dots,N-1, \tag{2.1}$$

we observe that in the $M/M/1-N$ queue (cf. [5]) and hence also in Q_1 of the $M/M/N^{(C)}$ model,

$$z_i = \begin{cases} \dfrac{1-a}{1-a^N} a^i, & a \neq 1, \\ 1/N, & a=1, \end{cases} \quad i=0,1,\dots,N-1, \tag{2.2}$$

with

$$a \stackrel{\text{def}}{=} \beta/\alpha. \tag{2.3}$$

By z we denote in the sequel an s.v. with distribution z_i, $i=0,1,\dots,N-1$. Denote by $s_n^{(1)}$ the sojourn time which the customer leaving Q_1 at r_n has just spent in Q_1, and by $s_n^{(2)}$ the sojourn time this customer will subsequently spend in Q_2. It is easy to see that a joint stationary distribution of $s_n^{(1)}$, $s_n^{(2)}$ does exist, too. By $s^{(1)}$, $s^{(2)}$ we denote in the sequel stochastic variables with joint distribution

$$\Pr\{s^{(1)}<s_1, s^{(2)}<s_2\} = \lim_{n\to\infty} \Pr\{s_n^{(1)}<s_1, s_n^{(2)}<s_2 \mid s_1^{(1)}=t_1, s_1^{(2)}=t_2\}, \quad s_1, s_2>0.$$

Restricting ourselves in the rest of this section to the $M/M/N^{(C)}$ model we now prove the following theorem.

Theorem 2.1. *In the* $M/M/N^{(C)}$ *model with mean service times* β *and* α *in* Q_1 *and* Q_2, *respectively, for* $\operatorname{Re}\rho_1, \rho_2 \geq 0$,

$$\mathrm{E}\left[e^{-\rho_1 s^{(1)} - \rho_2 s^{(2)}}\right] = \begin{cases} \displaystyle\sum_{k=0}^{N-1} \frac{1-a}{1-a^N} a^k \left(\frac{1}{1+\beta\rho_1}\right)^{k+1} \left(\frac{1}{1+\alpha\rho_2}\right)^{N-k}, & a \neq 1, \\ \displaystyle\sum_{k=0}^{N-1} \frac{1}{N} \left(\frac{1}{1+\beta\rho_1}\right)^{k+1} \left(\frac{1}{1+\alpha\rho_2}\right)^{N-k}, & a=1. \end{cases} \tag{2.4}$$

Proof. The key idea is to consider the system at departure epochs of Q_1, and then to look back at the past sojourn time of the leaving customer in Q_1 and to look forward to his sojourn time in Q_2. Now

$$\begin{aligned} \mathrm{E}\left[e^{-\rho_1 s^{(1)} - \rho_2 s^{(2)}}\right] &= \sum_{k=0}^{N-1} z_k \mathrm{E}\left[e^{-\rho_1 s^{(1)} - \rho_2 s^{(2)}} \mid z=k\right] \\ &= \sum_{k=0}^{N-1} z_k \mathrm{E}\left[e^{-\rho_1 s^{(1)}} \mid z=k\right] \mathrm{E}\left[e^{-\rho_2 s^{(2)}} \mid z=k\right], \quad \operatorname{Re}\rho_1, \rho_2 \geq 0. \end{aligned} \tag{2.5}$$

using the obvious fact, based on the memoryless property of the negative exponential distribution, that

$$\Pr\{s_n^{(2)}<s_2 \mid z_n=k, s_n^{(1)}<s_1\} = \Pr\{s_n^{(2)}<s_2 \mid z_n=k\}. \tag{2.6}$$

Obviously

$$\mathrm{E}\left[e^{-\rho_2 s^{(2)}} \mid z=k\right] = \left(\frac{1}{1+\alpha\rho_2}\right)^{N-k}, \quad \operatorname{Re}\rho_2 \geq 0, \tag{2.7}$$

since a customer who leaves k customers behind in Q_1 finds $(N-k-1)$ customers present in Q_2 and hence has a sojourn time in Q_2 consisting of $(N-k-1)+1$ exponentially distributed phases.

Next consider $\mathrm{E}[\mathrm{e}^{-\rho_1 s^{(1)}}|z=k]$. It is well known that the queue length process $\{y(t),\ t\geq 0\}$ in an $M/M/1-N$ queue is a birth- and death-process. A birth- and death-process is a reversible stochastic process (see Kelly [6]), i.e., for the stochastic process $y(t)$ holds:

$$(y(t_1), y(t_2),\dots,y(t_n)) \quad \text{and} \quad (y(\tau-t_1), y(\tau-t_2),\dots,y(\tau-t_n))$$

have the same distribution for all $t_1, t_2,\dots,t_n, \tau$. (Or, as Kelly remarks, "*speaking intuitively, if we take a film of such a process and then run the film backwards, the resulting process will be statistically indistinguishable from the original process*".)

Denoting by x an s.v. with distribution the limiting distribution of the number of customers seen by an arriving customer who is admitted to the $M/M/1-N$ queue, it is in particular seen that because of the reversibility of the queue length process,

$$\mathrm{E}\left[\mathrm{e}^{-\rho_1 s^{(1)}}|z=k\right] = \mathrm{E}\left[\mathrm{e}^{-\rho_1 s^{(1)}}|x=k\right], \quad \mathrm{Re}\,\rho_1\geq 0,\ k=0,1,\dots,N-1 \tag{2.8}$$

(see [2] for a more rigorous proof of the similar result for an $M/M/1$ queue with infinite capacity).

Now (2.4) follows from (2.5), (2.7), (2.8) and the following simple relation,

$$\mathrm{E}\left[\mathrm{e}^{-\rho_1 s^{(1)}}|x=k\right] = \beta^{k+1}(\rho_1) = \left(\frac{1}{1+\beta\rho_1}\right)^{k+1}, \quad \mathrm{Re}\,\rho_1\geq 0. \tag{2.9}$$

(In fact it is possible to prove directly that

$$\mathrm{E}\left[\mathrm{e}^{-\rho_1 s^{(1)}}|z=k\right] = \left(\frac{1}{1+\beta\rho_1}\right)^{k+1},$$

but we prefer the intuitive reversibility argument above a very lengthy calculation.) □

Remark 2.2. Performing the summation in (2.4) we obtain, for $\mathrm{Re}\,\rho_1, \rho_2\geq 0$,

$$\mathrm{E}\left[\mathrm{e}^{-\rho_1 s^{(1)}-\rho_2 s^{(2)}}\right] = \begin{cases} \dfrac{1-a}{1-a^N}\,\dfrac{\left(\dfrac{1}{1+\alpha\rho_2}\right)^N - \left(\dfrac{a}{1+\beta\rho_1}\right)^N}{1+\beta\rho_1-a(1+\alpha\rho_2)}, & a\neq 1,\\[2ex] \dfrac{1}{N}\,\dfrac{\left(\dfrac{1}{1+\beta\rho_2}\right)^N - \left(\dfrac{1}{1+\beta\rho_1}\right)^N}{\beta(\rho_1-\rho_2)}, & a=1. \end{cases} \tag{2.10}$$

However, the form (2.4) seems to be more useful in general; in particular we find, for $s_1, s_2>0$,

$$\frac{\partial^2}{\partial s_1 \partial s_2}\Pr\{s^{(1)}<s_1, s^{(2)}<s_2\} =$$

$$= \sum_{k=0}^{N-1}\frac{1-a}{1-a^N}a^k\frac{1}{\beta}\,\mathrm{e}^{-s_1/\beta}\frac{(s_1/\beta)^k}{k!}\,\frac{1}{\alpha}\,\mathrm{e}^{-s_2/\alpha}\frac{(s_2/\alpha)^{N-k-1}}{(N-k-1)!}$$

$$= \begin{cases} \dfrac{1-a}{1-a^N}\,\dfrac{1}{\beta}\,\mathrm{e}^{-s_1/\beta}\dfrac{1}{\alpha}\,\mathrm{e}^{-s_2/\alpha}\left(\dfrac{s_1+s_2}{\alpha}\right)^{N-1}\dfrac{1}{(N-1)!}, & a\neq 1,\\[2ex] \dfrac{1}{N}\,\dfrac{1}{\alpha^2}\,\mathrm{e}^{-(s_1+s_2)/\alpha}\left(\dfrac{s_1+s_2}{\alpha}\right)^{N-1}\dfrac{1}{(N-1)!}, & a=1. \end{cases} \tag{2.11}$$

Remark 2.3. Putting $\rho_1=\rho_2=\rho$ in (2.4) we obtain an expression for the LST of the distribution of

$c = s^{(1)} + s^{(2)}$, the cycle time in the $M/M/N^{(C)}$ model,

$$E[e^{-\rho c}] = \begin{cases} \dfrac{1}{1-a^N}\left(\dfrac{1}{1+\alpha\rho}\right)^N - \dfrac{a^N}{1-a^N}\left(\dfrac{1}{1+\beta\rho}\right)^N, & a \neq 1, \\ \left(\dfrac{1}{1+\beta\rho}\right)^{N+1}, & a = 1, \end{cases} \quad \operatorname{Re}\rho \geq 0; \tag{2.12}$$

hence, for $t > 0$,

$$\begin{aligned} \frac{d}{dt}\Pr\{c < t\} &= \frac{1}{1-a^N}\frac{1}{\alpha}e^{-t/\alpha}\frac{(t/\alpha)^{N-1}}{(N-1)!} - \frac{a^N}{1-a^N}\frac{1}{\beta}e^{-t/\beta}\frac{(t/\beta)^{N-1}}{(N-1)!} \\ &= \begin{cases} \dfrac{1}{1-a^N}\dfrac{1}{\alpha}\dfrac{(t/\alpha)^{N-1}}{(N-1)!}(e^{-t/\alpha} - e^{-t/\beta}), & a \neq 1, \\ \dfrac{1}{\beta}e^{-t/\beta}\dfrac{(t/\beta)^N}{N!}, & a = 1, \end{cases} \end{aligned} \tag{2.13}$$

a result found by Chow [3] (see his extensive discussion of (2.13)). (2.13) also follows from (2.11) after a simple integration.

After sometimes lengthy calculations we obtain expressions for moments, covariances and correlation coefficients of the response times in Q_1 and Q_2.

Corollary 2.4.

$$E[s^{(1)}] = \begin{cases} \beta N + \dfrac{\beta}{1-a} - \dfrac{\beta N}{1-a^N}, & a \neq 1, \\ \frac{1}{2}\beta(N+1), & a = 1; \end{cases} \tag{2.14}$$

$$E[s^{(2)}] = \begin{cases} \alpha N - \dfrac{\alpha a}{1-a} + \dfrac{\alpha N a^N}{1-a^N}, & a \neq 1, \\ \frac{1}{2}\alpha(N+1), & a = 1; \end{cases} \tag{2.15}$$

$$E[c] = E[s^{(1)} + s^{(2)}] = \begin{cases} N\dfrac{\beta^{N+1} - \alpha^{N+1}}{\beta^N - \alpha^N}, & a \neq 1, \\ (N+1)\alpha, & a = 1; \end{cases} \tag{2.16}$$

$$E[s^{(1)}s^{(2)}] = \begin{cases} \dfrac{\alpha\beta}{1-a}\left[\dfrac{N(1+a^{N+1})}{1-a^N} - \dfrac{2a}{1-a}\right], & a \neq 1, \\ \frac{1}{6}\alpha^2(N+1)(N+2), & a = 1; \end{cases} \tag{2.17}$$

$$\operatorname{cov}(s^{(1)}, s^{(2)}) = \begin{cases} \alpha\beta\left[-\dfrac{a}{(1-a)^2} + \dfrac{N^2a^N}{(1-a^N)^2}\right], & a \neq 1, \\ \frac{1}{12}\alpha^2(1-N^2), & a = 1; \end{cases} \tag{2.18}$$

$$\rho(s^{(1)}, s^{(2)}) \overset{\text{def}}{=} \operatorname{corr}(s^{(1)}, s^{(2)}) =$$

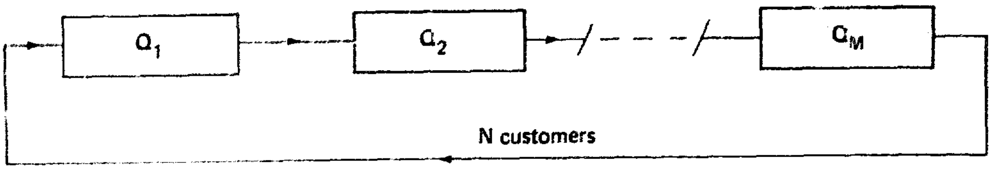

Fig. 2.

$$
=\begin{cases}
\left[\left[-\dfrac{a}{(1-a)^2}+\dfrac{N^2a^N}{(1-a^N)^2}\right]\Big/\left[\left\{-\dfrac{Na^N}{1-a^N}+\dfrac{1}{(1-a)^2}-\dfrac{N^2a^N}{(1-a^N)^2}\right\}^{1/2}\right.\right. & \\
\qquad\left.\times\left\{\dfrac{N}{1-a^N}+\dfrac{a^2}{(1-a)^2}-\dfrac{N^2a^N}{(1-a^N)^2}\right\}^{1/2}\right], & a\neq 1,\\
\dfrac{1-N}{N+5}, & a=1.
\end{cases} \tag{2.19}
$$

Remark 2.5. (2.16) has already been obtained by Chow [3]; (2.15) follows from (2.14) by interchanging α and β.

It can be seen from (2.18) that, for each $a>0$,

$$\operatorname{cov}(s^{(1)}, s^{(2)}) \leqslant 0, \tag{2.20}$$

with equality iff $N=1$. Since the first expression in (2.18) does not change if we interchange α and β, it suffices to consider the case $a<1$, and to prove that, for each $0<a<1$,

$$\frac{1}{(1-a)^2} \geqslant \frac{N^2a^{N-1}}{(1-a^N)^2} \quad \text{or} \quad \frac{1-a^N}{1-a}=1+a+\cdots+a^{N-1} \geqslant Na^{(N-1)/2}. \tag{2.21}$$

(2.21) follows immediately from the basic arithmetic mean – geometric mean inequality (Beckenbach and Bellman [1, p. 4]) – with equality iff $N=1$.

The fact that $\operatorname{cov}(s^{(1)}, s^{(2)}) \leqslant 0$ and $\rho(s^{(1)}, s^{(2)}) \leqslant 0$ agrees with our intuitive expectation. Further, numerical calculations point out that $\rho(s^{(1)}, s^{(2)})$ reaches its minimum at $a=1$ for fixed N.

Finally note that from (2.18) and (2.19) it follows that

$$\lim_{N\to\infty} \operatorname{cov}(s^{(1)}, s^{(2)}) = \begin{cases} -\dfrac{\beta^2}{(1-a)^2}, & a\neq 1,\\ -\infty, & a=1;\end{cases} \tag{2.22}$$

$$\lim_{N\to\infty} \rho(s^{(1)}, s^{(2)}) = \begin{cases} 0, & a\neq 1,\\ -1, & a=1.\end{cases} \tag{2.23}$$

Remark 2.6. In [4] Chow generalizes his exponential cyclic model to the case of K parallel servers replacing Q_2. He again obtains the cycle time distribution. In a future study we plan to show how one can extend his results to the joint sojourn time distribution at Q_1 and the parallel system.

In another future study [1] the case of a cyclic model of M ($\geqslant 3$) exponential servers, depicted in Fig. 2, will be considered.

[1] *Note added in proof*: O.J. Boxma, F.P. Kelly and A.G. Konheim, The product form for sojourn time distributions in cyclic exponential queues, Rept., Univ. of Utrecht, February 1982.

Konheim, from the I.B.M. Thomas J. Watson Research Center, made the following *conjecture*, after seeing Theorem 2.1.

For the model of Fig. 2, with mean service time α_i in Q_i, $i = 1, 2, \ldots, M$, and with

$$p(j_1, \ldots, j_M) = \prod_{i=1}^{M} \alpha_i^{j_i} \Bigg/ \left[\sum_{\substack{k_1, \ldots, k_M \geq 0 \\ k_1 + \cdots + k_M = N-1}} \prod_{i=1}^{M} \alpha_i^{k_i} \right] \tag{2.24}$$

the probability that immediately after the departure of a customer from Q_M there are $j_1, \ldots, j_M$ other customers in $Q_1, \ldots, Q_M$, the LST of the joint distribution of successive response times $s^{(1)}, \ldots, s^{(M)}$ in $Q_1, \ldots, Q_M$ has the following product form:

$$\mathrm{E}\left[e^{-\rho_1 s^{(1)} - \cdots - \rho_M s^{(M)}}\right] = \sum_{\substack{j_1, \ldots, j_M \geq 0 \\ j_1 + \cdots + j_M = N-1}} p(j_1, \ldots, j_M) \prod_{i=1}^{M} \left(\frac{1}{1 + \alpha_i \rho_i} \right)^{j_i + 1},$$

$$\mathrm{Re}\, \rho_1, \ldots, \rho_M \geq 0, \; M = 1, 2, \ldots. \tag{2.25}$$

The proof proceeds by induction, using reversibility.

The applicability of the reversibility concept is, however, less obvious than it is in the proof of Theorem 2.1 (except for the case that $\alpha_i \equiv \alpha$, $i = 1, \ldots, M$).

Recently Schassberger and Daduna [9] proved, using a nice recurrence relation between LST's, that the LST of the cycle time distribution in the model of Fig. 2 is given by (2.25) with $\rho_i \equiv \rho$, $i = 1, \ldots, M$.

3. The $G/M/N^{(C)}$ model

Consider the model described in Section 1 (see Fig. 1), with general service time distribution $B(\cdot)$ in Q_1. (2.5) and (2.7) are still valid for this model, hence

$$\mathrm{E}\left[e^{-\rho_1 s^{(1)} - \rho_2 s^{(2)}}\right] = \sum_{k=0}^{N-1} z_k \left(\frac{1}{1 + \alpha \rho_2} \right)^{N-k} \mathrm{E}\left[e^{-\rho_1 s^{(1)}} \mid z = k\right], \quad \mathrm{Re}\, \rho_1, \rho_2 \geq 0. \tag{3.1}$$

$\{z_k, k = 0, 1, \ldots, N-1\}$, the stationary distribution of the number of customers left behind in Q_1 (or in the equivalent $M/G/1-N$ queue) by a departing customer, is known: Cohen [5, p. 576] derives an expression for z_k involving a contour integral, while Lavenberg [8] gives a recursive algorithm for the determination of the z_k.

A more complicated problem is, however, the determination of $\mathrm{E}[e^{-\rho_1 s^{(1)}} \mid z = k]$ in an $M/G/1-N$ queue. The fact that customers can be refused admission makes a straightforward analysis extremely difficult, while the queue length process is not reversible when $G \neq M$, thus rendering (2.8) invalid.

Still, there is some support for an introduction of the following.

Approximation Assumption (cf. (2.8))

$$\mathrm{E}\left[e^{-\rho_1 s^{(1)}} \mid z = k\right] = \mathrm{E}\left[e^{-\rho_1 s^{(1)}} \mid x = k\right], \quad k = 0, 1, \ldots, N-1, \mathrm{Re}\, \rho_1 \geq 0. \tag{3.2}$$

Firstly, (3.2) is not only exact for $G \equiv M$, but also for $N = 1$. Secondly, consider the case $N = \infty$. In [2] we have derived the joint distribution of $s^{(1)}$, z and x in the $M/G/1-(\infty)$ queue. Although $\mathrm{E}[s^{(1)}x] \neq \mathrm{E}[s^{(1)}z]$ if $G \not\equiv M$, the difference is often rather small, and in particular both tend to zero for $a \to 0$ (with one exception) and their quotient tends to one for $a \to 1$.

Thirdly, there is at least some symmetry in the $M/G/1-N$ queue w.r.t. arrival and departure epochs: Cohen [5, p. 577] remarks that $z_k = \Pr\{z = k\}$ is equal to $x_k \stackrel{\text{def}}{=} \Pr\{x = k\}$, with x the number of customers seen by an arriving customer who is admitted. This implies in particular the following. Substituting the approximation (3.2) in (3.1) and using the fact that $z_k = x_k$ we have (with subscript A denoting approximation and $(\cdot)$ denoting an indicator function)

$$\mathrm{E}_A\left[e^{-\rho_1 s^{(1)} - \rho_2 s^{(2)}}\right] = \sum_{k=0}^{N-1} \left(\frac{1}{1+\alpha\rho_2}\right)^{N-k} \mathrm{E}\left[e^{-\rho_1 s^{(1)}}(x=k)\right], \quad \mathrm{Re}\,\rho_1, \rho_2 \geq 0. \tag{3.3}$$

Hence the LST of the marginal distribution $\mathrm{E}[e^{-\rho_1 s^{(1)}}]$ is exact in this approximation (as is $\mathrm{E}[e^{-\rho_2 s^{(2)}}]$).

We now have to consider the joint distribution of $s^{(1)}$ and x in the M/G/1 – N queue. It can be determined exactly [2], the main problem being the determination of $\mathrm{E}[e^{-\rho_1 \zeta}(x=k)]$ with ζ an s.v. with distribution the stationary distribution of the residual service time considered at the epoch a customer is admitted to the system. It is proved in [2] that

$$\mathrm{E}\left[s^x e^{-\rho_1 s^{(1)}}\right] = z_0 \beta(\rho_1) + P(s\beta(\rho_1), \rho_1), \quad \mathrm{Re}\,\rho_1 \geq 0, \tag{3.4}$$

with

$$P(s,\rho) \stackrel{\mathrm{def}}{=} \mathrm{E}\left[s^x e^{-\rho\zeta}(x>0)\right]$$

$$= z_0 \frac{1}{2\pi i} \int_{D_\omega} \frac{1-\omega}{\omega - \beta((1-\omega)/\alpha)} \frac{\beta((1-\omega)/\alpha) - \beta(\rho)}{\alpha\rho + \omega - 1} \left[1 - (s/\omega)^{N-1}\right] \frac{s}{s-\omega} \mathrm{d}\omega,$$

$$\mathrm{Re}\,\rho \geq 0, |\omega| < \mu_0; \tag{3.5}$$

here D_ω is a circle with center at zero and radius ω, μ_0 is the smallest zero (in absolute value) of $p - \beta((1-p)/\alpha)$,

$$z_0 = \left[\frac{1}{2\pi i} \int_{D_\omega} \frac{1}{\beta((1-\omega)/\alpha) - \omega} \frac{\mathrm{d}\omega}{\omega^{N-1}}\right]^{-1}, \quad |\omega| < \mu_0. \tag{3.6}$$

From (3.3) and (3.4) we finally obtain the following approximation for the LST of the joint distribution of the response times at Q_1 and Q_2:

$$\mathrm{E}_A\left[e^{-\rho_1 s^{(1)} - \rho_2 s^{(2)}}\right] = \frac{1}{(1+\alpha\rho_2)^N} \mathrm{E}\left[(1+\alpha\rho_2)^x e^{-\rho_1 s^{(1)}}\right]$$

$$= z_0\beta(\rho_1)\frac{1}{(1+\alpha\rho_2)^N} + \frac{1}{(1+\alpha\rho_2)^N} P((1+\alpha\rho_2)\beta(\rho_1), \rho_1), \quad \mathrm{Re}\,\rho_1, \rho_2 \geq 0. \tag{3.7}$$

In particular, the approximation of the LST of the cycle time distribution is given by

$$\mathrm{E}_A[e^{-\rho c}] = z_0\beta(\rho)\frac{1}{(1+\alpha\rho)^N} + \frac{1}{(1+\alpha\rho)^N} P((1+\alpha\rho)\beta(\rho), \rho), \quad \mathrm{Re}\,\rho \geq 0. \tag{3.8}$$

Now consider moment approximations based on (3.7). It has already been noted that the marginal response time distributions are exact. Obviously,

$$\mathrm{E}[s^{(2)}] = \sum_{k=0}^{N-1} z_k \mathrm{E}\left[s^{(2)} | z=k\right] = \sum_{k=0}^{N-1} z_k (N-k)\alpha. \tag{3.9}$$

$\mathrm{E}[s^{(1)}]$ can be obtained from [5, p. 577]; here we shall use Lavenberg's simple expression [8]:

$$\mathrm{E}[s^{(1)}] = N\beta - \sum_{k=1}^{N-1} z_k (N-k)\alpha. \tag{3.10}$$

Hence

$$\mathrm{E}[c] = N\beta + N\alpha z_0, \tag{3.11}$$

(which yields (2.16) if G ≡ M).

Furthermore,

$$\begin{aligned} \mathrm{E}[s^{(1)}s^{(2)}] &= \sum_{k=0}^{N-1} z_k \mathrm{E}[s^{(2)} | z=k] \mathrm{E}[s^{(1)} | z=k] \\ &\approx \sum_{k=0}^{N-1} x_k (N-k)\alpha \mathrm{E}[s^{(1)} | x=k] \\ &= N\alpha \mathrm{E}[s^{(1)}] - \alpha \mathrm{E}[xs^{(1)}] = \mathrm{E}_{\mathrm{A}}[s^{(1)}s^{(2)}]. \end{aligned} \tag{3.12}$$

Hence

$$\mathrm{E}[s^{(1)}s^{(2)}] - \mathrm{E}_{\mathrm{A}}[s^{(1)}s^{(2)}] = \tfrac{1}{2}\{\mathrm{E}[c^2] - \mathrm{E}_{\mathrm{A}}[c^2]\} = \alpha\mathrm{E}[xs^{(1)}] - \alpha\mathrm{E}[zs^{(1)}]. \tag{3.13}$$

It is shown in [2] that in the $M/G/1-N$ queue

$$\begin{aligned} \mathrm{E}[xs^{(1)}] = \beta \mathrm{E}[x^2] + z_0 \frac{1}{2\pi i} \int_{D_\omega} \frac{1}{\omega - \beta((1-\omega)/\alpha)} \left[-\beta + \alpha \frac{1-\beta((1-\omega)/\alpha)}{1-\omega} \right] \\ \times \left[\left(1-(1/\omega)^{N-1}\right) \frac{\omega}{(1-\omega)^2} + (N-1)(1/\omega)^{N-1} \frac{1}{1-\omega} \right] \mathrm{d}\omega, \quad |\omega| < \mu_0. \end{aligned} \tag{3.14}$$

Evaluating this expression and using (3.10) and (3.12) we find an approximation for $\mathrm{E}[s^{(1)}s^{(2)}]$.

A drawback of the approach presented above is that some contour integrals have to be evaluated; this may be quite cumbersome if N is not too small and if $\beta(\cdot)$ does not have a simple form (although it is in general not difficult to calculate the values of these integrals numerically, see Appendix A). Therefore we now present a very simple approximation, which does not differ much from the approximation (3.7) if the distribution of the residual service time ζ is rather insensitive to the value of $x = k$ (some thinking will make clear that ζ will be rather insensitive to the value of x unless $\alpha << \beta$ and N is small). Returning to (3.2) and using the well-known result

$$\mathrm{E}[\mathrm{e}^{-\rho\zeta}] = (1-\beta(\rho))/\beta\rho,$$

we put for $\operatorname{Re} \rho_1 \geq 0$

$$\begin{aligned} &\mathrm{E}\left[\mathrm{e}^{-\rho_1 s^{(1)}} | x=k\right] = \beta^k(\rho_1) \mathrm{E}\left[\mathrm{e}^{-\rho_1\zeta} | x=k\right] \approx \beta^k(\rho_1) \frac{1-\beta(\rho_1)}{\beta\rho_1}, \quad k=1,2,\ldots, \\ &\mathrm{E}\left[\mathrm{e}^{-\rho_1 s^{(1)}} | x=0\right] = \beta(\rho_1). \end{aligned} \tag{3.15}$$

Hence our second approximation (indicated by subscript A2) reads

$$\begin{aligned} &\mathrm{E}_{\mathrm{A}2}\left[\mathrm{e}^{-\rho_1 s^{(1)} - \rho_2 s^{(2)}}\right] = \\ &= z_0 \frac{1}{(1+\alpha\rho_2)^N} \beta(\rho_1) + \sum_{k=1}^{N-1} z_k \frac{1}{(1+\alpha\rho_2)^{N-k}} \beta^k(\rho_1) \frac{1-\beta(\rho_1)}{\beta\rho_1}, \quad \operatorname{Re} \rho_1, \rho_2 \geq 0. \end{aligned} \tag{3.16}$$

This transform can easily be inverted. Note that (3.15) and hence also (3.16) is exact for G ≡ M.

From (3.16)

$$\mathrm{E}_{\mathrm{A}2}[\mathrm{e}^{-\rho c}] = z_0 \frac{1}{(1+\alpha\rho)^N} \beta(\rho) + \sum_{k=1}^{N-1} z_k \frac{1}{(1+\alpha\rho)^{N-k}} \beta^k(\rho) \frac{1-\beta(\rho)}{\beta\rho}, \quad \operatorname{Re} \rho \geq 0, \tag{3.17}$$

$$\mathrm{E}_{\mathrm{A}2}[c] = N\alpha + \beta z_0 + \frac{\beta_2}{2\beta}(1-z_0) + (\beta-\alpha)\mathrm{E}[z], \tag{3.18}$$

$$\mathrm{E}_{\mathrm{A}2}[s^{(1)}s^{(2)}] = N\alpha\left[\beta\mathrm{E}[z] + \frac{\beta_2}{2\beta}(1-z_0) + \beta z_0\right] - \alpha\beta\mathrm{E}[z^2] - \alpha\mathrm{E}[z]\frac{\beta_2}{2\beta}. \tag{3.19}$$

In Appendix A we compare our approximations with some simulation results. Presently we only consider a special case, for which the exact joint response time distribution can be obtained, viz. the case $N=2$, $\mathrm{G}\equiv\mathrm{D}$ (constant service times β at Q_1). In this particular case $s_n^{(1)}$ and $s_n^{(2)}$ are independent, as can easily be seen in either of the following ways:

(i) A straightforward calculation of $\mathrm{E}[\mathrm{e}^{-\rho_1 s^{(1)}} | z=k]$ by conditioning on x (a calculation which can be made in this simple case).

(ii) Reasoning that for the nth customer K_n in Q_1 after $t=0$, $s_n^{(1)}=\beta$ or $\beta<s_n^{(1)}<2\beta$, but in both cases a service in Q_2 starts at the same epoch that K_n's service in Q_1 starts; hence $\Pr\{z=1 | s_n^{(1)}\}=\mathrm{e}^{-\beta/\alpha}$ independent of whether $s_n^{(1)}=\beta$ or $\beta<s_n^{(1)}<2\beta$. This implies that $s_n^{(2)}$, too, is independent of $s_n^{(1)}$.

As a consequence of the independence we have

$$\mathrm{E}[s^{(1)}s^{(2)}]=\mathrm{E}[s^{(1)}]\mathrm{E}[s^{(2)}]. \tag{3.20}$$

$P(s,\rho)$, occurring in the approximation formulae, can easily be evaluated in this case. A contour integration yields for $N=2$ (cf. (3.5), (3.6) and [2])

$$z_0=\beta(1/\alpha), \qquad z_1=1-\beta(1/\alpha), \qquad P(s,\rho)=\frac{\beta(1/\alpha)-\beta(\rho)}{\alpha\rho-1}s,$$

$$\mathrm{E}\left[s^x \mathrm{e}^{-\rho s^{(1)}}\right]=\beta(\rho)\beta(1/\alpha)+\frac{\beta(1/\alpha)-\beta(\rho)}{\alpha\rho-1}s\beta(\rho), \tag{3.21}$$

$$\mathrm{E}[xs^{(1)}]=2\beta-\alpha+(\alpha-\beta)\beta(1/\alpha).$$

In particular for $\mathrm{G}\equiv\mathrm{D}$

$$\mathrm{E_A}\left[\mathrm{e}^{-\rho_1 s^{(1)}-\rho_2 s^{(2)}}\right]=\mathrm{e}^{-\beta(\rho_1+1/\alpha)}\frac{1}{(1+\alpha\rho_2)^2}+\mathrm{e}^{-\beta\rho_1}\frac{\mathrm{e}^{-a}-\mathrm{e}^{-\beta\rho_1}}{\alpha\rho_1-1}\frac{1}{1+\alpha\rho_2}, \quad \mathrm{Re}\,\rho_1,\rho_2\geqslant 0, \tag{3.22}$$

$$\mathrm{E_A}[s^{(1)}s^{(2)}]=\alpha^2[(a+1)\,\mathrm{e}^{-a}+2a-1], \tag{3.23}$$

while from (3.9), (3.10), (3.20) and (3.21) the exact result is

$$\mathrm{E}[s^{(1)}s^{(2)}]=\alpha^2[2a-1+\mathrm{e}^{-a}][1+\mathrm{e}^{-a}]. \tag{3.24}$$

Our second approximation yields for this case

$$\mathrm{E_{A2}}[s^{(1)}s^{(2)}]=\alpha^2\left[\tfrac{1}{2}a\,\mathrm{e}^{-a}+\tfrac{3}{2}a\right]. \tag{3.25}$$

Note that $\mathrm{E_A}[s^{(1)}s^{(2)}]/\mathrm{E}[s^{(1)}s^{(2)}]$ tends to one for both $a\to 0$ and $a\to\infty$, whereas $\mathrm{E_{A2}}[s^{(1)}s^{(2)}]/\mathrm{E}[s^{(1)}s^{(2)}]$ tends to one for $a\to 0$ but to $\frac{3}{4}$ for $a\to\infty$.

Table 4 contains a comparison of (3.24) with the approximations (3.23) and (3.25).

Remark 3.1. From (3.11) and (3.18)

$$\mathrm{E_{A2}}[c]/\mathrm{E}[c]\to\begin{cases}1, & a\to 0\\ 1+\dfrac{\beta_2/(2\beta^2)-1}{N}, & a\to\infty.\end{cases} \tag{3.26}$$

As remarked above (3.15) ζ will not be insensitive to the value of x if $a>>1$ and N is small; moreover if N is small, the contribution of ζ to $s^{(1)}$ will be substantial and therefore in this case the second approximation will yield bad results, as is illustrated above.

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Appendix A. Numerical results

We now describe the numerical experiments, performed to test the accuracy of the two approximations A (cf. (3.7)) and A2 (cf. (3.16)); we also present four tables with numerical results and error percentages ($\{(E_A[\cdot]/E[\cdot]) - 1\} * 100\%$, etc.).

In the present study the emphasis has been on the joint distribution of the response times at Q_1 and Q_2, rather than on the cycle time distribution (which may be of greater practical interest). However, we now compare the approximations of both $E[s^{(1)}s^{(2)}]$ and $E[c^2]$ for A and A2 with simulation results.

The numerical experiments are organized as follows.

(1) The service time distribution G at Q_1 is chosen to be E_2, E_3, H_2 (with

$$B(t) = q(1 - e^{-t/m_1}) + (1 - q)(1 - e^{-t/m_2});$$

$$q = \tfrac{1}{2}(1 + \sqrt{0.6}), \qquad m_1 = \beta/(1 + \sqrt{0.6}), \qquad m_2 = \beta/(1 - \sqrt{0.6});$$

hence $E\{\tau\} = \beta$, $E\{\tau^2\} = 5\beta^2$) and D; $\alpha = 1$ (fixed) and $a = \beta/\alpha = \frac{1}{4}, \frac{1}{2}, 1, 2, 4$; $N = 2, 3, 4, 6, 10$.

(2) Simulation results for, among others, mean and standard deviation of c, $s^{(1)}$, $s^{(2)}$ have been obtained; we have used the regenerative method of simulation, using the IBM RESQ2 package. The accuracy of the simulated means of c, $s^{(1)}$, $s^{(2)}$ could be checked with exact results; errors are generally very small. From the

Table 1
$G \equiv E_2$

N	a	$E[s^{(1)}s^{(2)}]$ (sim.)	% error in $E[s^{(1)}s^{(2)}]$ (A)	% error in $E[s^{(1)}s^{(2)}]$ (A2)	$E[c^2]$ (sim.)	% error in $E[c^2]$ (A)	% error in $E[c^2]$ (A2)
2	$\frac{1}{4}$	0.50	−2.01	−2.17	6.34	−0.31	−1.28
	$\frac{1}{2}$	0.99	−3.53	−4.03	7.28	−0.96	−2.97
	1	1.95	−3.44	−4.86	10.77	−1.25	−3.57
	2	3.79	−1.23	−4.52	25.16	−0.37	−5.63
	4	7.57	−0.24	−6.08	84.09	−0.04	−15.43
3	$\frac{1}{4}$	0.77	−0.82	−0.95	12.27	−0.10	−1.68
	$\frac{1}{2}$	1.65	−5.05	−5.47	13.38	−1.25	−5.18
	1	3.33	−6.39	−7.57	18.32	−2.33	−4.41
	2	6.32	−3.89	−6.52	45.87	−1.07	−3.73
	4	12.08	−1.65	−5.97	169.68	−0.24	−6.70
4	$\frac{1}{4}$	1.08	−0.87	−0.96	20.10	−0.09	−0.55
	$\frac{1}{2}$	2.39	−4.71	−5.02	21.02	−1.07	−3.12
	1	4.86	−3.45	−4.38	27.17	−1.23	−1.71
	2	9.27	−4.94	−6.90	77.36	−1.19	−5.92
	4	16.56	−0.74	−3.96	292.93	−0.08	−6.45
6	$\frac{1}{4}$	1.70	−0.49	−0.55	42.04	−0.04	−0.19
	$\frac{1}{2}$	4.03	−3.33	−3.52	42.46	−0.63	−1.18
	1	9.36	−5.28	−5.83	52.04	−1.90	−1.71
	2	15.57	−3.92	−5.14	157.56	−0.85	−1.76
	4	25.88	−0.84	−2.91	626.96	−0.07	−3.91
10	$\frac{1}{4}$	2.93	0.47	0.43	109.92	0.02	0.03
	$\frac{1}{2}$	7.46	−1.46	−1.56	108.92	−0.20	0.78
	1	21.87	−2.88	−3.14	127.45	−0.99	−2.63
	2	29.04	−4.49	−5.14	417.36	−0.62	−0.22
	4	45.24	−2.45	−3.63	1673.74	−0.13	−1.80

Table 2
$G \equiv E_3$

N	a	$E[s^{(1)}s^{(2)}]$ (sim.)	% error in $E[s^{(1)}s^{(2)}]$ (A)	% error in $E[s^{(1)}s^{(2)}]$ (A2)	$E[c^2]$ (sim.)	% error in $E[c^2]$ (A)	% error in $E[c^2]$ (A2)
2	$\frac{1}{4}$	0.50	−3.71	−3.91	6.40	−0.58	−2.87
	$\frac{1}{2}$	0.98	−3.80	−4.44	7.11	−1.05	−2.71
	1	1.91	−3.58	−5.49	10.28	−1.33	−4.40
	2	3.77	−3.25	−7.78	23.91	−1.02	−9.44
	4	7.49	−1.41	−9.70	80.45	−0.26	−15.78
3	$\frac{1}{4}$	0.78	−2.82	−2.97	12.08	−0.40	−0.45
	$\frac{1}{2}$	1.59	−3.75	−4.27	12.64	−0.95	−1.13
	1	3.24	−6.40	−7.94	17.35	−2.40	−3.97
	2	6.25	−5.73	−9.21	44.47	−1.61	−7.29
	4	12.20	−5.02	−10.72	165.55	−0.74	−11.59
4	$\frac{1}{4}$	1.08	−1.65	−1.76	20.47	−0.18	−2.46
	$\frac{1}{2}$	2.35	−4.98	−5.36	20.93	−1.12	−3.64
	1	4.87	−5.80	−6.96	26.39	−2.14	−3.06
	2	8.80	−3.20	−5.85	74.09	−0.76	−6.65
	4	16.71	−4.43	−8.65	280.00	−0.53	−7.75
6	$\frac{1}{4}$	1.71	−2.26	−2.33	43.23	−0.18	−2.80
	$\frac{1}{2}$	3.95	−3.84	−4.08	42.33	−0.71	−1.19
	1	9.56	−8.96	−9.62	51.83	−3.31	−4.50
	2	14.97	−4.73	−6.33	156.31	−0.91	−4.15
	4	24.74	0.18	−2.68	619.25	−0.01	−6.55
10	$\frac{1}{4}$	2.94	−1.71	−1.74	114.45	−0.09	−3.94
	$\frac{1}{2}$	7.44	−4.46	−4.59	112.57	−0.59	−2.56
	1	22.49	−6.72	−7.02	123.78	−2.44	−1.88
	2	26.56	−1.50	−2.40	423.78	−0.19	−3.64
	4	43.69	−2.90	−4.53	1696.75	−3.32	−5.49

simulation results mentioned above we have calculated

$$E[c^2], \quad E\left[(s^{(1)})^2\right], \quad E\left[(s^{(2)})^2\right], \quad \operatorname{cov}(s^{(1)}, s^{(2)}), \quad \rho(s^{(1)}, s^{(2)}).$$

These calculations may propagate simulation errors somewhat; still it is safe to say that the percentage errors in the tables give a good idea of the accuracy of the approximations.

(3) $E_{A2}[s^{(1)}s^{(2)}]$ and $E_{A2}[c^2]$ are calculated in a straightforward manner from (3.19) and (3.17); the z_k's are evaluated by the recursion scheme of Lavenberg [8].

(4) Because $E_A[(s^{(1)})^2]$ and $E_A[(s^{(2)})^2]$ are exact, we have not bothered to calculate them by contour integrations; $E_A[c^2]$ is determined by summing the simulation results for those two second moments to twice the numerically determined value of $E_A[s^{(1)}s^{(2)}]$.

$E_A[s^{(1)}s^{(2)}]$ is evaluated using (3.14) (see also (3.12)); the contour integral in (3.14) is analyzed by determining the residues for $|\omega| \geq \mu_0$. For E_2, E_3 and H_2 this is simple, because all zeros of $\omega - \beta((1-\omega)/\alpha)$ are easily found. For D a complication arises (unless $N = 2$, see at the end of Section 3).

$\omega - \beta((1-\omega)/\alpha)$ now has two real zeros $\omega_0 = 1$, ω_1, and an infinite set of pairs of complex conjugate zeros $\omega_{21}, \omega_{22}, \omega_{31}, \omega_{32}, \ldots$, (written in nondecreasing order of modulus) all of which have real part greater than ω_0, ω_1. We have calculated the residues of $\omega_{21}, \ldots, \omega_{42}$, and have incorporated these contributions in the contour integral evaluation. If $a = 1$, all zeros $\omega \neq 1$ have residue zero. If $a > 1$, the contributions of even ω_{21}, ω_{22} are almost negligible. If $a < 1$ the contributions of the residues decrease more slowly. The sum of the contributions of $\omega_{51}, \ldots$ has been estimated by using theoretical results about the position of these

Table 3
$G \equiv H_2$

N	a	$E[s^{(1)}s^{(2)}]$ (sim.)	% error in $E[s^{(1)}s^{(2)}]$ (A)	% error in $E[s^{(1)}s^{(2)}]$ (A2)	$E[c^2]$ (sim.)	% error in $E[c^2]$ (A)	% error in $E[c^2]$ (A2)
2	$\frac{1}{4}$	0.52	3.42	8.08	7.02	0.51	0.08
	$\frac{1}{2}$	1.07	1.26	11.82	9.64	0.28	6.46
	1	2.20	−1.12	18.92	19.35	−0.25	28.59
	2	4.50	−4.82	26.90	57.58	−0.75	60.27
	4	8.56	−2.05	43.86	210.94	−0.17	87.86
3	$\frac{1}{4}$	0.84	5.81	10.49	13.08	0.75	−0.15
	$\frac{1}{2}$	1.73	7.40	19.19	16.07	1.60	8.02
	1	3.77	0.03	22.09	31.81	0.01	21.74
	2	7.12	2.52	38.52	91.41	0.39	56.17
	4	13.23	3.93	50.53	325.66	0.32	87.55
4	$\frac{1}{4}$	1.20	6.32	10.30	21.60	0.70	−2.65
	$\frac{1}{2}$	2.59	7.03	17.54	25.26	1.44	3.30
	1	5.34	7.89	29.76	43.47	1.94	23.38
	2	10.51	4.34	37.12	130.94	0.70	48.95
	4	19.62	0.21	37.52	504.11	0.02	65.35
6	$\frac{1}{4}$	1.96	7.47	10.28	43.37	0.68	−1.16
	$\frac{1}{2}$	4.54	8.61	16.52	47.70	1.64	2.43
	1	9.57	13.06	30.70	75.46	3.31	17.54
	2	19.07	4.23	28.43	249.92	0.64	27.28
	4	31.19	3.23	29.29	924.23	0.22	47.53
10	$\frac{1}{4}$	3.94	−1.03	0.44	112.28	−0.07	−1.47
	$\frac{1}{2}$	9.54	7.20	11.73	118.65	1.16	−1.62
	1	20.99	18.59	29.58	160.31	4.87	13.79
	2	38.40	7.73	22.11	554.08	1.07	18.86
	4	55.77	3.82	18.73	2137.84	0.20	31.33

zeros (see Wright [11]). This extra contribution to $E_A[s^{(1)}s^{(2)}]$ varies between 1% ($a=\frac{1}{4}$, $N=2$) and less than 0.1% ($a=\frac{1}{2}$, $N=10$).

Apparently in most practical cases its suffices to consider only a few zeros in evaluating contour integrals like (3.14).

Numerical results are presented in Tables 1, 2, 3 and 4. These and other results yield the following.

A.1. Conclusions

(1) The 'A' approximation is very accurate: marginal response time distributions (and hence their moments) are exact, errors in $E_A[c^2]$ are generally well below 5%, errors in $E_A[s^{(1)}s^{(2)}]$ are generally below 10% (in particular, results are very good for $a \to 0$, $a \to \infty$).

(2) The 'A2' approximation is accurate unless $a >> 1$ or N is small (in which cases the distribution of $s^{(1)}$ is badly approximated).

Further we have performed some calculations on $\text{cov}(s^{(1)}, s^{(2)})$ and $\rho(s^{(1)}, s^{(2)})$, based on simulation results, and these show the following.

(3) In all cases of Tables 1, 2, 3 and 4 $\text{cov}(s^{(1)}, s^{(2)}) \leq 0$, and $\rho(s^{(1)}, s^{(2)}) \leq 0$; $\rho(s^{(1)}, s^{(2)})$ tends to zero for $N \to \infty$, $a \neq 1$, whereas it seems to tend to -1 for $N \to \infty$, $a = 1$ (cf. (2.20), (2.23) for $G \equiv M$); in fact our numerical results suggest that $\rho(s^{(1)}, s^{(2)})$ is rather insensitive to the choice of service time distribution G.

Table 4
G≡D

N	a	$E[s^{(1)}s^{(2)}]$ (sim.)	% error in $E[s^{(1)}s^{(2)}]$ (A)	% error in $E[s^{(1)}s^{(2)}]$ (A2)	$E[c^2]$ (sim.)	% error in $E[c^2]$ (A)	% error in $E[c^2]$ (A2)
2	$\frac{1}{4}$	0.49	−4.05	−4.28	6.20	−0.65	−1.02
	$\frac{1}{2}$	0.96	−5.90	−6.74	6.82	−1.67	−3.09
	1	1.85	−6.51	−9.30	9.18	−2.63	−5.43
	2	3.48	−2.29	−10.05	19.80	−0.80	−12.01
	4	7.14	−0.73	−15.50	66.58	−0.16	−23.27
3	$\frac{1}{4}$	0.78	−3.59	−4.92	12.20	−0.46	−1.91
	$\frac{1}{2}$	1.60	−7.02	−8.29	12.25	−1.83	−1.11
	1	3.21	−11.18	−13.22	16.03	−4.49	−6.98
	2	5.87	−6.31	−11.66	39.41	−1.88	−9.94
	4	11.36	−2.11	−11.63	146.21	−0.33	−16.41
4	$\frac{1}{4}$	1.08	−3.78	−4.73	20.68	−0.40	−3.66
	$\frac{1}{2}$	2.31	−6.86	−7.74	20.24	−1.57	−2.01
	1	4.99	−13.04	−14.45	25.16	−5.17	−7.46
	2	8.21	−4.16	−8.04	67.03	−1.02	−7.35
	4	15.11	0.55	−6.60	257.98	0.06	−12.51
6	$\frac{1}{4}$	1.64	−1.23	−1.86	42.01	−0.10	−0.23
	$\frac{1}{2}$	3.75	−3.87	−4.41	41.93	−0.69	−0.73
	1	9.50	−12.07	−12.86	48.67	−4.71	−5.13
	2	13.21	−2.81	−5.23	147.08	−0.56	−4.90
	4	22.99	1.58	−3.12	577.81	0.13	−8.49
10	$\frac{1}{4}$	2.81	−0.84	−1.21	109.84	−0.04	0.07
	$\frac{1}{2}$	6.83	−3.43	−3.73	110.28	−0.43	−0.62
	1	22.62	−9.63	−9.97	120.84	−4.08	−3.93
	2	23.50	−2.65	−4.01	403.16	−0.31	−2.83
	4	40.00	−0.80	−3.50	1601.34	−0.06	−5.15

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