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PROCESSES WITH DELTA-CORRELATED CUMULANTS

N.G. van KAMPEN

Institute for Theoretical Physics of the University, Utrecht, The Netherlands

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In a recent paper[1]) a differential equation was studied which involves a stochastic process having the property that all its cumulants are delta-correlated. It is here shown that such processes consist of a random sequence of delta functions with random coefficients. As a consequence the solutions of the differential equation are Markov processes, whose master equation can be constructed. From it closed equations for the successive moments may be obtained, and the auto-correlation is determined, in agreement with the results of reference 1. Some generalizations are given in Appendices B and C.

In a recent paper West, Lindenberg, and Seshadri[1]) studied the following randomly perturbed damped harmonic oscillator

$$\ddot{x} + 2\lambda\dot{x} + \{\Omega^2 + \gamma(t)\}x = f(t). \tag{1}$$

Here f is a Gaussian Langevin term with

$$\langle f(t)\rangle = 0, \quad \langle f(t_1)f(t_2)\rangle = 2\tilde{D}\delta(t_1 - t_2). \tag{2}$$

The frequency perturbation $\gamma(t)$ is also random with zero average, and is assumed to have delta-correlated cumulants ($m \geqslant 2$)

$$\langle\langle \gamma(t_1)\gamma(t_2)\dots\gamma(t_m)\rangle\rangle = 2D_m\delta(t_1 - t_2)\,\delta(t_1 - t_3)\dots\delta(t_1 - t_m). \tag{3}$$

By explicit calculation they found the first and second moments of x and $p \equiv \dot{x}$ and showed that these do not depend on $D_3, D_4, \dots$.

A process $\gamma(t)$ with the property (3) has a characteristic functional with test function $k(t)$

$$G_\gamma[k] = \langle e^{i\int k(t)\gamma(t)dt}\rangle = \exp\left[\sum_{m=1}^{\infty} \frac{i^m}{m!} 2D_m \int \{k(t)\}^m \,dt\right]. \tag{4}$$

When $k(t)$ consists of two parts with non-overlapping supports this functional factorizes. It follows that the values of γ at different time points are statistically independent. Hence γ is a completely random function[2]) and the integrated process

$$y(t) = \int_0^t \gamma(t')\,dt' \tag{5}$$

has independent increments. According to a theorem of Feller[3]) it is therefore equivalent with a compound Poisson process. That means that it can be represented by a random sequence of instantaneous jumps, whose size ξ has a certain probability density $\varphi(\xi)$–or at least as a limiting form thereof. Such a process, however, is governed by a master equation for its transition probability $P(y, t\,|\,y_0, t_0) \equiv P(y, t)$

$$\frac{\partial P(y,t)}{\partial t} = \rho \int P(y-\xi, t)\,\varphi(\xi)\,\mathrm{d}\xi - \rho P(y,t), \tag{6}$$

where ρ is the average number of events per time unit.

The derivative process γ itself has a delta peak at each jump of y and is otherwise zero. Of course that is not a properly defined stochastic process, but it can be used as an approximate description of a sequence of short pulses, as shown in Appendix A. To study its effect on the oscillator write (1) in the form

$$\dot{x} = p, \tag{7a}$$

$$\dot{p} = -2\lambda p - \Omega^2 x - \gamma(t)x + f(t). \tag{7b}$$

Each delta peak in γ, corresponding to a jump ξ of y, creates in p a jump $-\xi x$. Per unit time there is a probability $\rho\varphi(\xi)$ to undergo such a jump. Hence the probability density $P(x, p, t)$ obeys

$$\frac{\partial P(x,p,t)}{\partial t} = -p\frac{\partial P(x,p,t)}{\partial x} + \frac{\partial}{\partial p}\{2\lambda p + \Omega^2 x\}P(x,p,t)$$
$$+ \tilde{D}\frac{\partial^2 P(x,p,t)}{\partial p^2} + \rho \int P(x, p+\xi x, t)\,\varphi(\xi)\,\mathrm{d}\xi - \rho P(x,p,t). \tag{8}$$

The first three terms on the right are the familiar ones and the last two are analogous to (6). We shall now derive from it the results of West et al.

Multiply with x and p respectively and integrate:

$$\partial_t\langle x\rangle = \langle p\rangle, \tag{9a}$$

$$\begin{aligned}\partial_t\langle p\rangle &= -2\lambda\langle p\rangle - \Omega^2\langle x\rangle \\ &\quad + \rho\int \varphi(\xi)\,\mathrm{d}\xi \int \mathrm{d}x \int p\{P(x, p+\xi x, t) - P(x,p,t)\}\,\mathrm{d}p \\ &= -2\lambda\langle p\rangle - \Omega^2\langle x\rangle - \rho\bar{\xi}\langle x\rangle.\end{aligned} \tag{9b}$$

The bar denotes the average over φ. Since $\gamma(t)$ was supposed to have zero average one has $\bar{\xi} = 0$. The remaining terms in (9) are the same as for the unperturbed oscillator. The average motion is therefore neither affected by γ nor by f, as found by West et al. The solution is simply that of the damped harmonic oscillator, which we write in abbreviated form in terms of an evolution matrix U

$$\begin{bmatrix} \langle x \rangle_t \\ \langle p \rangle_t \end{bmatrix} = U(t) \begin{bmatrix} \langle x \rangle_0 \\ \langle p \rangle_0 \end{bmatrix}. \tag{10}$$

In the same way one obtains for the second moments

$$\partial_t \langle x^2 \rangle = 2\langle xp \rangle, \tag{11a}$$

$$\partial_t \langle xp \rangle = \langle p^2 \rangle - 2\lambda \langle xp \rangle - \Omega^2 \langle x^2 \rangle, \tag{11b}$$

$$\partial_t \langle p^2 \rangle = -4\lambda \langle p^2 \rangle - 2\Omega^2 \langle xp \rangle + 2\tilde{D} + \rho\overline{\xi^2}\langle x^2 \rangle. \tag{11c}$$

In order to find the correlation functions in equilibrium only the stationary solution of these equations is required

$$\langle xp \rangle^s = 0, \qquad \langle p^2 \rangle^s = \Omega^2 \langle x^2 \rangle^s = \frac{2\tilde{D}\Omega^2}{4\lambda\Omega^2 - \rho\overline{\xi^2}}. \tag{12}$$

The correlation matrix C_τ is then found from

$$C_\tau = \begin{bmatrix} \langle x(0)x(\tau) \rangle^s & \langle x(0)p(\tau) \rangle^s \\ \langle p(0)x(\tau) \rangle^s & \langle p(0)p(\tau) \rangle^s \end{bmatrix} = C_0 U^\dagger(\tau), \tag{13}$$

where $U^\dagger$ is the transpose of the evolution matrix (10) and C_0 is the matrix of the quantities (12).

Our result (13) is readily seen to coincide with the result (4.24) of West et al., provided that our $\frac{1}{2}\rho\overline{\xi^2}$ is identical with their D, which is the D_2 in our equation (3). To show that, derive from (6) for the change in y in a short time Δt

$$\langle (\Delta y)^2 \rangle = \rho \Delta t \overline{\xi^2}.$$

On the other hand using (5) one has

$$\langle (\Delta y)^2 \rangle = \left\langle \left\{ \int_0^{\Delta t} \gamma(t')\, dt' \right\}^2 \right\rangle = 2D_2 \Delta t,$$

q.e.d. The general connection between the coefficients D_n in (3) and the moments of ξ is derived in Appendix A.

Appendix A

Following Campbell[4]) we construct a stationary random process $x(t)$ by starting from a Poisson distribution of time points τ (density ρ) and attaching to each τ a pulse $\psi(t-\tau)$. More generally we allow the pulse shape to be selected at random from a set of functions ψ_j and will ultimately average over

j. A sample function $x(t)$ is then specified in $0<t<T$ choosing an $s=0,1,2,\dots$, and subsequently s time points $\tau_1, \tau_2, \dots, \tau_s$ and s values of the index j:

$$x(t)=\sum_{\sigma=1}^{s} \psi_{j_\sigma}(t-\tau_\sigma).$$

The characteristic functional, with test function k, is

$$\langle e^{i\int_0^T k(t)x(t)dt}\rangle = \sum_{s=0}^{\infty} \frac{(\rho T)^s}{s!} e^{-\rho T} \int_0^T \frac{d\tau_1 \dots d\tau_s}{T^s} \overline{\exp\left[i\sum_{\sigma=1}^{s} \int_0^T k(t)\psi_{j_\sigma}(t-\tau_\sigma)\, dt\right]},$$

where the bar averages over the j_σ. As the τ and j are independent the sum over s can be carried out to give

$$\exp\left[\rho \int_0^T \{e^{i\int_0^T k(t)\psi(t-\tau)\, dt} - 1\}\, d\tau\right].$$

One may now extend the interval from $-\infty$ to $+\infty$, so that the logarithm of the characteristic functional becomes

$$\rho \int_{-\infty}^{\infty} d\tau \left\{\exp\left[i\int_{-\infty}^{\infty} k(t)\psi(t-\tau)\, d\tau\right] - 1\right\}.$$

Expansion in powers of k gives

$$\rho \sum_{m=1}^{\infty} \frac{i^m}{m!} \int_{-\infty}^{\infty} k(t_1)\dots k(t_m)\, dt_1 \dots dt_m \int_{-\infty}^{\infty} \overline{\psi(t_1-\tau)\dots\psi(t_m-\tau)}\, d\tau. \qquad (14)$$

The final integral, together with the factor ρ, is the m-th cumulant of $x(t)$.

Now take for ψ a narrow rectangular peak,

$$\psi_j(t) = \begin{cases} 0 & \text{for } |t| > \frac{1}{2}\epsilon \\ A_j & \text{for } |t| < \frac{1}{2}\epsilon \end{cases}.$$

In the limit $\epsilon \to 0$, $A_j \to \infty$ with ϵA_j constant, (14) reduces to

$$\rho \sum_{m=1}^{\infty} \frac{i^m}{m!} k(\tau)^m\, d\tau \epsilon^m \overline{A^m}.$$

This is identical with (4) if one sets $\rho\epsilon^m \overline{A^m} = 2D_m$. It is also clear that ϵA is identical with the size ξ of the jump in (6), so that $\rho\overline{\xi^m} = 2D_m$ (of which the case $n=2$ was used in the text). Finally one obtains Gaussian white noise in the limit $\xi \to 0$, $\rho \to \infty$ with constant $\rho\overline{\xi^2} = 2D_2$.

Appendix B

The result can be expanded to more variables. Consider the set of equations

$$\dot{u}_\upsilon = \sum_\mu A_{\upsilon\mu} u_\mu + \gamma(t) \sum_\mu B_{\upsilon\mu} u_\mu + f_\upsilon, \tag{15}$$

where $\upsilon, \mu = 1, 2, \ldots, n$ and the f_υ are independent white noise processes with

$$\langle f_\upsilon(t_1) f_\mu(t_2) \rangle = \tilde{D}_{\upsilon\mu} \delta(t_1 - t_2).$$

Each delta peak of γ causes the vector u_υ to jump by an amount proportional to $\Sigma B_{\upsilon\mu} u_\mu$. Therefore, in order that the equation has a well-defined meaning it is necessary that this jump does not affect the quantities $\Sigma B_{\upsilon\mu} u_\mu$ themselves. That requirement states

$$\sum_{\upsilon\mu} B_{\upsilon\mu} u_\mu \frac{\partial}{\partial u_\upsilon} \left[\sum_\kappa B_{\lambda\kappa} u_\kappa \right] = 0$$

for all λ and all u_μ; or

$$\sum_\upsilon B_{\upsilon\mu} B_{\lambda\upsilon} = 0 \quad \text{for all } \lambda, \mu.$$

Thus $B^2 = 0$, i.e. B is nilpotent. By a suitable linear transformation of variables it can be cast in the form

$$\left(\begin{array}{c|c} 0 & 0 \\ \hline E & 0 \end{array} \right). \tag{16}$$

This is the form it has in (7) if one sets $u_1 = x$, $u_2 = p$.

If this condition is satisfied $u_\upsilon(t)$ is a multivariate Markov process and the same method as used in the text leads to the master equation

$$\frac{\partial P(u,t)}{\partial t} = -\sum_{\upsilon,\mu} A_{\upsilon\mu} \frac{\partial}{\partial u_\upsilon} u_\mu P(u,t) + \sum_{\upsilon\mu} \tilde{D}_{\upsilon\mu} \frac{\partial^2 P(u,t)}{\partial u_\upsilon \partial u_\mu}$$
$$+ \rho \int \varphi(\xi)\, \mathrm{d}\xi P(u - \xi B u, t) - \rho P(u,t).$$

In analogy with (9) one obtains for the first moments

$$\partial_t \langle u_\alpha \rangle = \sum_\mu A_{\alpha\mu} \langle u_\mu \rangle.$$

In analogy with (11) one obtains for the second moments

$$\partial_t \langle u_\alpha u_\beta \rangle = \sum_\mu A_{\alpha\mu} \langle u_\mu u_\beta \rangle + \sum_\mu A_{\beta\mu} \langle u_\alpha u_\mu \rangle + \rho \overline{\xi^2} \sum_{\mu\upsilon} B_{\alpha\mu} B_{\beta\upsilon} \langle u_\mu u_\upsilon \rangle.$$

For higher moments closed equations are obtained as well.

It is easily seen that nothing changes if A and B are allowed to depend on time. Finally it is possible to write an analog of (15) with several independent processes γ, but that makes not much difference either.

Appendix C

Nonlinear equations involving one or more stochastic functions with the property (3) have been considered by Deker in a somewhat opaque article[5]). We consider

$$\dot{u}_\upsilon = F_\upsilon(u_\lambda) + \gamma(t) G_\upsilon(u_\lambda) + f_\upsilon(t). \tag{17}$$

In order that this equation has a meaning it is necessary that the $G_\upsilon(u)$ are insensitive to the jumps in u_υ, that is,

$$G_\upsilon(u_\lambda + \xi G_\lambda(u_\mu)) = G_\upsilon(u_\lambda) \tag{18}$$

for all υ and all possible magnitudes ξ of the jumps in the process (5). For instance, this condition is satisfied if for each υ it is true that either $G_\upsilon = 0$ or u_υ does not occur as an argument in any of the functions G–which is the analog of (16). In the limit of Gaussian white noise (18) reduces to

$$\sum_\lambda G_\lambda \frac{\partial G_\upsilon}{\partial u_\lambda} = 0 \quad \text{for all } \upsilon.$$

If the condition (18) is satisfied $u_\upsilon(t)$ is a multivariate Markov process governed by the master equation

$$\frac{\partial P(u_\lambda, t)}{\partial t} = -\sum_\upsilon \frac{\partial}{\partial u_\upsilon} F_\upsilon(u_\lambda) P(u_\lambda, t) + \sum_{\upsilon\mu} \tilde{D}_{\upsilon\mu} \frac{\partial^2 P(u_\lambda, t)}{\partial u_\upsilon \partial u_\mu}$$
$$+ \rho \int \varphi(\xi) \, d\xi \, P(u_\lambda - \xi G_\lambda(u_\mu), t) - \rho P(u_\lambda, t).$$

In contrast with the linear case it is now impossible to extract closed equations for the successive moments.

Again it is easy to include an explicit time dependence in F, G and $\tilde{D}$, and to replace the single term $\gamma(t) G_\upsilon(u_\lambda)$ in (17) by a sum of such terms with different processes γ, as in reference 5.

References

1) B.J. West, K. Lindenberg and V. Seshadri, Physica **102A** (1980) 470. See also K. Lindenberg, V. Seshadri, K.E. Shuler and B.J. West, J. Stat. Phys., to be published. We follow their notation.

2) Such processes are mentioned incidentally in the proof of Doob's theorem, see J.L. Doob, Annals Math. **43** (1942) 351; reprinted in N. Wax, Selected Papers on Noise and Stochastic Processes (Dover, New York, 1954).
3) W. Feller, An Introduction to Probability Theory and Its Applications, vol. II (Wiley, New York, 1966), p. 294.
4) Previous reference, p. 176. See also P. Hänggi, Z. Phys. **B36** (1980) 271.
5) U. Deker, Phys. Rev. **A19** (1979) 846.