

ON A NEW FORMULATION OF THE CONTINUUM HEISENBERG SPIN SYSTEM IN A SPACE OF ARBITRARY DIMENSIONALITY*

TH.W. RUIJGROK and J. JURKIEWICZ†

Instituut voor Theoretische Fysica, Rijksuniversiteit Utrecht, The Netherlands‡

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The classical equations of motion of a continuum Heisenberg spin system in N dimensions are written in the form of equations for a particle field and a gauge field. Known results, such as the exact solution for $N = 1$ and the self-dual solution for $N = 2$, are recovered. Of the new results we mention: (a) velocity independence of the soliton-energy for $N = 1$; (b) equivalence with a nonlinear Schrödinger equation in N dimensions; (c) relation to the static sine-Gordon equation; (d) numerical calculations of finite-energy solutions in two and three dimensions.

1. Introduction

The purpose of this paper is to get a better understanding of the N -dimensional continuum Heisenberg spin system, by deriving and studying a new form of the equations of motion. Our starting point is the partial differential equation

$$\partial_0 \mathbf{S} = \mathbf{S} \times \Delta \mathbf{S} \quad (1)$$

for the three-dimensional vector $\mathbf{S}(x_1, \dots, x_N, t)$ in an $(N + 1)$ -dimensional space. The Laplace operator is $\Delta = \partial_k \partial_k$ (summation over k from 1 to N is implied), while the time derivative is denoted by ∂_0 . The (conserved) total energy of the system is $E = \int \mathcal{E}(\mathbf{r}, t) d_N r$, where the energy density is given by

$$\mathcal{E}(\mathbf{r}, t) = \partial_k \mathbf{S} \cdot \partial_k \mathbf{S}. \quad (2)$$

Using the Pauli matrices we introduce the 2×2 hermitian matrix $S(\mathbf{r}, t) = \mathbf{S} \cdot \boldsymbol{\sigma}$, which satisfies the equation

$$\partial_0 S = \frac{1}{2i} [S, \Delta S]. \quad (3)$$

Since in each point \mathbf{r} the length of the spinvector is conserved, the solution of

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† Permanent address: Instytut Fizyki UJ, Reymonta 4, 30–59 Kraków, Poland.

‡ Mailing address: Princetonplein 5, P.O. Box 80006, 3508 TA Utrecht, The Netherlands.

eq. (3) can be written in the form

$$S(\mathbf{r}, t) = U(\mathbf{r}, t)S_0 U^{-1}(\mathbf{r}, t), \tag{4}$$

where $U(\mathbf{r}, t)$ is a unitary matrix and S_0 is independent of \mathbf{r} and t . We also assume that in each point \mathbf{r} the spin has a length $|S_0|$. Using this U -matrix we define $N + 1$ hermitian matrices A_α (Greek indices will always run through $\alpha = 0, 1, 2, \dots, N$) by

$$\partial_\alpha U = iUA_\alpha \quad \text{and} \quad \partial_\alpha U^+ = -iA_\alpha U^+. \tag{5}$$

In order that the compatibility condition $\partial_\alpha(\partial_\beta U) = \partial_\beta(\partial_\alpha U)$ be satisfied the following relation for the ‘‘potentials’’ A_α must hold:

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha + i[A_\alpha, A_\beta] = 0. \tag{6}$$

On substitution of eq. (4) into the equation of motion (3) it follows that

$$A_0 = -\frac{1}{2}[A_k, [A_k, S_0]] + \frac{i}{2}[\partial_k A_k, S_0] + \lambda S_0, \tag{7}$$

where $\lambda(\mathbf{r}, t)$ is an arbitrary real function. The definition of $U(\mathbf{r}, t)$ is not unique, because

$$\tilde{U}(\mathbf{r}, t) = U(\mathbf{r}, t)e^{if(\mathbf{r}, t)S_0}, \tag{8}$$

with an arbitrary real function $f(\mathbf{r}, t)$, leads to the same spinvector. In terms of the potentials this means that the eqs. (6) and (7) are invariant under the gauge transformation

$$A_\alpha \rightarrow \tilde{A}_\alpha(\mathbf{r}, t) = e^{-ifS_0} A_\alpha e^{ifS_0} + \partial_\alpha f \cdot S_0. \tag{9}$$

We now return to real vector functions $A_\alpha(\mathbf{r}, t)$, defined by $A_\alpha(\mathbf{r}, t) = A_\alpha \cdot \boldsymbol{\sigma}$. That A_α is indeed traceless follows from the unitarity of U . Each of the vectors A_α can be decomposed into a component parallel to S_0 and a component perpendicular to S_0 :

$$A_\alpha(\mathbf{r}, t) = B_\alpha(\mathbf{r}, t) + a_\alpha(\mathbf{r}, t)S_0, \tag{10}$$

with $B_\alpha \cdot S_0 = 0$. If $b_\alpha^{(1)}$ and $b_\alpha^{(2)}$ are the two components of B_α in the plane perpendicular to S_0 we can introduce the complex functions $b_\alpha(\mathbf{r}, t) = b_\alpha^{(1)} + ib_\alpha^{(2)}$. In terms of these new functions $a_\alpha(\mathbf{r}, t)$ (real) and $b_\alpha(\mathbf{r}, t)$ (complex) the eqs. (7) and (6) take the following form

$$b_0 = 2a_k b_k + i\partial_k b_k \quad \text{eq. of motion} \tag{11a}$$

and

$$\begin{aligned} \partial_\alpha b_\beta - \partial_\beta b_\alpha - 2i(a_\alpha b_\beta - a_\beta b_\alpha) &= 0 \\ \partial_\alpha a_\beta - \partial_\beta a_\alpha + i(b_\alpha^* b_\beta - b_\alpha b_\beta^*) &= 0 \end{aligned} \quad \text{compatibility equations.} \tag{11b}$$

The gauge transformation (9) now becomes

$$a_\alpha \rightarrow \tilde{a}_\alpha = a_\alpha + \partial_\alpha f, \quad (12a)$$

$$b_\alpha \rightarrow \tilde{b}_\alpha = e^{2if} b_\alpha \quad (12b)$$

and it can be shown directly that the eqs. (11) are indeed invariant under this transformation.

Observe that a_α transforms like an electromagnetic potential and b_α like a particle wave function. The flux through a closed contour, defined as $\oint a_k dx_k$, is invariant under gauge transformations, which shows that it will not always be possible to transform away the functions $a_\alpha(r, t)$.

The eqs. (11) are also invariant under scale transformations defined by

$$\begin{aligned} x'_k &= \lambda x_k, & t' &= \lambda^2 t, \\ \tilde{a}_k(r', t') &= \frac{1}{\lambda} a_k(r, t), & \tilde{a}_0(r', t') &= \frac{1}{\lambda^2} a_0(r, t), \\ \tilde{b}_k(r', t') &= \frac{1}{\lambda} b_k(r, t), & \tilde{b}_0(r', t') &= \frac{1}{\lambda^2} b_0(r, t). \end{aligned} \quad (13)$$

It is possible to write down the equations for a similarity solution

$$a_0 = \frac{1}{r^2} A_0(\boldsymbol{\eta}); \quad a_k = \frac{1}{r} A_k(\boldsymbol{\eta}); \quad b_0 = \frac{1}{r^2} B_0(\boldsymbol{\eta}); \quad b_k = \frac{1}{r} B_k(\boldsymbol{\eta})$$

with $\eta_k = x_k/\sqrt{t}$. This will not be done, however, since we have not made a systematic study of these equations.

The energy density eq. (2) can also be expressed in terms of the new fields and we find

$$\mathcal{E}(r, t) = 4 \sum_{k=1}^N b_k^* b_k. \quad (14)$$

The fact that the total spin $\int S(r, t) d_N r$ is conserved can be shown to be equivalent to

$$\int \partial_k b_k d_N r = 0, \quad (15)$$

which says that there is no source or sink of the b -field.

In the remaining sections we will study special cases of the eqs. (11).

2. The linear chain

We first consider the case where $a_k = 0$ ($k = 1, \dots, N$), so that the total flux through any closed curve is zero.

With

$$\rho(\mathbf{r}, t) \equiv \partial_k b_k \quad (16)$$

the eqs. (11) then become

$$b_0 = i\rho; \quad \partial_j b_k = \partial_k b_j; \quad b_j^* b_k = b_j b_k^*; \quad (17)$$

$$i b_k = -(\partial_k \rho + 2a_0 b_k); \quad \partial_k a_0 = b_k \rho^* + b_k^* \rho.$$

In section 4 we will study the case where all functions are time independent. At this moment, however, we want to restrict ourselves to solutions of the form

$$b_k = \frac{x_k}{r} \psi(\mathbf{r}, t), \quad a_0 = a_0(\mathbf{r}, t) \quad \text{and} \quad b_0 = b_0(\mathbf{r}, t),$$

where $r^2 = x_1^2 + \dots + x_N^2$. On substitution into the above equations we obtain

$$i \frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial r^2} - \frac{N-1}{r} \frac{\partial \psi}{\partial r} + \left(\frac{N-1}{r^2} - 2a_0 \right) \psi, \quad (18)$$

$$\frac{\partial a_0}{\partial r} = \frac{\partial |\psi|^2}{\partial r} + \frac{2(N-1)}{r} |\psi|^2. \quad (19)$$

Eq. (18) is a nonlinear Schrödinger equation for a radially symmetric wavefunction, in which the potential $-2a_0$ is determined by eq. (19). So far we have not succeeded in finding solutions for $N > 1$. For $N = 1$ it follows from eq. (19) that $a_0 = |\psi|^2$ and eq. (18) becomes

$$i \frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} - 2|\psi|^2 \psi(x, t), \quad (20)$$

which is the nonlinear Schrödinger equation as derived by Lakshmanan for the Heisenberg chain. This equation has been solved completely and not much new can be added to its understanding. We would like, however, to make the following remark. If we consider a Galilei-transformation $x' = x + vt$; $t' = t$ and define the new wavefunction as

$$\bar{\psi}(x', t') = e^{(i/2)v(x' - (1/2)vt')} \psi(x' - vt', t'),$$

eq. (20) will be satisfied again. The energy density, which was $\mathcal{E}(x, t) = 4|\psi(x, t)|^2$, becomes after the transformation $\bar{\mathcal{E}}(x', t') = \mathcal{E}(x, t)$. From this we see that the total energy is invariant under a Galilei-transformation. In particular, if a moving soliton solution is obtained from a soliton at rest, by boosting it, but not changing its width, the energy is not changed. This

statement has been verified by using the explicit expression for the energy of a solitary wave. If nevertheless a relation between energy and momentum exists²⁾, it is because not the velocity but the total spin is kept constant after a change in momentum. For a constant width, i.e., for a constant energy the relation between momentum and velocity is given by

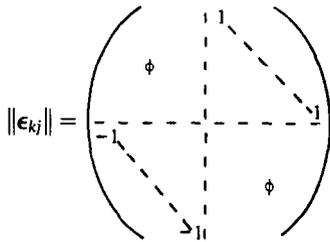
$$P = 4 \operatorname{arctg} \frac{E}{2v}.$$

3. Duality

For a two-dimensional coordinate space Belavin and Polyakov³⁾ have defined a dual derivative by

$$\partial_k \tilde{S} = \frac{1}{2i} \epsilon_{kj} [\partial_j S, S]. \tag{21}$$

This definition can also be used for an arbitrary even-dimensional coordinate space, where now ϵ_{kj} is given by



also their definition of “degree of mapping” can be extended to this case:

$$q = -\frac{1}{16\pi} \int \operatorname{Tr}(\partial_k \tilde{S} \cdot \partial_k S) d_{Nr}. \tag{22}$$

This formula can be written as

$$q = -\frac{1}{8\pi} \epsilon_{\alpha\beta\gamma} \epsilon_{kj} \int S^\alpha \partial_k S^\beta \partial_j S^\gamma d_{Nr}, \tag{23}$$

or, after introducing the polar and azimuthal angles $\theta(r)$ and $\phi(r)$, as

$$q = -\frac{1}{4\pi} \epsilon_{kj} \int \sin \theta \cdot \partial_k \theta \cdot \partial_j \phi \cdot d_{Nr}. \tag{24}$$

Only for $N = 2$, can this q really be interpreted as the degree of mapping, and only in that case, therefore, it must necessarily be an integer, if moreover we

require the spin to be a continuous function of the coordinates. The energy

$$E = \frac{1}{2} \int \text{Tr}(\partial_k S \cdot \partial_k S) d_N r, \quad (25)$$

however, still satisfies the inequality

$$E \geq 8\pi|q| \quad (26)$$

for arbitrary even N , with equality only for a self-dual or anti-self-dual solution:

$$\partial_k \tilde{S} = \pm \partial_k S. \quad (27)$$

For such an (anti)-self-dual solution it follows from (21), after applying $\epsilon_{ik} \partial_l$, that

$$[S, \Delta S] = 0. \quad (28)$$

Comparison with eq. (3) shows that it is a time independent solution of the original equation of motion.

We now want to translate these duality properties in terms of the new fields a_α and b_α .

In analogy with

$$\partial_k S = i U [A_k, S_0] U^{-1}, \quad (29)$$

which follows from eqs. (4) and (5), we define the dual field \tilde{A}_k by

$$\partial_k \tilde{S} = i U [\tilde{A}_k, S_0] U^{-1}. \quad (30)$$

With the definition (21) and after some formal manipulations it can then be shown that the dual of the "particle field" is given by

$$\tilde{b}_k = -i \epsilon_{kj} b_j. \quad (31)$$

(Anti)-self-duality is expressed by $\tilde{b}_k = \pm b_k$.

The "degree of mapping" becomes

$$q = \frac{i}{4\pi} \epsilon_{kj} \int (b_k^* b_j - b_k b_j^*) d_N r = -\frac{1}{4\pi} \epsilon_{kj} \int (\partial_k a_j - \partial_j a_k) d_N r, \quad (32)$$

where the last equality follows from eq. (11b). This q can also be written as

$$q = -\frac{1}{4\pi} \int (b_k^* \tilde{b}_k + b_k \tilde{b}_k^*) d_N r, \quad (33)$$

from which we see by comparing with eq. (14) that for an (anti)-self-dual solution $E = 8\pi|q|$.

In the two-dimensional case the expression (32) can be reduced a little

further. It may happen that in a number of points in the coordinate plane, r_1, \dots, r_p , the functions $a_1(r)$ and $a_2(r)$ are singular or that for some physical reason neighborhoods of these points are inaccessible and must be excluded, so that the space is not simply connected. In that case we find from eq. (32)

$$q = -\frac{1}{2\pi} \int (\partial_1 a_2 - \partial_2 a_1) dx_1 dx_2, \tag{34}$$

which, using Stokes' theorem, can be written as

$$q = -\frac{1}{2\pi} \int_{C_\infty} a \cdot d\ell + \frac{1}{2\pi} \sum_{j=1}^p \int_{C_j} a \cdot d\ell, \tag{35}$$

where the contour C_∞ must be taken to infinity. See fig. 1.

We have not found self-dual solutions for $N \neq 2$. For $N = 2$, however, we have found that the eqs. (11) can be satisfied by an anti-self-dual solution of the form $a_0 = b_0 = 0$ and

$$a_j = -\frac{\epsilon_{jk} x_k}{2r^2} A(r); \quad b_j = \frac{x_j + i\epsilon_{jk} x_k}{2r^2} B(r). \tag{36}$$

with $r^2 = x_1^2 + x_2^2$, provided $A(r)$ and $B(r)$ satisfy

$$\frac{dA}{dr} = -\frac{B^2}{r} \quad \text{and} \quad \frac{dB}{dr} = \frac{AB}{r}. \tag{37}$$

These equations can easily be solved by observing that $A^2 + B^2 = q'^2$ is a constant. The solution is

$$A(r) = q' \frac{1 - x^{2q'}}{1 + x^{2q'}} \quad \text{and} \quad B(r) = 2q' \frac{x^{q'}}{1 + x^{2q'}}, \tag{38}$$

with $x = r/r_0$ and r_0 an integration constant. The value of q can be calculated from eq. (35), where now C_1 is a contour around $r = 0$, which shrinks to zero.

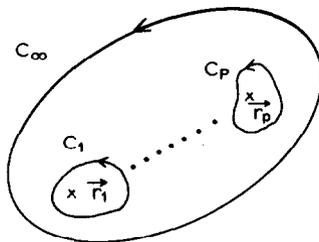


Fig. 1. Contours and singularities for eq. (35).

We find

$$q = q', \quad (39)$$

in which half the contribution comes from C_1 , the other half from C_∞ .

If b_1 and b_2 and all their derivatives with respect to r are to be continuous along any straight line through the origin, then q must necessarily be an integer and our solution reduces to the one found by Belavin and Polyakov³. If, however, a circle of radius r_0 around the origin is excluded, q need not be an integer. We find from eq. (35) that $q = \frac{1}{2}q'$ and $E = 4\pi q'$, with q' an arbitrary positive real number. In all cases the energy density, given by the expression

$$\mathcal{E} = \frac{8q'^2}{r_0^2} \frac{x^{2q'-2}}{(1+x^{2q'})^2}, \quad \left(x = \frac{r}{r_0}\right) \quad (40)$$

is a continuous function of the coordinates.

4. Relation to the static sine-Gordon equation

In this section we want to return to the eqs. (16) and (17) of section 2. A solution of b_k with a harmonic time dependence can, by a space independent gauge transformation, always be changed into a time independent solution. Restricting ourselves therefore to that case, the equations for $b_k(r)$, $a_0(r)$ and $b_0(r)$ now become:

$$\partial_k b_k = \rho(r); \quad b_0 = i\rho; \quad \partial_j b_k = \partial_k b_j; \quad b^* b_k = b_j b^* \ddot{x}; \quad (41a)$$

$$\partial_k \rho + 2a_0 b_k = 0; \quad \partial_k a_0 = b_k \rho^* + b^* \ddot{x} \rho. \quad (41b)$$

The fact that $a_0(r)$ and $b_0(r)$ do not vanish means that, although these fields are stationary, the spin as calculated via eqs. (4) and (5), does depend on time.

Using eq. (41) it is easy to show that

$$\partial_k (\rho^* \rho + a_0^2) = 0. \quad (42)$$

A further restriction is to consider only real functions $b_k(r)$. In this case it follows from eq. (42) that there exists a real function $\phi(r)$, such that

$$a_0 = -K \cos \phi(r) \quad \text{and} \quad \rho(r) = K \sin \phi(r), \quad (43)$$

where K is an arbitrary real constant. From (41) it then follows that

$$b_k = \frac{1}{2} \partial_k \phi(r), \quad (44)$$

where ϕ is a solution of the static sine-Gordon equation

$$\Delta\phi(r) = 2K \sin \phi(r), \quad (45)$$

Δ being the Laplace operator in N dimensions. The total energy is

$$E = 4 \sum_{k=1}^N \int |b_k|^2 d_N r = \sum_{k=1}^N \int (\partial_k \phi)^2 d_N r, \quad (46)$$

which, after partial integration and assuming that for $r \rightarrow \infty$ the limit of $\phi(r)$ is reached fast enough, can also be written as

$$E = -2K \int \phi(r) \sin \phi(r) d_N r. \quad (47)$$

The conservation of total spin, expressed by eq. (15), now reads

$$\int \sin \phi(r) d_N r = 0, \quad (48)$$

which is the same as to say that the total "charge" is zero:

$$\int \rho(r) d_N r = 0. \quad (49)$$

Although for $N = 1, 2$ and 3 Bäcklund transformations for the sine-Gordon equation have been constructed^{4,5}, no stationary solutions with finite energy have been found for $N > 1$. For radially symmetric solutions it can be shown that, if $b_k = \frac{1}{2} \partial_k \phi = \frac{1}{2} (x_k/r) d\phi/dr$ is required to be continuous in $r = 0$, i.e. for $d\phi/dr|_{r=0} = 0$, the energy can only be infinite. The argument runs as follows.

In the radially symmetric case eq. (45) can be written as

$$\frac{d^2 \phi}{dr^2} = -\frac{dV}{d\phi} - \frac{N-1}{r} \frac{d\phi}{dr}, \quad (50)$$

with $V = 2K \cos \phi$. This equation can be interpreted as describing the motion of a particle moving in a potential V and feeling an additional friction proportional to its velocity. Since the initial velocity is zero and the motion is damped, the particle will approach one of the equilibrium positions $\phi_\ell = (2\ell + 1)\pi$. Since $\sin \phi_\ell = 0$, eq. (50) can be linearized for large r , leading to

$$\frac{d^2 \chi}{dr^2} + \frac{N-1}{r} \frac{d\chi}{dr} + \chi = 0 \quad (51)$$

for $\chi = \phi - \phi_\ell$. The solution of eq. (51) is proportional to a Bessel function, and its asymptotic behavior is given by

$$\chi(r) \approx C \frac{\sin(r + \delta)}{r^{(N-1)/2}}. \quad (52)$$

The energy density, defined by eq. (14), now becomes

$$\mathcal{E}(r) \sim \left(\frac{d\chi}{dr}\right)^2 = C^2 \frac{\cos^2(r + \delta)}{r^{N-1}} \quad (53)$$

and the total energy $E = \int \mathcal{E}(r) d_N r$ diverges.

Finite energy solutions of the radially symmetric type therefore only exist if the continuity condition for b_k is dropped. Or, putting it a different way, if an N -dimensional sphere, of radius r_0 say, is cut out around the origin of the coordinate space. Then $d\phi/dr|_{r=r_0}$ need not be zero and this initial velocity can then be chosen in such a way that for $r \rightarrow \infty$ the particle just reaches one of the points $\phi_\ell = 2\ell\pi$, i.e., a maximum of the potential V .

For ϕ close to ϕ_ℓ eq. (50) can again be linearized, which then leads to

$$\frac{d^2\chi}{dr^2} + \frac{N-1}{r} \frac{d\chi}{dr} - \chi = 0 \quad (54)$$

for $\chi = \phi_\ell - \phi$. The solution is now proportional to a modified Bessel function and its asymptotic behaviour is given by

$$\chi(r) \approx C \frac{e^{-r}}{r^{(N-1)/2}}.$$

The total energy is proportional to

$$\int_{r_0}^{\infty} \left(\frac{d\chi}{dr}\right)^2 r^{N-1} dr,$$

which is a finite number.

For $\ell = 1, 2$ and 3 we have performed a numerical calculation of $E_\ell(r_0)$ and the results are shown in figs. 2 and 3 for $N = 2$ and $N = 3$, respectively. The figures show that in the limit $r_0 \rightarrow 0$ the energy $E_\ell(r_0)$ tends to zero.

For $N = 2$ and for two different values of r_0 we have plotted $\phi(r)$ in fig. 4. The curves labeled with $\ell = 1, 2, 3$ correspond to the limiting values $\lim_{r \rightarrow \infty} \phi(r) = 2\pi\ell$. In the limit $r_0 \rightarrow 0$ the function $\phi(r)$ approaches a step function, which is zero for $r = 0$ and $2\pi\ell$ for $r > 0$.

For a discrete N -dimensional lattice of spins there may be a nonvanishing spectrum of localized states and it is to be expected that the ratio's E_ℓ/E_1 will not differ very much from the corresponding ratio's as calculated in the continuum limit. These ratio's are shown in fig. 5.

5. Conclusions

In this paper we have reformulated the Heisenberg problem for a continuous spin system in N dimensions and have shown that it can be written in the form of a set of gauge invariant equations for a particle field and a gauge field, eq. (11). The remaining part of the paper is devoted to a study of special cases. In section 2 we derived a nonlinear Schrödinger equation for radially

symmetric wavefunctions. No exact solutions have been found, but it is very probable that the eqs. (18) and (19) allow ring-shaped quasi-soliton solutions, which are reflected a number of times before tunneling through the potential, which is built up by the wavefunction itself. We believe in the existence of such solutions because they have been demonstrated⁴⁾ to exist for the sine-Gordon equation, which is related to our equations (section 4). In the one-dimensional case the nonlinear Schrödinger equation reduces to the one derived by Lakshmanan¹⁾. In that case the equations are invariant under a Galilei transformation, from which we derive that the total energy of any solution is not changed by a boost of the coordinate system.

In section 3 the concept of dual field is derived from the dual spin derivative as defined by Belavin and Polyakov³⁾. The anti-self-dual solution which they found for $N = 2$ is recovered and extended to the case where the space is not simply connected.

In section 4 it is shown that time independent solutions satisfy the static sine-Gordon equation.

Leibbrandt⁵⁾ has constructed a Bäcklund transformation for this equation, but no solutions with finite energy are known. By omitting a sphere around the origin from the coordinate space, we have shown, however, that a whole

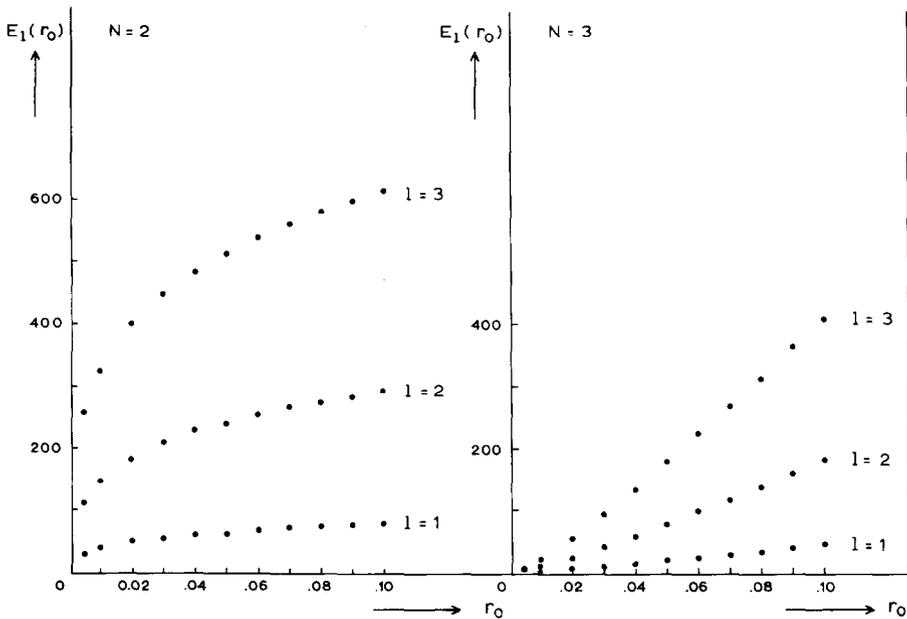


Fig. 2. Spectrum of finite-energy solutions of eq. (50) as a function of the radius of the excluded ball. Two-dimensional case.

Fig. 3. Same as fig. 2. Three-dimensional case.

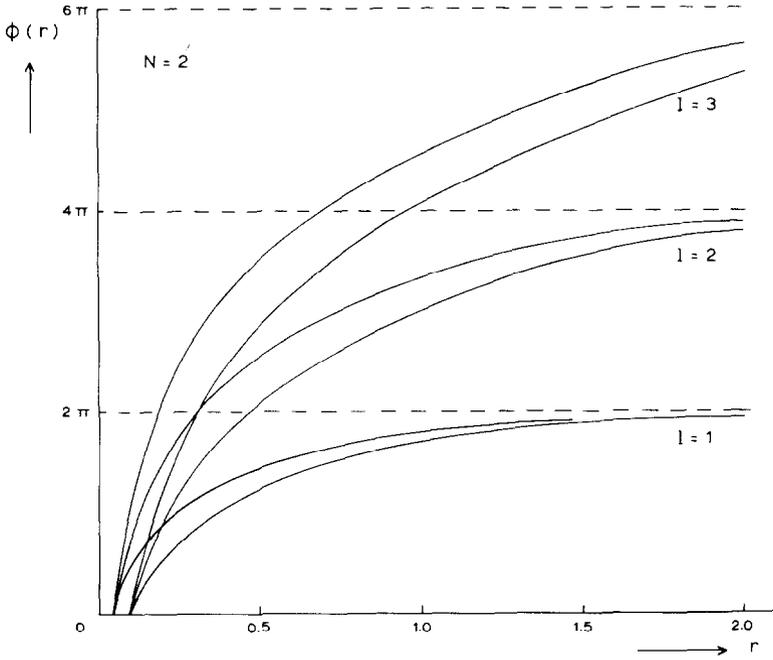


Fig. 4. Solution of eq. (50) for three lowest states and two values of r_0 . Two-dimensional case.

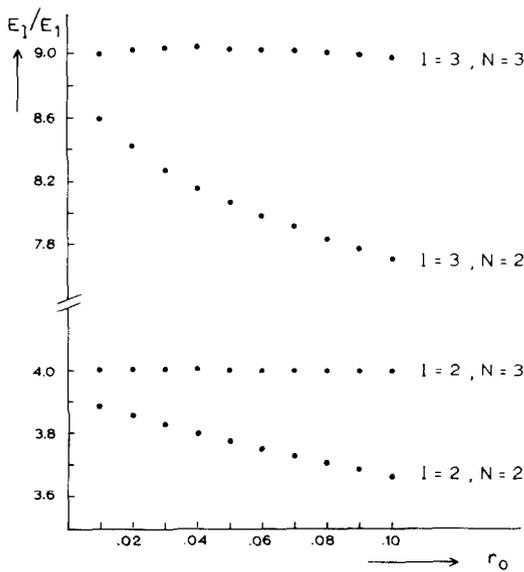


Fig. 5. Ratio of first two excited state- to ground state energy as function of r_0 for two- and three-dimensional case.

spectrum of such solutions exists. The detailed results of the numerical calculation are exhibited in figs. 2–5.

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