Proof of a conjecture of A. Haefliger

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This paper is concerned with the cohomology of étale groupoids. These are topological groupoids whose source and target maps are local homeomorphisms. Etale groupoids play a central rôle in foliation theory (cf. e.g. [C, H84, BN, vE] and many others). Indeed, the main examples of étale groupoids include the Haefliger groupoid , ^{q}, as well as the holonomy groupoid of any foliation. Any orbifold also gives rise to an étale groupoid in a natural way.

Each étale groupoid G defines an abelian category of G-sheaves, and a natural sheaf cohomology $H^*(G, A)$ with coefficients in an arbitrary G-sheaf A. On the other hand, each such sheaf A also induces an (ordinary) sheaf $\tilde{A}$ on the classifying space BG of the groupoid G, and hence defines the usual sheaf cohomology groups $H^*(BG, \tilde{A})$ of the space BG. I will prove the following theorem, conjectured by Haefliger (see e.g. [H76, H92]).

Theorem. *For any étale topological groupoid G and any abelian G-sheaf A, there is an isomorphism*

$$H^*(G, A) \xrightarrow{\sim} H^*(BG, \tilde{A}),$$

natural in G and A.

In section 1 of this paper, I will recall all the necessary definitions. The proof of the theorem will be divided into several steps, and occupies the rest of the paper. The method of the proof is taken from [M95], but the presentation here is to a large extent independent. In [M91], the theorem was proved for the special (and easier) case where the sheaf A is locally constant.

§1. Definitions. In this section I will recall some standard definitions, and I will describe the sheaf $\tilde{A}$ occuring in the statement of the theorem.

A *groupoid* is a small category G in which all arrows are invertible. Thus G is given by two sets G_0 (of objects) and G_1 (of arrows), and structure maps $s, t : G_1 \to G_0$ (source and target), $u : G_0 \to G_1$ (the units, $u(x) = 1_x$), $m : G_1 \times_{G_0} G_1 \to G_1$ (composition, $m(f, g) = f \circ g$) and $i : G_1 \to G_1$ (inverse, $i(x) = x^{-1}$). A *topological groupoid* is a groupoid G where G_0 and G_1 are both equipped with a topology, making the structure maps continuous. An *étale groupoid* is a topological groupoid in which the source map $s : G_1 \to G_0$ is a local homeomorphism; this implies that all other structure maps are local homeomorphisms as well.

Many examples of étale groupoids arise in foliation theory: Haefliger's groupoid , ^{q}, or the holonomy groupoid $Hol(M, \mathcal{F})$ of a foliated manifold $(M, \mathcal{F})$. (The latter is étale if one reduces the space of objects to a complete transversal.) See e.g. the references quoted in the introduction.

A *G-sheaf* is a sheaf of abelian groups A on the space G_0, equipped with a continuous right action by G. This means that for any arrow $g : x \to y$ in G, there is a group homomorphism between the stalks $A_y \to A_x$, $a \mapsto a \cdot g$, satisfying the usual identities for an action; moreover, when A is viewed as an étale space $A \to G_0$, this action should be continuous, as a map $A \times_{G_0} G_1 \to A$. A

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in G; the group of all such invariant sections is denoted

$$,_{inv}(G, A).$$

The category of such G-sheaves, and action preserving maps between them, has enough injectives, and the sheaf cohomology groups are defined as the right derived functors of $,_{inv}$; i.e.

$$H^n(G, A) \overset{def}{=} H^n(,_{inv}(G, I^{\cdot})),$$

where $0 \to A \to I^0 \to I^1 \to \dots$ is an injective resolution. These cohomology groups were introduced in [H76] and further discussed in [H92]; they can also be viewed as the cohomology of the topos of G-sheaves (of sets), so that the SGA-machinery applies. (They are discussed from this point of view in [M91].)

In relation to what follows, it should be emphasized that in the definition of G-sheaf and of the cohomology groups $H^*(G, A)$, the inverse $i : G \to G$ plays no rôle; thus, the definitions apply equally well to an arbitrary topological category.

Any (étale) topological groupoid G gives rise to a simplicial space $G_{\cdot}$, the *nerve* of G, whose space G_n of n-simplices is the space $G_1 \times_{G_0} G_1 \times \cdots \times_{G_0} G_1$ of composable strings of arrows $g = (x_0 \overset{g_1}{\leftarrow} x_1 \leftarrow \cdots \overset{g_n}{\leftarrow} x_n)$, with the fibered product topology. The *classifying space* BG is the geometric realization of $G_{\cdot}$. This should be interpreted as the "thick" realization of [S], so that BG is obtained from the disjoint sum $\sum_{n \geq 0} G_n \times \Delta^n$ by the identifications $(g, \alpha_* t) \sim (\alpha^* g, t)$, for any $g \in G_n, t \in \Delta^m$, and *injective* monotone function $\alpha : [m] \to [n]$. One can construct BG by iterated adjunction spaces, starting with $BG^{(0)} = G_0 = \Delta^0 \times G_0$, and constructing $BG^{(n)}$ as the adjunction space

$$BG^{(n)} = BG^{(n-1)} \cup_{(G_n \times \partial \Delta^n)} (G_n \times \Delta^n). \tag{1}$$

Note that in the relevant examples, BG is usually neither Hausdorff nor paracompact.

All this is completely standard and well-known. Perhaps less well-known is the fact that each G-sheaf A gives rise to a sheaf $\tilde{A}$ on the classifying space BG, constructed as follows. For a point (equivalence class) $[g, t] \in BG$, where $g = (x_0 \overset{g_1}{\leftarrow} x_1 \leftarrow \cdots \overset{g_n}{\leftarrow} x_n)$ and $t \in \Delta^n$ is an interior point, the stalk of $\tilde{A}$ is defined as

$$\tilde{A}_{[g,t]} = A_{x_n}.$$

Thus $\tilde{A}|g \times Int(\Delta^n)$ is the constant sheaf A_{x_n}, which is glued to the parts $\tilde{A}|(g \times \partial_i \Delta^n)$ over the faces by the identity map $A_{x_n} \to A_{x_n}$ for $i < n$, and by the action of $(g_{k+1} \circ \cdots \circ g_n)$: $A_{x_k} \to A_{x_n}$ for $i = n$. More formally, one constructs $\tilde{A}^{(n)}$ over $BG^{(n)}$ by induction, gluing over the adjunctions (1). I will only use the following two evident properties of the sheaf $\tilde{A}$:

1.1 Lemma. *For the n-simplex $\Delta^n = \Delta^n_g \subseteq BG$ corresponding to $g = (x_0 \overset{g_1}{\leftarrow} x_1 \leftarrow \cdots \overset{g_n}{\leftarrow} x_n)$,*
(a) $,(\Delta^n_g, \tilde{A}) = A_{x_0}$,
(b) *$\tilde{A}$ is constant on the simplex Δ^n_g.*

A sheaf F on the classifying space BG will be called *constructible* if it satisfies the property as in part (b) of the lemma. (Note that such a constructible sheaf need not be locally constant, but can depend on the variable g in an arbitrary way.) The next section provides a description of the cohomology $H^*(BG, F)$ of BG with coefficients in a constructible sheaf F.

§2. Constructible sheaves on the classifying space. Let G be a fixed étale groupoid. Let *simp*(G) be the associated "category of simplices" (or "subdivision") of G : an object of *simp*(G) is a pair (n, g) where $n \geq 0$ and $g = (x_0 \overset{g_1}{\leftarrow} \cdots \overset{g_n}{\leftarrow} x_n)$ is a point in G_n. An arrow $\alpha : (m, h) \to (n, g)$

This category $simp(G)$ inherits a natural topology: the space of objects $simp(G)_0$ is topologized as the topological sum of all spaces $G_n, n \geq 0$, while the space of arrows is the topological sum $\sum_{\alpha:[m]\to[n]} G_n$, with α ranging over all injective monotone functions, for all $n, m \geq 0$. Observe that the source and target maps of this topological category are again étale (local homeomorphisms).

A $simp(G)$-sheaf B, as defined in §1, can equivalently be described by the following data: For each $n \geq 0$, a sheaf $B^{(n)}$ on the space G_n; and for each injective $\alpha : [m] \to [n]$ as above, a sheaf map

$$B(\alpha) : A^{(n)} \to \alpha_!(B^{(m)})\ ;$$

here $\alpha_!(B^{(m)})$ denotes the pullback of the sheaf $B^{(m)}$ along the map $\alpha^* : G_n \to G_m$. These sheaf maps should be functorial in α, i.e. $B(\alpha\beta) = B(\beta) \circ B(\alpha)$ and $B(id) = id$.

Now let F be a sheaf on the classifying space BG. Then one can construct a $simp(G)$-sheaf $\gamma(F)$ as follows. For a point $g = (x_0 \overset{g_1}{\leftarrow} \cdots \overset{g_n}{\leftarrow} x_n)$ in G_n, the stalk $\gamma(F)^{(n)}_g$ is defined in terms of the simplex $\Delta^n_g \subset BG$, by

$$\gamma(F)^{(n)}_g = ,\ (\Delta^n_g, F). \tag{1}$$

These stalks form a well-defined sheaf on G_n, in fact

$$\gamma(F)^{(n)} = (\pi_n)_*(i_n)^*(F),$$

where $\pi_n : \Delta^n \times G_n \to G_n$ and $i_n : \Delta^n \times G_n \hookrightarrow BG$ are the obvious maps. Next, for $\alpha : [m] \to [n]$ as above, the sheaf map $\gamma(F)(\alpha)$ is defined at the level of the stalks as the map

$$\gamma(F)(\alpha)_g : ,\ (\Delta^n_g, F) \to ,\ (\Delta^m_{\alpha^*(g)}, F),$$

which is simply the restriction along $\Delta^m_{\alpha^*(g)} \subseteq \Delta^n_g$.

This construction in fact defines a left exact functor γ from the category of sheaves on the classifying space to the category of $simp(G)$-sheaves.

The following lemma relates the cohomology of the space BG to the cohomology (as defined in §1) of the topological category $simp(G)$.

2.1. Lemma. *Let F be a constructible sheaf on the classifying space. Then the functor γ induces a natural isomorphism*

$$H^*(BG, F) \xrightarrow{\sim} H^*(simp(G), \gamma(F)).$$

Proof. First, I claim that for *any* sheaf F on BG (not necessarily constructible), the right derived functors $R^q\gamma$ can be described at the level of the stalks by

$$R^q\gamma(F)^{(n)}_g = H^q(\Delta^n_g, F) \tag{2}$$

Indeed, this identity holds for $q = 0$ by definition (1). Furthermore, if F is injective, then its restriction to $\Delta^n_g \subset BG$ is soft, so that $H^q(\Delta^n_g, F) = 0$. The identity (2) now readily follows from the uniqueness of derived functors. (See also [M95, III.2.6].)

Next, it is clear from (2) that $R^q\gamma(F) = 0$ for $q > 0$ if F is constructible. This proves the lemma.

§3. Proof of theorem. Again, G is a fixed étale topological groupoid, and $simp(G)$ is the associated topological category of simplices. There is a natural continuous functor

$$\varphi : simp(G) \to G \tag{1}$$

pulls back along φ to a $simp(G)$-sheaf $\varphi^*(A)$. Recall also from §1 and §2 that a G-sheaf A induces first a sheaf $\tilde{A}$ on BG, and then a $simp(G)$-sheaf $\gamma(\tilde{A})$. Using Lemma 1.1, it readily follows that there is a natural isomorphism of $simp(G)$-sheaves

$$\varphi^*(A) \cong \gamma(\tilde{A}). \tag{2}$$

Since $\tilde{A}$ is a constructible sheaf on BG, the theorem now follows from Lemma 2.1 together with the following lemma.

3.1. Lemma. *For any G-sheaf A, there is a natural isomorphism*

$$H^*(G, A) \tilde{\to} H^*(simp(G), \varphi^*(A)).$$

Proof. The proof makes use of an auxiliary topological category φ/U, for each open subset $U \subseteq G_0$. The objects of φ/U are strings of arrows in G,

$$(n, z \stackrel{u}{\leftarrow} x_0 \stackrel{g_1}{\leftarrow} x_1 \leftarrow \cdots \stackrel{g_n}{\leftarrow} x_n), \tag{3}$$

where $z \in U$; we briefly denote such an object by (n, g, u). An arrow $(m, h, v) \to (n, g, u)$ in φ/U is an arrow $\alpha : (m, h) \to (n, g)$ in $simp(G)$ so that $u \circ \varphi(\alpha) = v$. The topology on φ/U is defined in the same way as for $simp(G)$. There is an obvious continuous projection functor

$$\pi_U : \varphi/U \to simp(G), \tag{4}$$

sending (n, g, u) to (n, g). Thus, any $simp(G)$-sheaf B induces a φ/U-sheaf $\pi_U^*(B)$. Also, if F is any (ordinary) sheaf on the space $U \subseteq G_0$, there is an obvious induced φ/U-sheaf on which all the arrows of φ/U act trivially; this sheaf can formally be defined as $\varepsilon_U^*(F)$, where

$$\varepsilon_U : \varphi/U \to U \ , \ \varepsilon_U(n, g, u) = z \tag{5}$$

is the evident continuous functor into U (viewed as a topological category with identity arrows only).

Using these categories φ/U, the lemma is now proved in three steps.

First, the functor φ^* has a right adjoint

$$\varphi_* : (simp(G)\text{-sheaves}) \to (G\text{-sheaves})$$

whose right derived functors can be described at the level of the stalks by

$$R^q\varphi_*(B)_x = \varinjlim_{x \in U} H^q(\varphi/U, \pi_U^*(B)); \tag{6}$$

here B is any $simp(G)$-sheaf, x is a point in G_0, and U ranges over all neighbourhoods of x in G_0. (See the Appendix.)

Next, for a G-sheaf A, one can forget the G-action and restrict A to a sheaf $A|U$ on $U \subseteq G_0$. There is a natural isomorphism

$$\pi_U^*\varphi^*(A) \cong \varepsilon_U^*(A|U). \tag{7}$$

To prove (7), observe that at a point (n, g, u) in φ/U as in (3), the stalk of $\pi_U^*\varphi^*(A)$ is A_{x_0} and that of $\varepsilon_U^*(A|U)$ is A_z; these are isomorphic by the action $A_z \tilde{\to} A_{x_0}$ of the arrow $u : x_0 \to z$. Thus, (6) and (7) together give

$$R^q\varphi_*(\varphi^*A)_x = \varinjlim_{x \in U} H^q(\varphi/U, \varepsilon_U^*(A|U)). \tag{8}$$

$$\nu(z) = (0, z \xleftarrow{1} x_0)\ ,\ z \in U,\ \ x_0 := z. \tag{9}$$

Clearly $\varepsilon_U \circ \nu$ is the identity. On the other hand, the composition $\nu \circ \varepsilon_U$ is homotopic to the identity by a homotopy *over* U, as shown by the continuous natural transformations whose components for any object (3) of φ/U are pictured by the chain

$$(n, z \xleftarrow{u} x_0 \leftarrow \cdots \leftarrow x_n) \Leftarrow (0, z \xleftarrow{u} x_0) \Rightarrow (1, z \xleftarrow{1} z \xleftarrow{u} x_0) \Leftarrow (0, z \xleftarrow{1} z),$$

connecting $(n, z \leftarrow x_0 \leftarrow \cdots \leftarrow x_n)$ to $\nu\varepsilon_U(n, z \leftarrow x_0 \leftarrow \cdots \leftarrow x_n) = (0, z \xleftarrow{1} z)$. By homotopy invariance (Proposition A2 in the Appendix), ε_U^* induces an isomorphism

$$H^q(\varphi/U, \varepsilon_U^*(A|U)) \cong H^q(U, A|U). \tag{10}$$

But $\varinjlim_{x \in U} H^q(U, A|U)$ is the derived functor of the exact functor which takes the stalk at x, hence vanishes for $q > 0$. Thus (8) and (10) together yield $R^q\varphi_*(\varphi^*A) = 0$ for $q > 0$. The proof is now completed by the Leray spectral sequence of Proposition A1.

Appendix. In this appendix I will outline the basic functoriality properties of the sheaf cohomology $H^*(G, A)$ of étale topological groupoids (or topological categories) – in particular, the Leray spectral sequence and the homotopy invariance used in the proof of Lemma 3.1.

Let G and K be fixed topological categories. In the relevant example, G is an étale groupoid and K is the (étale) topological category *simp*(G). Let $\varphi : K \to G$ be a continuous functor. Then φ induces by pullback a functor

$$\varphi^* : (G\text{-sheaves}) \to (K\text{-sheaves}).$$

This functor always has a right adjoint $\varphi_* : (K\text{-sheaves}) \to (G\text{-sheaves})$. At least if the source map of G is étale, $\varphi_*(A)$ can be described explicitly for any K-sheaf A, as the associated sheaf of the presheaf

$$U \mapsto \Gamma_{inv}(\varphi/U, \pi_U^*(A)) \tag{1}$$

on G_0. Here U ranges over all open subsets of G_0, and φ/U is the topological category whose objects are arrows $(y, g : \varphi(y) \to x)$ with $y \in K_0, g \in G_1, x \in U$; there are arrows $(y, g : \varphi(y) \to x) \to (y', g' : \varphi(y') \to x')$ in φ/U only if $x = x'$, and in that case they are the arrows $h : y \to y'$ in K so that $g' \circ \varphi(k) = g$. This category φ/U has a natural topology; in fact, $(\varphi/U)_0 = K_0 \times_{G_0} G_1$ and $(\varphi/U)_1$ is constructed similarly as a fibered product. There is an evident projection functor

$$\pi_U : \varphi/U \to K.$$

This defines all the terms in (1). The associated sheaf of this presheaf (1) carries the natural structure of a G-sheaf.

A1. Proposition. *There is a natural Leray spectral sequence*

$$E_2^{p,q} = H^p(K, R^q\varphi_*(A)) \Rightarrow H^{p+q}(G, A). \tag{2}$$

If the source map of G is étale, then $R^q\varphi_*(A)$ is the associated sheaf of the presheaf

$$U \mapsto H^q(\varphi/U, \pi_U^*(A)).$$

(This associated sheaf has the natural structure of a G-sheaf.)

Proof. This spectral sequence is a special case of the Leray spectral sequence of [SGA4, vol.

Next, we turn to homotopy invariance. Let I denote the unit interval [0,1]. If K is a topological category, $K \times I$ will denote the topological category with $K_0 \times I$ as space of objects and $K_1 \times I$ as space of arrows. If $\varphi, \psi : K \to L$ are two continuous functors between topological categories, a *homotopy* between φ and ψ is a continuous functor $\eta : K \times I \to L$ with $\eta(-,0) = \varphi$ and $\eta(-,1) = \psi$. Now suppose K and L are equipped with continuous functors $\alpha : K \to G$ and $\beta : L \to G$ into a "base" category G, and $\beta\varphi = \alpha = \beta\psi$. The homotopy η is said to be *over* G if $\beta\eta = \alpha\pi$ (where $\pi : K \times I \to K$ is the projection). Such a homotopy η induces an exact functor

$$\eta^* : (L\text{-sheaves}) \to (K \times I\text{-sheaves}), \tag{3}$$

and $\eta^*\beta^* = \pi^*\alpha^*$ when the homotopy is over G, as above.

Now suppose, instead, that η is a continuous natural transformation from φ to ψ. Then η does not directly induce a homotopy $K \times I \to L$, but it does induce an exact functor η^* as in (3), defined for any K-sheaf A by $\eta^*(A)_{(x,t)} = \varphi^*(A)_x = A_{\varphi(x)}$, for $0 \le t < 1$ and $\eta^*(A)_{(x,t)} = \psi^*(A)_x = A_{\psi(x)}$ for $t = 1$. The stalk $A_{\psi(x)}$ at $(x,1)$ is glued to the stalks $A_{\varphi(x)}$ at (x,t) for $t < 1$ by means of the map $A_{\psi(x)} \to A_{\varphi(x)}$ given by the action of the arrow $\eta(x) : \varphi(x) \to \psi(x)$ on A. For this reason, we will refer to such a transformation $\eta : \varphi \to \psi$ as a *categorical homotopy* between φ and ψ.

If φ and ψ are functors over a base category G as above, we say that η is a categorical homotopy *over* G if $\beta(\eta(x) : \varphi(x) \to \psi(x))$ is the identity arrow at $\alpha(x)$, for any $x \in G_0$. In this case, the induced functor η^* again satisfies $\eta^*\beta^* = \pi^*\alpha^*$.

The following proposition and its proof apply equally to homotopies and to categorical homotopies.

A2. Proposition. *(Homotopy invariance) Let η be a (categorical) homotopy between φ and ψ over the base G, as above. Then for any G-sheaf A,*

$$\varphi^* = \psi^* : H^*(L, \beta^*(A)) \to H^*(K, \alpha^*(A)).$$

Proof. Again, this is a special case of homotopy invariance for topos cohomology; but it follows relatively easily from Proposition A1 in case the source map of K is étale, and this case suffices for the present applications. In detail, let $\pi : K \times I \to K$ be the projection and let B be any K-sheaf. The main thing is to show that π induces an isomorphism

$$\pi^* : H^*(K, B) \tilde{\to} H^*(K \times I, \pi^* B). \tag{4}$$

To see that this is the case, note that for any open $U \subseteq K_0$, and any $(K \times I)$-sheaf F, one has $\pi/U = (id/U) \times I$ where $id : K \to K$ is the identity functor, and hence $\Gamma_{inv}(\pi/U, F) \cong \Gamma(U \times I, F)$. Thus $\pi_*(F)_x = \Gamma(I, F_x)$ where F_x is the sheaf on I obtained by restricting F to $\{x\} \times I$. By ordinary proper base change [Godement, p. 202; SGA4, vol. 2, p. 141], the derived functor of π_* is given by $(R^q\pi_*)(F)_x = H^q(I, F_x)$. In particular, $R^q\pi_*(\pi^*B) = 0$ for any K-sheaf B and any $q > 0$, whence the isomorphism (4).

The proposition now follows formally from (4), in the usual way. First, for the two inclusions $i_0, i_1 : K \rightrightarrows K \times I$ one has $\pi i_0 = \pi i_1$, hence $i_0^* = i_1^* : H^*(K \times I, \pi^*(B)) \to H^*(K, B)$ since π^* is an isomorphism. Thus, for $\varphi, \psi, \alpha, \beta, \eta$ and A as above,

$$\varphi^* = i_0^*\eta^* = i_1^*\eta^* = \psi^* :$$

$$H^*(L, \beta^*A) \xrightarrow{\eta^*} H^*(K \times I, \eta^*\beta^*(A)) = H^*(K \times I, \pi^*\alpha^*(A)) \underset{i_1^*}{\overset{i_0^*}{\rightrightarrows}} H^*(K, \alpha^*A).$$

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