On the Validity of the Degenerate Ginzburg-Landau Equation.

A. Shepeleva

Department of Mathematics
University of Utrecht, P.O. Box 80.010
3508 TA Utrecht, The Netherlands

Abstract

The Ginzburg–Landau equation which describes nonlinear modulation of the amplitude of the basic pattern does not give a good approximation when the Landau constant (which describes the influence of the nonlinearity) is small. In this paper a derivation of the so–called degenerate (or generalized) Ginzburg–Landau (dGL) equation is given. It turns out that one can understand the dGL–equation as an example of a normal form of a co–dimension two bifurcation for parabolic PDEs. The main body of the paper is devoted to the proof of the validity of the dGL as an equation whose solution approximate the solution of the original problem.

1 Introduction

In many experimental situations in fluid dynamics [4], [36], [9], [35], chemical reactions, combustion and population dynamics [22], [2], [16] it was observed that when the control parameter R lies close to its critical value R_c (i.e. when the basic state loses its stability), a state with a periodic or quasiperiodic structure appears.
The dynamics of the bifurcation are described by the following amplitude (Landau) equation:

$$\frac{\partial U_1}{\partial t} = \mu(k_c)U_1 + \beta(k_c)U_1|U_1|^2 + \gamma(k_c)U_1|U_1|^4 + \dots \tag{1}$$

This equation is obtained by imposing in the original system the periodicity condition $U(x,t) = \sum_{n=-\infty}^{\infty} U_n(t)\exp(ink_cx)$, using the fact that U_1 is the most unstable mode ($\mathrm{Re}[\mu(k_c)]$ is small; $\mathrm{Re}[\mu(k_c)] > \mathrm{Re}[\mu(k)]$) and applying e.g. the center manifold reduction. The coefficients of (1) can be explicitly found in any particular problem under consideration (see for instance [4]). In the classical (supercritical) case one assumes the real part of $\mu(k)$ to be positive and of order ϵ^2 around the critical wave number k_c ($\mu(k_c) = \check{\mu}_1\epsilon^2 + i\omega_0$, $\epsilon \ll 1$) and the so–called Landau constant $\beta(k_c)$ to be negative, of order one. Then in the original equation using the natural scaling and slightly varying the amplitude

$$U = U_b + \sum_{m\neq 0} \epsilon^{|m|} U_m(\epsilon x, \epsilon^2 t) e^{im(k_c x + \omega_0 t)} + \epsilon^2 U_0(\epsilon x, \epsilon^2 t), \tag{2}$$

one finds the Ginzburg-Landau equation[1]

$$\frac{\partial U_1}{\partial T} = \check{\mu}_1 U_1 + \alpha \frac{\partial^2 U_1}{\partial X^2} + \beta(k_c) U_1 |U_1|^2 + \dots , \tag{3}$$

where the coefficients are in general complex, U_b is the basic state; X and T are long space and time variables ($T = \epsilon^2 t$, $X = \epsilon x$).

This equation is well–known [13], [25], [33] and its solutions have been extensively studied (see [7] and references therein). Much work has been done on the question of the validity of (3) in one– and multidimensional situations [5], [17], [28], [29], [30], [1], [32]. Equation (3) can be viewed as the normal form for a co–dimension one bifurcation for parabolic PDEs.

From (1) it is easy to see that when $Re[\beta]$ is small, higher order terms should be taken into consideration [14]. This situation is not exceptional; evidence of this kind was found in [26] — for the Plane Poiseuille flow, in [12] — for the Jeffrey–Hamel flow in a divergence channel, in [6] — for the tree–dimensional Poiseuille flow, in [34] — in the Blasius boundary–layer flow, in [23], [4] — for the Taylor–Couette flow between counter–rotating cylinders, in [21] — for the double diffusive convection. For more examples see [14] and [15].

Varying a second control parameter ν (the angle of divergence of a channel in the Jeffrey–Hamel flow or the Prandtl number in the Poiseuille flow) in the system one reaches the situation when zeros of $Re[\beta(k)]$ lie as close to the critical wave number k_c as one would wish.

In [15] Fujimura and Kelly investigated stably stratified plane Poiseuille flow in a horizontal channel of infinite extent in both horizontal directions. It is remarkable that in this case the sub–critical bifurcation persists for most values of the Prandtl number and by changing the Rayleigh number one can get even higher order degenerations.

It is clear from (1) that in order to get a full bifurcation picture when the Landau constant is small ($Re[\beta] = \check{\beta}_0\delta^2$, where δ is a small parameter and $\check{\beta}_0$ is a constant) one has to take ϵ to be of order δ^2. In this paper the case of real coefficients will

[1] Here we suppose that a transformation $x \to x + \epsilon\omega_1 t$ was made, with $\omega_1 = \frac{d}{dk} Im[\mu(k)]|_{k=k_c}$.

be considered which arises when the original system is invariant under the reflection ($x \to -x$). This is for example the case in the reaction–diffusion systems. Using natural scaling in this kind of systems and varying the amplitude

$$U = U_b + \sum_{m\neq 0} \delta^{|m|} U_m(\delta^2 x, \delta^4 t) e^{imk_c x} + \delta^2 U_0(\delta^2 x, \delta^4 t), \quad (4)$$

one gets

$$\frac{\partial U_1}{\partial T} = \check{\mu}_1 U_1 + \mu_1'' \frac{\partial^2 U_1}{\partial X^2} + \check{\beta}_0 |U_1|^2 U_1 + i\,(\check{\beta}_1 U_1 \frac{\partial |U_1|^2}{\partial X} + \check{\beta}_2 U_{-1} \frac{\partial U_1^2}{\partial X}) + \gamma_1 |U_1|^4 U_1. \quad (5)$$

This equation was first proposed in [14] and was called the degenerate Ginsburg–Landau equation (dGL). A lot of work was done to study the solutions of the equations of this type. Periodic solutions and their stability in sub–critical ($Re[\check{\mu}_0] < 0$, $Re[\check{\beta}_0] > 0$) and super–critical ($Re[\check{\mu}_0] > 0$, $Re[\check{\beta}_0] < 0$) cases were considered in [14]; periodic and quasiperiodic solutions in the sub–critical case and $\check{\beta}_0 = 0$ in [8]; singular heteroclinic orbits — in [19]; plane waves and their stability — in [18]. There is a number of papers investigating the solutions of (5) without gradient terms (i.e. when $\check{\beta}_1 = \check{\beta}_2 = 0$), see for instance [3] and review paper of Saarloos and Hohenberg [27] and references therein.

The aim of this paper is to prove the validity of this kind of equations in the case of real coefficients. Let us note that the case when the coefficients are not real brings out additional difficulties involving for instance different time scaling for the phase and the amplitude (see [14]).

Remark. One can skip this difficulty and automatically extend our validity result by weakening the condition for system being reflection symmetric to demanding both: real and imaginary parts of the Landau constant to be of order δ^2 which can be achieved by e.g. varying some extra parameter in the system.

In the second section of the paper the derivation of the dGL–equation in the Fourier space will be given and the nature of all, including the "hidden" [14] coefficients $\check{\beta}_1$, $\check{\beta}_2$ in front of the gradient terms, will be demonstrated. In the third section the functional spaces to work in will be discussed. Next the existence and uniqueness of the solution for the Fourier transformed version of the dGL–equation will be shown. The question of global existence of solutions of the Cauchy problem for the equation of the type (5) were considered in [11], [10].

Finally the main theorem will be formulated which states that if the solution of the Fourier transformed dGL exists until the time T_0, then there exists a unique solution of the Fourier transformed version of the original problem and it is well approximated by the solution of the Fourier transformed dGL in the chosen norm which immediately can be reduced to the long–term validity result in the physical space.

To prove the validity the fixed point argument for the map defined by the equation for the error will be used, which is a standard method in dealing with this kind of problems [17].

Let us mention some additional complications on the way:

- The absence of the spectral gap due to the continuity of the spectrum. To deal with it we will use the technique developed in [1], [28], [29].
- The mere distinction between critical and non–critical modes as was done in the proofs for the validity of the GL equation (see for example [28], [1]), is not good enough anymore.
- The order of the arising time–dependent linear terms must be refined.
- The more complicated structure of the nonlinearity in the error equation must be taken into consideration.

2 Derivation of the amplitude equation.

We will study here the solutions of the class of nonlinear PDEs in one space variable $x \in (-\infty, \infty)$, $t \geq 0$ which are of the form

$$\frac{\partial U}{\partial t} = L(U) + N(U) \tag{6}$$

where L is a real linear elliptic differential operator of order $2d$ in x, with constant coefficients containing some control parameters R and ν. $N(U)$ are quadratic nonlinear terms. They are of the structure

$$N(U) = P(U^2) \tag{7}$$

where P is again an real linear differential operator of order ℓ in x, with constant coefficients. This choice of nonlinear terms is made to demonstrate the essence of the phenomena and to avoid some computational complications. The generalization of $N(U)$ to an operator of the form

$$\sum_{j=1}^{m}(P_1^{(j)}U)\,(P_2^{(j)}U)\;+\;(Q_1^{(j)}U)\,(Q_2^{(j)}U)\,(Q_3^{(j)}U). \tag{8}$$

with $P_k^{(j)}$, $Q_k^{(j)}$ liner operators, can be easily done. The only requirement made here is that the symbols of linear operators are polynomial in k with real coefficients smoothly dependent on control parameters R and ν, and the natural requirement that the symbol of the linear operator is of higher order polynomial than the multiplication of symbols of operators P_j and Q_j in the nonlinear part. Let U_b be a basic state of (6) (We can take $U_b = 0$). Linearization around this state gives in the (k, R)–plane a

neutral parabola–like stability curve $\mu(k,R)=0$ with a minimum in (k_c,R_c). We are interested in what is going on around this minimum $R=R_c\pm\epsilon^2$, i.e. in the case of super– or of sub–critical bifurcations.
We will work with the Fourier–transformed version of (6)

$$\frac{\partial\Phi}{\partial t}=\mu(k)\ \Phi+\rho(k)\ \Phi*\Phi, \tag{9}$$

where $\Phi(t,k)$ denotes the Fourier–transform of $U(x,t):\ \Phi=\mathcal{F}(U)$, and $\mu(k)$, $\rho(k)$ are symbols of the operators L and P respectively.

As mentioned in the introduction an appropriate approximation in physical space is of the form

$$U_{GL}\simeq\sum_{m\neq 0}\delta^{|m|}\ U_m(\delta^2x,\delta^4t)\ e^{imk_cx}\ +\ \delta^2\ U_0(\delta^2x,\delta^4t), \tag{10}$$

and correspondingly in Fourier space:

$$\Phi_{GL}\simeq\sum_{m\neq 0}\delta^{|m|}\ A_m(K_m,T)\ +\ \delta^2A_0(K_0,T), \tag{11}$$

where $A_m(K_m,T)=\frac{1}{\delta^2}a_m(K_m,T)$, $T=\delta^4t$, $K_m=\frac{k-mk_c}{\delta^2}$ and $\delta\ll 1$ or

$$\Phi_{GL}\simeq\sum_{m\neq 0}\delta^{|m|}\ \left[\frac{1}{\delta^2}S^m a_m(K,T)\right]+\ a_0(K,T) \tag{12}$$

with S a unit shift operator and K is defined by k/δ^2. $A_m(K,T)$ can be expanded as $\sum_{j=0}^{\infty}A_m^j(K,T)\delta^{2j}$.
Let us define $f_j(k)=-\frac{\rho_j(k)}{\mu_j(k)}$ where $\rho_j(k)$, $\mu_j(k)$ are Taylor expansions of corresponding symbols around the point $j\cdot k_c$:

$$\begin{array}{rcl}\rho_j(k) &=& \rho_j\ +\ \rho_j'K_j\delta^2\ +\ \rho_j''K_j^2\delta^4+\ldots\\ \mu_j(k) &=& \mu_j\ +\ \mu_j'K_j\delta^2\ +\ \mu_j''K_j^2\delta^4+\ldots\\ f_j(k) &=& f_j\ +\ f_j'K_j\delta^2\ +\ f_j''K_j^2\delta^4+\ldots\end{array} \tag{13}$$

with μ_j, μ_j', μ_j'', ρ_j, ρ_j', ρ_j'', f_j, f_j' and f_j'' constants. Substitution of the formal approximation (11) into (9) provides us with the following relations for the first four amplitudes :

$$\begin{array}{rcl}A_0 &=& f_0(k)\,[2A_1*A_{-1}\ +\ \delta^2\ (A_0*A_0\ +\ 2\ A_{-2}*A_2)]\ +\delta^4\ Res_0\\ \frac{\partial A_1}{\partial T} &=& \frac{\mu_1(k)}{\delta^4}A_1\ +\ \frac{2\rho_1(k)}{\delta^4}\,[A_1*A_0\ +\ A_2*A_{-1}\ +\ \delta^2\ A_3*A_{-2}]+Res_1\\ A_2 &=& f_2(k)\,[A_1*A_1\ +\ \delta^2\ (2\ A_0*A_2\ +\ 2\ A_{-1}*A_3)]\ +\delta^4\ Res_2\\ A_3 &=& 2\ f_3(k)\ A_1*A_2\ +\ \delta^2\ Res_3\end{array} \tag{14}$$

with $A_{-m}(K,T)=\overline{A}_m(-K,T)$; Res_j are of order one and express the corresponding higher order terms. One can easily write down the equations for A_{-1}, A_{-2}, A_{-3} simply by changing signs of the indexes. Through this paper $*$ should be understood as a convolution in the natural variable domain, i.e. K in the last formula. Expanding (14) and expressing everything in terms of the critical amplitude $A_{\pm 1}$ we get :

$$\begin{aligned} A_0 &= 2f_0 A_1 * A_{-1} \;+\; 2\delta^2\,[f_0' K_0 A_1 * A_{-1} \;+\; (f_0 f_2 f_{-2} + 2f_0^3) A_1 * A_1 * A_{-1} * A_{-1}] \\ \tfrac{\partial A_1}{\partial T} &= \tfrac{\mu_1(k)}{\delta^4} A_1 \;+\; 2\rho_1 \delta^{-2} (f_2 + 2f_0) A_1 * A_1 * A_{-1} \;+\; 2\,[2\rho_1 f_0' A_1 * (K_0 A_1 * A_{-1}) \\ &+\; \rho_1 f_2' A_{-1} * (K_2 A_1 * A_1) \;+\; 2\rho_1'\,(2f_0 + f_2)\,K_1 * (A_1 * A_1 * A_{-1})] \\ &+\; 4\rho_1\,(f_0 f_2 f_{-2} + f_3 f_2 f_{-2} + 2f_0^3 + 2f_0 f_2^2 + 2f_3 f_2^2)\,A_1 * A_1 * A_1 * A_{-1} * A_{-1} \\ A_2 &= f_2 A_1 * A_1 \;+\; \delta^2\,[f_2' K_2 A_1 * A_1 \;+\; 4(f_0 f_2^2 + f_2^2 f_3) A_1 * A_1 * A_1 * A_{-1})] \\ A_3 &= 2\;f_3 f_2\;A_1 * A_1 * A_1 \end{aligned} \tag{15}$$

From (15) one sees that all non–critical modes are slaved to the critical one $A_{\pm 1}$. The equation for the critical mode shows that the condition for the Landau constant $\beta = 2\rho_1(2f_0 + f_2)$ to be small can be expressed as $2f_0 + f_2 = \check{\beta}\,\delta^2$ with $\delta \ll 1$ and $\check{\beta}$ an order one constant.
The expansion of $\mu_1(k)$ near k_c is $\mu_1(k) = \delta^4\left(\frac{\epsilon^2}{\delta^4}\check{\mu}_1 - K_1^2\mu_1'' + ...\right)$, with $\mu_1 = \epsilon^2\check{\mu}_1$ from (13) and $\check{\mu}_1$ an order one constant. From the equations for $A_{\pm 1}$ it follows that in order to get a full bifurcation picture one has to take ϵ of order δ^2. Then with the natural scaling of the time $T = \delta^4 t$ one obtains

$$\begin{aligned} \tfrac{\partial A_1}{\partial T} &= \check{\mu}_1 A_1 - \mu_1'' {K_1}^2 A_1 + 2\rho_1 \left\{\check{\beta} A_1 * A_1 * A_{-1} + 2f_0' A_1 * (K_0 A_1 * A_{-1}) \right. \\ &+ \left. f_2' A_{-1} * (K_2 A_1 * A_1)\right\} + \gamma_1 A_1 * A_1 * A_1 * A_{-1} * A_{-1}, \end{aligned} \tag{16}$$

with $\gamma_1 = 4\rho_1\,(f_{-2} f_2 f_0 + 2f_0^3 + f_{-2} f_2 f_3 + 2f_0 f_2^2 + 2f_3 f_2^2)$.

In the physical space it corresponds to the so–called degenerate Ginzburg–Landau equation (dGL) [14], [8] of the form:

$$\frac{\partial U_1}{\partial T} = \check{\mu}_1 U_1 + \mu_1'' \frac{\partial^2 U_1}{\partial X^2} + 2\rho_1 \left\{ \check{\beta}|U_1|^2 U_1 - 2f_0' i U_1 \frac{\partial |U_1|^2}{\partial X} - f_2' i U_{-1} \frac{\partial U_1^2}{\partial X} \right\} + \gamma_1 |U_1|^4 U_1 \tag{17}$$

The coefficients in front of the nonlinear gradient terms can be called "hidden" coefficients in a sense that the knowledge of all coefficients of the Landau amplitude equation does not permit to define all coefficients of the modulation equation through the algorithmic derivation [14]. It becomes obvious now what kind of properties one has to ask from the operators L and N in order to get a certain type of degenerate equation. For example the equation without gradient terms, whose solutions were widely studied in [27] corresponds to (in some sense) symmetry conditions $f_0' = f_2' = 0$.

Remark. Let us note that using the smallness of the Landau constant we could write A_1^0 instead of A_1 not carrying out the expansion $\sum_{j=0}^{\infty} A_m^j(K,T)\delta^{2j}$ from the very beginning. This can be easily checked by straightforward calculations substituting $\sum_{j=0}^{\infty} A_m^j(K,T)\delta^{2j}$ in (16) and using the fact that the Landau constant is of order δ^2. From now on we will write A_1 instead of A_1^0.

3 Functional spaces.

The choice of the functional spaces to work in plays an important role in the proof of the validity of the GL–type equations. They should be easy to handle and be large enough to contain all interesting solutions such as periodic and quasiperiodic solutions found in [14], [8], [18], pulse and kink like solutions found in [27], [24]. The earlier results for the validity of GL–type equations show the development in the conditions for the dimension of the problem and in the order of estimates as well as in the choice of the functional spaces. See for instance [5], [17], [20], [28], [30], [29], [32], [1], [31].

Let us introduce a space of analytic functions, bounded in a strip S_α with $S_\alpha = \{z \in \mathbb{C} : |Imz| \leq \alpha\}$, which was used in [17], [1]

$$\begin{array}{rcl} \mathbf{H}_\alpha & = & \{\Phi \in C\,(S_\alpha \to \mathbb{C}) : \Phi \text{ is analytic in } S\} \\ \mathbf{B}_\alpha & = & \{U \in S'(\mathbb{R}) : \exists \Phi \in \mathbf{H}_\alpha \text{such that } U = \mathcal{F}\Phi|_{y=0}\} \end{array} \tag{18}$$

$||\Phi||_{\mathbf{H}_\alpha} := \sup_{S_\alpha}|\Phi| < \infty$; $\mathbf{B}_\alpha = \mathcal{F}\mathbf{H}_\alpha$ is a Fourier image with the natural norm $||\cdot||_{\mathbf{B}_\alpha} =: ||\mathcal{F}^{-1}\cdot||_{\mathbf{H}_\alpha}$. The requirement of analyticity is not a strong restriction, as it was shown in [37] that all solutions of semi–linear parabolic systems with essentially bounded initial conditions become analytic in the strip with width $\vartheta(t)$ which for $t < t_0$ grows proportionally to t^{2d} (with $2d$ the degree of the liner operator) and then remains bounded. In this paper we will work with these spaces as far as they satisfy both of our two requirements : they are convenient to work with and they are big enough to include the most interesting solutions (see for instance [17]).

Let us define so–called mode filters [1], [28] as follows. Let ξ be a smooth cut function:

$$\xi(k) = \begin{cases} 1 & \text{if } 0 \leq k \leq \frac{1}{m}|k_c| \\ 0 & \text{if } k > \frac{1}{m-1}|k_c| \\ \in \mathbb{C}^\infty\,(\mathbb{R} \to [0,1]) & \text{elsewhere} \end{cases} \tag{19}$$

with some fixed integer $m \geq 4$. Using $\xi(k)$ one can extract important modes and in some sense localize the problem of validity by introducing the following filters:

$$\begin{array}{rcl} \mathcal{P}_0 & = & \xi(k) \\ \mathcal{P}_1 & = & \xi(k-k_c) + \xi(k+k_c) \\ \mathcal{P}_2 & = & \xi(k-2k_c) + \xi(k+2k_c) \\ \mathcal{P}_3 & = & \mathrm{Id} - \mathcal{P}_1 - \mathcal{P}_2 - \mathcal{P}_0 \end{array} \tag{20}$$

It is obvious that it is not a projection and consequently the decomposition is not unique. We will use the following properties.

Lemma 1 *Let $f \in \mathbf{B}_\alpha$, $k_0 \in \mathbb{R}$, $\epsilon \ll 1$ and $f_\epsilon(k) = \frac{1}{\epsilon}f((k-k_0)/\epsilon)$, then $f_\epsilon \in \mathbf{B}_{\alpha/\epsilon} \subset \mathbf{B}_\alpha$ and $||f_\epsilon||_{\mathbf{B}_\alpha} \leq e^{\alpha|k_0|}||f||_{\mathbf{B}_\alpha}$.*

Lemma 2 *Let* $f \in \mathbf{B}_\alpha$, $k_0 \in \mathbb{R}$, $\epsilon \ll 1$ *and* $f_\epsilon(k) = \frac{1}{\epsilon} f((k-k_0)/\epsilon)$; $\xi \in C^\infty[\mathbb{R} \to [0,1]]$ *with* $supp(\xi) \subset \{|k| < r\}$. *Then* $\exists C(\xi) : ||\xi f_\epsilon||_{\mathbf{B}_\alpha} \leq \frac{C}{\epsilon} e^{(2r-|k_0|)\alpha/\epsilon} ||f_\epsilon||_{\mathbf{B}_\alpha}$.

Lemma 3 *Let* $f \in \mathbf{B}_\alpha$, $k_0 \in \mathbb{R}$, $\epsilon \ll 1$ *and* $\xi = 1$ *if* $|k| < r_0$, $r/2 < r_0 < r$ *and* $g_\epsilon = \frac{1}{\epsilon} f(k/\epsilon)$, *then* $||(1-\xi)g_\epsilon||_{\mathbf{B}_\alpha} < \frac{C}{\epsilon} e^{(r-2r_0)\alpha/\epsilon} ||g_\epsilon||_{\mathbf{B}_\alpha}$.

Let us define $\mu_j(k)$ as the following C^∞ function: it is equal to $\mathcal{P}_j\mu(k)$ on the intervals where $\mathcal{P}_j\mu(k) = \mu(k)$; there exists a constant $\tilde{\mu}_j > 0$ such that $\mu_j(k) \leq -\tilde{\mu}_j$ outside these intervals and for $j = 0; 1; 2$ it decays faster than any polynomial at infinity. The notation $\rho_i(k)$ will be used simply for $\mathcal{P}_j\rho(k)$. Note that $f_j(k) = -\rho_j(k)/\mu_j(k)$ is defined correctly as far as we are not going to use it for $j = 1$.

Lemma 4 *Let* $f \in \mathbf{B}_\alpha$. *Then there exists constants* $\tilde{\mu}_j$ *and* c *such that*

$$||e^{\mu_j(k)t}\mathcal{P}_j f||_{\mathbf{B}_\alpha} \leq \begin{cases} ce^{-\tilde{\mu}_j t}||\mathcal{P}_j f||_{\mathbf{B}_\alpha} & j \neq 1 \\ ce^{\delta^4 \tilde{\mu}_j t}||\mathcal{P}_j f||_{\mathbf{B}_\alpha} & j = 1 \end{cases} \tag{21}$$

$$||e^{\mu_j(k)t}\rho_j\mathcal{P}_j f||_{\mathbf{B}_\alpha} \leq \begin{cases} ce^{-\tilde{\mu}_j t}||\mathcal{P}_j f||_{\mathbf{B}_\alpha} & j = 0,2 \\ ce^{\delta^4 \tilde{\mu}_j t}||\mathcal{P}_j f||_{\mathbf{B}_\alpha} & j = 1 \\ ce^{-\tilde{\mu}_j t}t^{-\frac{\ell}{2d}}||\mathcal{P}_j f||_{\mathbf{B}_\alpha} & j = 3 \end{cases} \tag{22}$$

The proofs are straightforward. For details one can consult [1], [17].
Let us define the functional spaces $\mathbf{B}_1 = \mathcal{P}_1\mathbf{B}_\alpha$, $\mathbf{B}_{nc} = \mathcal{P}_{nc}\mathbf{B}_\alpha$, $\mathbf{B} = \mathbf{B}_1 \otimes \mathbf{B}_{nc}$ with natural norm

$$||(w_1, w_{\rm nc})||_{\mathbf{B}} = ||w_1||_{\mathbf{B}_\alpha} + ||w_{\rm nc}||_{\mathbf{B}_\alpha} \tag{23}$$

where the index "nc" is used for the noncritical modes (0,2,3).

4 Existence of solutions for the $\mathcal{F}$dGL equation.

In order to prove the validity for the dGL equation we have to answer the question about solvability of the $\mathcal{F}$dGL equation, i.e. for (16) with initial conditions $A_1|_{t=0} = \tilde{A}_1$ from $\mathbf{B}_\alpha$ and consequently for (15) with the rest of the initial conditions for non–critical modes naturally produced by the system (15) (for instance $A_0|_{t=0} = 2f_0\tilde{A}_1 * \tilde{A}_1 + 2\delta^2\left[f_0'K_0\tilde{A}_1 * \tilde{A}_{-1} + (f_0f_2f_{-2} + 2f_0^3)\tilde{A}_1 * \tilde{A}_1 * \tilde{A}_{-1} * \tilde{A}_{-1}\right]$). Let us formulate

Theorem 1 *The Fourier transformed version of the dGL equation (16) with the initial conditions* $\tilde{A}_{\pm 1}$ *in* $\mathbf{B}_\alpha$, *has a unique solution in* $C([0,T_0] \to \mathbf{B}_\alpha)$ *for some* $T_0 > 0$

Proof of this statement can be given in a standard way, see e.g. [17], by rewriting (16) as an integral equation and applying the contracting mapping argument in $B = \{A_j | A_i \in C([0, T_0] \to \mathbf{B}_\alpha); j = \pm 1\}$.

As we already mentioned in the introduction the question of the global existence and uniqueness of the solutions for the dGL were considered for the Cauchy problem with periodic boundary conditions in [11] and for the Cauchy problem on the real line in [10] by applying the semi–group theory in the original physical space under suitable conditions. Let us note that for the validity result one does not need the global existence of solutions for which some special requirements for the coefficients of (66) were made in [11] and [10]. The validity statement only assures us that for the time interval in which the solution of (66) exists it will give a good approximation to the solution of the original problem.

5 The validity and the error estimates.

In this section the main theorem will be formulated and proved:

Theorem 2 *Let the system (15) have an unique solution for $T \in (0, T_0)$, $T_0 > 0$. Let*

$$\Phi_{GL} \simeq \sum_{m \neq 0, |m| \leq 3} \delta^{|m|} A_m(K_m, T) + \delta^2 A_0(K_0, T) \tag{24}$$

be our approximation, then there exist $C > 0$ and a unique solution $\Phi \in \mathbf{B}_\alpha$ of the Fourier–transformed original problem (9) with the same initial conditions $\Phi|_{t=0} = \Phi_{GL}|_{t=0}$ such that

$$sup_{t \leq \frac{T_0}{\delta^4}} ||\Phi - \Phi_{GL}||_{\mathbf{B}_\alpha} \leq C\delta^2. \tag{25}$$

As an obvious **corollary** of this theorem one gets the validity result for the equation in the physical space with the following estimate

$$\sup_{t \leq \frac{T_0}{\delta^4}} \sup_{\mathbb{R}} |U - U_{GL}| \leq C\delta^2. \tag{26}$$

Let us give the sketch of the **proof**. First we will write down the error equation induced by Φ_{GL}. We look for a solution of the form

$$\Phi = \Phi_{GL} + \delta^2 W, \quad W|_{t=0} = 0, \tag{27}$$

where Φ_{GL} has a structure given by (24). Substituting this in (9) one gets

$$\frac{\partial W}{\partial t} = \mu(k)\, W + 2\rho(k)\, \Phi_{GL} * W + \delta^2 \rho(k) W * W + \frac{1}{\delta^2} R(\Phi_{GL}) \tag{28}$$

Using properties of the filters (lemmas 1,2,3), one deduces that $\mathcal{P}_1(\Phi_{GL}) = O(\delta)$, $\mathcal{P}_2(\Phi_{GL}) = O(\delta^2)$, $\mathcal{P}_3(\Phi_{GL}) = O(\delta^3)$ in the $\mathbf{B}_\alpha$ norm, which immediately gives us the structure of the error : $W = W_1 + \delta W_2 + \delta^2 W_3$ with $\mathcal{P}_1 W = W_1$, $\mathcal{P}_2 W = \delta W_2$, $\mathcal{P}_3 W = \delta^2 W_3$. Then one has the following system:

$$\begin{aligned}
\frac{\partial W_0}{\partial t} &= \mu_0(k)\, W_0 + 2\rho_0(k)\, \mathcal{L}_{-1,1} + \delta^2 2\rho_0(k)\mathcal{M}_0 + \delta^2\mathcal{R}_0;\\
\frac{\partial W_1}{\partial t} &= \delta^4\check{\mu}_1(k)\, W_1 + 2\rho_1(k)\, \delta^2(\mathcal{L}^{1,-2}_{-1,2} + \mathcal{L}^{0,-1}_{0,1}) + \delta^4 2\rho_0(k)\mathcal{M}_1 + \delta^4\mathcal{R}_1;\\
\frac{\partial W_2}{\partial t} &= \mu_2(k)\, W_2 + \rho_2(k)\, \mathcal{L}^{-1,-1}_{1,1} + \delta^2 2\rho_2(k)\mathcal{M}_2 + \delta^2\mathcal{R}_2;\\
\frac{\partial W_3}{\partial t} &= \mu_3(k)\, W_3 + 2\rho_3(k)\, \mathcal{L}^{-1,-2}_{1,2} + \delta^2 2\rho_3(k)\mathcal{M}_3 + \delta^2\mathcal{R}_3
\end{aligned} \tag{29}$$

with $W_j|_{t=0} = 0$.

$$W_0 = w_0; \quad W_j = \begin{pmatrix} w_{-j} \\ w_j \end{pmatrix} \quad for\ j \neq 0 \tag{30}$$

$$\mathcal{L}^{l,m}_{i,j} = \begin{pmatrix} A_l * w_m + A_m * w_l \\ A_i * w_j + A_j * w_i \end{pmatrix} \quad \mathcal{N}^{l,m}_{i,j} = \begin{pmatrix} w_l * w_m \\ w_i * w_j \end{pmatrix} \tag{31}$$

$$\begin{aligned}
\mathcal{L}_{i,j} &= A_i * w_j + A_j * w_i\\
\mathcal{N}_{i,j} &= w_i * w_j\\
\mathcal{M}_0 &= \mathcal{L}_{-2,2} + \tfrac{1}{2}\,\mathcal{L}_{0,0} + \mathcal{N}_{-1,1} + \tfrac{\rho_0'(k)}{\rho_0(k)}K_0\mathcal{L}_{-1,1}\\
\mathcal{M}_1 &= \mathcal{L}^{2,-3}_{-2,3} + \mathcal{N}^{0,-1}_{0,1} + \mathcal{N}^{1,-2}_{-1,2} + \tfrac{\rho_1'(k)}{\rho_1(k)}K_1(\mathcal{L}^{1,-2}_{-1,2} + \mathcal{L}^{0,-1}_{0,1})\\
\mathcal{M}_2 &= \mathcal{L}^{1,-3}_{-1,3} + \mathcal{L}^{0,-2}_{0,2} + \tfrac{1}{2}\,\mathcal{N}^{-1,-1}_{1,1} + \tfrac{\rho_2'(k)}{2\rho_2(k)}K_2\mathcal{L}^{-1,-1}_{1,1}\\
\mathcal{M}_3 &= \mathcal{L}^{0,-3}_{0,3} + \mathcal{N}^{-1,-2}_{1,2} + \tfrac{\rho_3'(k)}{\rho_3(k)}K_3\mathcal{L}^{-1,-2}_{1,2}
\end{aligned} \tag{32}$$

From lemmas (1,2,3) and (14), (15) one obtains

Lemma 5 *The inhomogeneous terms are of order one, i.e.*

$$sup_{t \leq \frac{T_0}{\delta^4}} ||\mathcal{R}_j||_{\mathbf{B}_\alpha} = O(1). \tag{33}$$

Rewriting system (29) in integral form using the variation of constant formula and the notations defined for lemma 4, one gets

$$\begin{aligned}
W_0 &= \textstyle\int_0^t E_0(t-s)\, \left\{2\rho_0(k)\, \mathcal{L}_{-1,1} + \delta^2 2\rho_0(k)\mathcal{M}_0 + \delta^2\mathcal{R}_0\right\}\, ds;\\
W_1 &= \textstyle\int_0^t E_1(t-s)\, \left\{2\rho_1(k)\, \delta^2(\mathcal{L}_{-2,1} + \mathcal{L}_{0,1}) + \delta^4 2\rho_0(k)\mathcal{M}_1 + \delta^4\mathcal{R}_1\right\}\, ds;\\
W_2 &= \textstyle\int_0^t E_2(t-s)\, \left\{\rho_2(k)\, \mathcal{L}_{1,1} + \delta^2 2\rho_2(k)\mathcal{M}_2 + \delta^2\mathcal{R}_2\right\}\, ds;\\
W_3 &= \textstyle\int_0^t E_3(t-s)\, \left\{2\rho_3(k)\, \mathcal{L}_{1,2} + \delta^2 2\rho_3(k)\mathcal{M}_0 + \delta^2\mathcal{R}_3\right\}\, ds.
\end{aligned} \tag{34}$$

We have to prove now that this system has a bounded solution on the long time scale T_0/δ^4 (for some $T_0 > 0$) when the solution of the equation (16), and consequently of the system (15), exists and is bounded. In order to do this we will use a fixed point argument for a contractive map on a ball in $C\left(\left[0, \frac{T_0}{\delta^4}\right] \to \mathbf{B}\right)$ with the center defined by the inhomogeneous terms $\mathcal{R} = (\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3)$ which are of order one as specified in lemma 5.

For the equations of the non–critical modes $W_{nc} = (W_0, W_2, W_3)$ it is clear from the form of (29) that if we suppose long–time boundedness for the critical modes then it becomes easy to show that the rest of the modes will also be bounded due to the fact that we have separated ourselves from the small (positive or negative) problematic part of the spectrum. The restrictions : $\mu(k)|_{\text{range}\mathcal{P}_0}$, $\mu(k)|_{\text{range}\mathcal{P}_2}$, $\mu(k)|_{\text{range}\mathcal{P}_3}$ are generators of exponentially damped semi–group (lemma 4) and consequently W_0, W_2, W_3 will be of order one over any time–scale. From the equation for the critical mode it is obvious that in order to get the $\frac{T_0}{\delta^4}$ boundedness result for the solutions in the second equation (29) one needs at least to get rid of the term $\mathcal{L} = (\mathcal{L}_{-2,1} + \mathcal{L}_{0,1})$. The smallness of the Landau constant is going to play a crucial role in the proof of the following result:

Lemma 6 *In the system (29) with the Landau constant of order δ^2 the term $\mathcal{L} = (\mathcal{L}_{-1,2} + \mathcal{L}_{0,1})$ is of order δ^2 i.e.*

$$||(\mathcal{L}_{-2,1} + \mathcal{L}_{0,1})||_{B_\alpha} = O(\delta^2). \tag{35}$$

Proof To make notations of the proof easier we will split $W_j = (w_{-j}, w_j)$ and consider the system with the positive indices:

$$\begin{aligned}
\frac{\partial w_0}{\partial t} &= \mu_0(k)\, w_0 + 2\rho_0(k)\, \mathcal{L}_{-1,1} + \delta^2 2\rho_0(k)\mathcal{M}_0 + \delta^2\mathcal{R}_0;\\
\frac{\partial w_1}{\partial t} &= \delta^4\check{\mu}_1(k)\, w_1 + 2\rho_1(k)\, \delta^2(\mathcal{L}_{-1,2} + \mathcal{L}_{0,1}) + \delta^4 2\rho_1(k)\mathcal{M}_1 + \delta^4\mathcal{R}_1;\\
\frac{\partial w_2}{\partial t} &= \mu_2(k)\, w_2 + \rho_2(k)\, \mathcal{L}_{1,1} + \delta^2 2\rho_2(k)\mathcal{M}_2 + \delta^2\mathcal{R}_2;\\
\frac{\partial w_3}{\partial t} &= \mu_3(k)\, w_3 + 2\rho_3(k)\, \mathcal{L}_{1,2} + \delta^2 2\rho_3(k)\mathcal{M}_3 + \delta^2\mathcal{R}_3.
\end{aligned} \tag{36}$$

To get the result for the complete system one has to repeat the same procedure for the part with negative sign. Let us rewrite the equations for the zero– and the second–mode as integral equations

$$\begin{aligned}
w_0 &= \int_0^t E_0(t-s)\left(2\rho_0(k)\,\left(\mathcal{L}_{-1,1} + \delta^2\mathcal{M}_0\right) + \delta^2\mathcal{R}_0\right)\, ds;\\
w_2 &= \int_0^t E_2(t-s)\,\left(\rho_2(k)\,\left(\mathcal{L}_{1,1} + 2\delta^2\mathcal{M}_2\right) + \delta^2\mathcal{R}_2\right)\, ds.
\end{aligned} \tag{37}$$

Integrating the first terms by part one gets

$$\begin{aligned}
w_0 &= 2f_0(k)\,\left(\mathcal{L}_{-1,1}\right) - 2\int_0^t E_0(t-s)\, f_0(k)\, [\mathcal{L}_{-1,1}]'\, ds\\
&+ \delta^2\int_0^t E_0(t-s)\mathcal{R}_0\, ds + 2\delta^2\int_0^t E_0(t-s)\,\rho_0(k)\,\mathcal{M}_0\, ds;\\
w_2 &= f_2(k)\,\mathcal{L}_{1,1} - \int_0^t E_2(t-s)\, f_2(k)\, [\mathcal{L}_{1,1}]'\, ds\\
&+ \delta^2\int_0^t E_2(t-s)\,\mathcal{R}_2\, ds + 2\delta^2\int_0^t E_2(t-s)\,\rho_2(k)\,\mathcal{M}_2\, ds.
\end{aligned} \tag{38}$$

Using (15) and (36) we have

$$\begin{aligned} \left[\mathcal{L}_{1,1}\right]' &= 4\rho_1(k)\delta^2 A_1 * \mathcal{L} \ + \ 2\delta^4 \left[w_1 * S_1 + \check{\mu}_1(k) A_1 * w_1 + 2\rho_1(k) A_1 * \mathcal{M}_1 + A_1 * \mathcal{R}_1\right] \\ \left[\mathcal{L}_{-1,1}\right]' &= 2\rho_1(k)\delta^2 \left(A_{-1} * \mathcal{L} + A_1 * \overline{\mathcal{L}}\right) \ + \ \delta^4 \left[(w_{-1} * S_1 + w_1 * S_{-1}) + \check{\mu}_1(k)\mathcal{L}_{-1,1}\right. \\ &+ \left. 2\rho_1(k)\left(A_1 * \mathcal{M}_{-1} + A_{-1} * \mathcal{M}_1\right) + \left(A_{-1} * \mathcal{R}_1 + A_1 * \mathcal{R}_{-1}\right)\right], \end{aligned} \tag{39}$$

where $\mathcal{S}_1$ denotes the right hand side of (16), $\mathcal{S}_{-1} = \overline{\mathcal{S}}_1$ and $\bar{\mathcal{L}} = \mathcal{L}_{-2,1} + \mathcal{L}_{-1,0}$. Let us write

$$\begin{aligned} \Delta_2 &= 2\left[w_1 * S_1 + \check{\mu}_1(k) A_1 * w_1 + 2\rho_1(k) A_1 * \mathcal{M}_1 + A_1 * \mathcal{R}_1\right] \\ \Delta_0 &= (w_{-1} * S_1 + w_1 * S_{-1}) + \check{\mu}_1(k)\mathcal{L}_{-1,1} \\ &+ 2\rho_1(k)\left(A_1 * \mathcal{M}_{-1} + A_{-1} * \mathcal{M}_1\right) + \left(A_{-1} * \mathcal{R}_1 + A_1 * \mathcal{R}_{-1}\right) \end{aligned} \tag{40}$$

We are now in a position to consider $\mathcal{L} = A_{-1} * w_2 + A_2 * w_{-1} + A_1 * w_0 + A_0 * w_1$ which can be written using (15), (36), (38), (39), (40), as follows

$$\begin{aligned} \mathcal{L} &= f_2(k) A_{-1} * \mathcal{L}_{1,1} \ + \ 2f_0(k) A_1 * \mathcal{L}_{-1,1} \ + \ f_2 A_1 * A_1 * w_{-1} \ + \ 2f_0 A_{-1} * A_1 * w_1 \\ &+ \delta^2 A_{-1} * \left[\int_0^t E_2(t-s)\left\{2\rho_2(k)\mathcal{M}_2 - f_2(k)(4\rho_1(k) A_1 * \mathcal{L} + \delta^2\Delta_2\right\} ds\right] \\ &+ 2\delta^2 A_1 * \left[\int_0^t E_0(t-s)\left\{\rho_0(k)\mathcal{M}_0 - f_0(k)(2\rho_1(k)(A_1 * \overline{\mathcal{L}} + A_{-1}\mathcal{L}) + \delta^2\Delta_0)\right\} ds\right] \\ &+ \delta^2\left[(\mathcal{S}_2 * w_{-1} + \mathcal{S}_0 * w_1) + A_{-1} * \int_0^t E_2(t-s)\mathcal{R}_2\, ds + A_1 * \int_0^t E_0(t-s)\mathcal{R}_0 ds\right]. \end{aligned}$$

Here $\mathcal{S}_0$, $\mathcal{S}_2$ are terms in front of δ^2 in the first and the third equation in (15)

$$\begin{aligned} \mathcal{S}_0 &= 2\left[f_0' K_0 A_1 * A_{-1} \ + \ (f_0 f_2 f_{-2} + 2f_0^3) A_1 * A_1 * A_{-1} * A_{-1})\right]; \\ \mathcal{S}_2 &= f_2' K_2 A_1 * A_1 \ + \ 4(f_0 f_2^2 + f_2^2 f_3) A_1 * A_1 * A_1 * A_{-1}. \end{aligned} \tag{41}$$

Now use the fact that the Landau constant is small, i.e. $f_2 + 2f_0 = \check{\beta}\delta^2$, then applying the expansion of (13), one finds

$$\mathcal{L} = \delta^2 c\left[\mathcal{V}_1 + \delta^2\mathcal{V}_2 + \Upsilon(\mathcal{L})\right] \tag{42}$$

with

$$\begin{aligned} \mathcal{V}_1 &= 2\check{\beta} A_{-1} * A_1 * w_1 + \check{\beta} A_1 * A_1 * w_{-1} + \mathcal{S}_2 * w_{-1} + \mathcal{S}_0 * w_1 \\ &- 2A_{-1} * \int_0^t E_2(t-s)\rho_2(k)\mathcal{M}_2\, ds - 2A_1 * \int_0^t E_0(t-s)\rho_0(k)\mathcal{M}_0\, ds \\ &+ A_{-1} * \int_0^t E_2(t-s)\mathcal{R}_2\, ds + A_1 * \int_0^t E_0(t-s)\mathcal{R}_0\, ds \\ \mathcal{V}_2 &= -A_{-1} * \int_0^t E_2(t-s) f_2(k)\Delta_2\, ds - 2A_1 * \int_0^t E_0(t-s) f_0(k)\Delta_0\, ds \\ &+ 2A_1\mathcal{L}_{-1,1} f_0' K_0 + A_{-1}\mathcal{L}_{1,1} f_2' K_2 \\ \Upsilon(\mathcal{L}) &= -4\rho_1(k)\left[A_{-1} * \int_0^t E_2(t-s) f_2(k) A_1 * \mathcal{L}\, ds\right. \\ &+ \left. A_1 * \int_0^t E_0(t-s) f_0(k)\left(A_1 * \overline{\mathcal{L}} + A_{-1} * \mathcal{L}\right)\, ds\right] \end{aligned} \tag{43}$$

Iterating (42) and using estimates of lemma 4 one easily gets that $\mathcal{L} = \delta^2\mathcal{V}_1 + \delta^4\left(\mathcal{V}_2 + \Upsilon(\mathcal{V}_1)\right) +$ h.o.t. with $\mathcal{V}_1$ of order one which proves the lemma and gives us the

higher order terms.
Using the result of the lemma one gets the following equation for the critical mode

$$\frac{\partial W_1}{\partial t} = \delta^4 \check{\mu}_1(k)\, W_1 + 2\rho_1(k)\, \delta^4 \left(\mathcal{V}_1 + \delta^2 \left(\mathcal{V}_2 + \Upsilon(\mathcal{V}_1)\right)\right) + \delta^4 2\rho_1(k)\mathcal{M}_1 + \delta^4 \mathcal{R}_1 \tag{44}$$

or

$$\frac{\partial W_1}{\partial t} = \delta^4 \check{\mu}_1(k)\, W_1 + \delta^4 \left(\mathcal{B}_1(W) + \delta^2 \tilde{\mathcal{B}}_1(W) + \tilde{\mathcal{Q}}_1(W,W)\right) + \delta^4 \tilde{\mathcal{R}}_1, \tag{45}$$

where $\tilde{\mathcal{R}}_1$ is an order one inhomogeneous term consisting of inhomogeneous terms from (43) which are also of order one due to the lemma 4. In $\mathcal{B}_1(W), \tilde{\mathcal{B}}_1(W)$ we arrange time dependent linear parts according to corresponding orders and in $\tilde{Q}_1$ nonlinear terms which can be explicitly expressed as

$$\begin{aligned} \mathcal{B}_1(W) &= 2\rho_1 \left(\mathcal{V}_1^\ell + \mathcal{L}_{-2,3}\right) \\ \tilde{\mathcal{B}}_1(W) &= 2\rho_1 \left(\mathcal{V}_2^\ell + \Upsilon(\mathcal{V}_1^\ell) + \tfrac{\rho_1'(k)}{\rho_1(k)} K_1 \mathcal{V}_1^\ell\right) \\ \tilde{\mathcal{Q}}_1(W,W) &= 2\rho_1 \left(\mathcal{N}_{-1,2} + \mathcal{N}_{0,1} + \mathcal{V}_1^n\right), \end{aligned} \tag{46}$$

where $\mathcal{V}_j^\ell$ and $\mathcal{V}_j^n$ are respectively linear and nonlinear parts of $\mathcal{V}_j$. Let us now consider the nonlinear part. We are ready to prove the following result:

Lemma 7 *In the equation (45) due to the smallness of the Landau constant the nonlinear terms are of order δ^2:*

$$||\tilde{\mathcal{Q}}_1(W,W)||_{\mathbf{B}_\alpha} = \mathcal{O}(\delta^2) \tag{47}$$

Proof: Let us consider the highest order in the $\tilde{\mathcal{Q}}_1(W,W)$–term:

$$\begin{aligned} \tilde{\mathcal{Q}}_1(W,W) &= 2\rho_1(k)(\mathcal{N}_{-1,2} + \mathcal{N}_{0,1} + \mathcal{V}_1^n) + \ldots \stackrel{(15),(38)}{=} \\ &= 2\rho_1(k)(\mathcal{N}_{-1,2} + \mathcal{N}_{0,1} + f_2(k)A_{-1} * w_1 * w_1 + 2f_0(k)A_1 * w_1 * w_{-1}) + \ldots \\ &= 2\rho_1(k)(2f_0(k) + f_2(k))(A_1 * w_1 * w_{-1} + A_{-1} * w_1 * w_{-1}) + \ldots \\ &\le 2\rho_1(k)c(2f_0 + f_2)(A_1 * w_1 * w_{-1} + A_1 * w_1 * w_{-1}) + \ldots \end{aligned} \tag{48}$$

where $2f_0 + f_2 = \delta^2\check{\beta}$, which proves the lemma. Then one gets

$$\frac{\partial W_1}{\partial t} = \delta^4 \check{\mu}_1(k)\, W_1 + \delta^4 \mathcal{B}_1(W) + \delta^6 \mathcal{Q}_1(W,W) + \delta^4 \tilde{\mathcal{R}}_1, \tag{49}$$

where $\mathcal{B}_1$ is defined in (46) and in $\mathcal{Q}_1(W,W)$ the nonlinear terms and the higher order linear terms were collected using the results of the lemmas 6 and 7.
Using the notations introduced in lemma 4, (49) can be rewritten as follows:

$$W_1 = \delta^4 \mathcal{E}_1 \mathcal{B}_1(W) + \delta^4 \mathcal{E}_1 \left(\delta^2 \mathcal{Q}_1(W,W) + \mathcal{R}_1\right), \tag{50}$$

and for non–critical modes we have

$$W_{nc} = \mathcal{E}_{nc}\mathcal{B}_{nc}(W) + \delta^2\mathcal{E}_{nc}\left(\mathcal{Q}_{nc}(W,W) + \mathcal{R}_{nc}\right), \quad (51)$$

where the index "nc" corresponds to the vector with $(0,2,3)$–components, for example $E_{nc} = (E_0, E_2, E_3)$ and

$$\begin{array}{ll} \mathcal{B}_0 = 2\rho_0\mathcal{L}_{-1,1} & \mathcal{Q}_0 = 2\rho_0\mathcal{M}_0 \\ \mathcal{B}_2 = \rho_2\mathcal{L}_{1,1} & \mathcal{Q}_2 = 2\rho_2\mathcal{M}_2 \\ \mathcal{B}_3 = 2\rho_3\mathcal{L}_{1,2} & \mathcal{Q}_3 = 2\rho_3\mathcal{M}_3 \end{array} \quad (52)$$

Using the fact that non–critical modes are slaved by the critical ones (36,38), one can map $\mathcal{B}_1(W) \to \mathcal{B}'_1(W_1)$ and $\mathcal{B}_{\rm nc}(W) \to \mathcal{B}'_{\rm nc}(W_1)$. Skipping primes, one can consider $\mathcal{B}(W_1) = (\mathcal{B}_1(W_1), \mathcal{B}_{nc}(W_1))$ as operators of W_1. So one obtains a system

$$W = \mathcal{E}\mathcal{B}(W_1) + \mathcal{E}\left(\delta^2\mathcal{Q}(W,W) + \mathcal{R}\right) \quad (53)$$

with $W = (W_1, W_{nc}) = (W_1, W_0, W_2, W_3)$, $\mathcal{E} = (\delta^4\mathcal{E}_1, \mathcal{E}_{nc})$, $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_{nc})$ and $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_{nc})$.

From the form of our nonlinearity (32), (46) (52) and estimates of lemma 4 it is obvious that

Lemma 8 *Let $\mathcal{Q}$ be given by (46, 49 ,52); $j = 0,1,2,3$; $sup_{t\in[0,T_0/\delta^4]}||U_j||_{\mathbf{B}} \leq \rho$ and $sup_{t\in[0,T_0/\delta^4]}||W_j||_{\mathbf{B}} \leq \rho$ for some $0 < \rho < \infty$. Then there exists a constant $C(\rho)$ independent of δ such that*

$$\begin{array}{l} sup_{t\in[0,T_0/\delta^4]}||\mathcal{Q}(U,U) - \mathcal{Q}(W,W)||_{\mathbf{B}_\alpha} \leq \\ C\left(sup_{t\in[0,T_0/\delta^4]}||U_1 - W_1||_{\mathbf{B}_\alpha} + sup_{t\in[0,T_0/\delta^4]}||U_{nc} - W_{nc}||_{\mathbf{B}_\alpha}\right) \leq \\ C sup_{t\in[0,T_0/\delta^4]}||U - W||_{\mathbf{B}}. \end{array} \quad (54)$$

Note that C is also depending on our $\Phi_{\rm GL}$.

We have to show now that the operator $\mathrm{Id} - \mathcal{E}\mathcal{B}$ is long–time invertible with $\mathcal{E}\mathcal{B} = (\delta^4\mathcal{E}_1\mathcal{B}_1, \mathcal{E}_{nc}\mathcal{B}_{nc})$. This result will be formulated as follows.

Lemma 9 *There exists a constant c independent of δ such that*

$$sup_{t\leq\frac{T_0}{\delta^4}}||\left(Id - \mathcal{E}\mathcal{B}\right)^{-1}g||_{\mathbf{B}} \leq c||g||_{\mathbf{B}}. \quad (55)$$

with $g \in C\left([0, T_0/\delta^4] \to \mathbf{B}\right)$.

Proof of 9: We want to show that the operator $(\mathrm{Id}-\mathcal{E}\mathcal{B})$ is a long–time invertible operator, in other words, that for every given $g\in C\left([0,T_0/\delta^4]\to\mathbf{B}\right)$ there exists an unique $W=(W_1,W_{nc})$ which is the solution of

$$(\mathrm{Id}-\mathcal{E}\mathcal{B})\,W=g. \tag{56}$$

Let us construct the following iterative sequence

$$W^{n+1}=g+\mathcal{E}\mathcal{B}W^n;\qquad W^0=0;\qquad g=(g_1,g_{\mathrm{nc}}). \tag{57}$$

Our purpose is to show that this is a Cauchy sequence. Let us show first that the following estimate holds

$$||W_1^{n+1}-W_1^n||_{\mathbf{B}_\alpha}\le\left(c_1\delta^4\right)^n\max\left\{\frac{t^n}{n!};\frac{,\,(1-\frac{\ell}{2d})t^{n-\frac{\ell}{2d}}}{,\,(n+1-\frac{\ell}{2d})}\right\}\sup||g_1||_{\mathbf{B}_\alpha} \tag{58}$$

With $2d$ is the order of the linear part and ℓ is the maximal order of the nonlinearity. We prove this by induction:

$$\begin{aligned}||W_1^{n+1}-W_1^n||_{\mathbf{B}_\alpha}&=\delta^4||\mathcal{E}_1\mathcal{B}_1\left(W_1^n-W_1^{n-1}\right)||_{\mathbf{B}_\alpha}\\&\le\delta^4\textstyle\int_0^t||E_1(t-s)\mathcal{B}_1(W_1^n-W_1^{n-1})||_{\mathbf{B}_\alpha}\,ds\\&\le\delta^4c_0\textstyle\int_0^t\max\left\{1;(t-s)^{-\frac{\ell}{2d}}\right\}||W_1^n(s)-W_1^{n-1}(s)||_{\mathbf{B}_\alpha}\,ds\end{aligned} \tag{59}$$

$||W_1^1-W_1^0||_{\mathbf{B}_\alpha}=||g_1||_{\mathbf{B}_\alpha}$ and $||W_1^2-W_1^1||=(c_0\delta^4)\max\left\{t;\frac{t^{1-\frac{\ell}{2d}}}{1-\frac{\ell}{2d}}\right\}\sup_{t\le\frac{T_0}{\delta^4}}||g_1||_{\mathbf{B}_\alpha}$. The above inequalities are obtained by repeatedly applying lemma 4 to the components of the operator $\mathcal{B}_1$ given by (46). By integration one gets (58).
Hence we proved that (57) is a Cauchy sequence which implies that there exists $W_1^\star=\lim_{n\to\infty}W_1^n$ and

$$\begin{aligned}&\sup_{t\le\frac{T_0}{\delta^4}}||W_1^\star||_{\mathbf{B}_\alpha}=\lim_{n\to\infty}\sup_{t\le\frac{T_0}{\delta^4}}||W_1^n||_{\mathbf{B}_\alpha}\\&\le\lim_{n\to\infty}\textstyle\sum_{j=0}^n\sup_{t\le\frac{T_0}{\delta^4}}||W_1^j-W_1^{j-1}||_{\mathbf{B}_\alpha}\\&\le\lim_{n\to\infty}\sup_{t\le\frac{T_0}{\delta^4}}\left\{\sum_{m=0}^n\frac{\left(c_1\delta^4t\right)^m}{m!};\sum_{m=0}^n t^{-\frac{\ell}{2d}}\frac{\Gamma(1-\frac{\ell}{2d})(c_0t\delta^4)^m}{\Gamma(m+1-\frac{\ell}{2d})}\right\}||g||_{\mathbf{B}_\alpha}\\&\le c\,\sup_{t\le\frac{T_0}{\delta^4}}||g||_{\mathbf{B}}.\end{aligned} \tag{60}$$

This proves the statement of the lemma for the critical mode. To complete the proof for the slaved modes we set

$$W_{nc}=g_{nc}+\mathcal{E}_{nc}\mathcal{B}_{nc}(W_1)$$

and then the estimate

$$\sup_{t\le T_0/\delta^4}||W_{nc}||_{\mathbf{B}_\alpha}\le c\,\sup_{t\le T_0/\delta^4}||g||_{\mathbf{B}}$$

proves the lemma.

Now we are able to invert the time dependent linear part

$$W = (\mathrm{Id} - \mathcal{E}\mathcal{B})^{-1}\, \mathcal{E}\left(\delta^2 \mathcal{Q} + \mathcal{R}\right). \tag{61}$$

For this equation we can apply a fixed point argument. The map F_δ corresponding to (61) from a ball with the center $(\mathrm{Id} - \mathcal{E}\mathcal{B})^{-1}\, \mathcal{E}\mathcal{R}$ (which is of order one according to lemma 4,5 and 9)

$$F_\delta : C\left(\left[0, T_0/\delta^4\right] \to \mathbf{B}\right) \to C\left(\left[0, T_0/\delta^4\right] \to \mathbf{B}\right)$$

is contracting due to the smallness of δ and the results of lemmas 8 and 9.

By this we established the existence and uniqueness of order one solutions of (29). However, as we already mentioned the decomposition of W on (W_1, W_{nc}) is not unique and our results do not imply the uniqueness of the error W. The uniqueness of W follows from the uniqueness of the solutions of (9) with small initial conditions. This can be easily proven by supposing that two solutions exist and applying the same contracting argument to their difference.
Remark. As we already mentioned in section 3 one can easily extend the validity statement for a larger class of spaces using the results of Takáč et al.: weakening the assumption of analyticity in the strip to the essential boundedness. For details one can see [37].

6 Discussion

In this paper we proved the validity of the degenerate equation for the model problem (9). The choice of more general nonlinear terms will not give any essential difficulties apart from more complicated calculations. The most interesting extension would be to make the coefficients complex i.e. to consider the situation without reflection symmetry . The equation in this case looks as follows,

$$\begin{aligned} \frac{\partial U_1}{\partial t} &= i\delta^2 \beta_i |U|^2 U + \delta^4 [(\check{\mu}_1 U_1 + \check{\beta}_r |U_1|^2 + \gamma |U_1|^4) U_1 \\ &+ (\check{\beta}_1 |U_1|^2 \tfrac{\partial U_1}{\partial X} + \check{\beta}_2 U_1^2 \tfrac{\partial U_1^*}{\partial X}) - \mu_1'' \tfrac{\partial^2 U_1}{\partial X^2}]. \end{aligned} \tag{62}$$

with $\check{\mu}_1$, $\check{\beta}_r$, γ, $\check{\beta}_1$, $\check{\beta}_2$, $\mu_1'' \in \mathbb{C}$ and $\check{\beta}_i \in \mathbb{R}$, see [14]. Let us note that in this case one can get rid of the problematic δ^2–terms by the following transformation $U(X,t) = e^{i\delta^2 \Omega(X,t)}\, \tilde{U}(X,t)$ with $\Omega_t(X,t) = \beta_i |\tilde{U}(X,t)|^2$. After this one recovers on the long time $T = \delta^4 t$ an amplitude equation (5) with all coefficients in $\mathbb{C}$ and moreover one has to take care of the short–time behavior of Ω.

Furthermore it would be interesting to extend the proof to the spatial multidimensional case which would allow us to treat the important physical problems mentioned in the introduction.

7 Appendix.

7.1 The amplitude equation in the cubic case

Let us show now how the coefficients of the amplitude equation will be changed if the cubic term in the nonlinearity of original equation is present, i.e. instead of (7) one has $N(U) = P(U^2) + Q(U^3)$. Which will be the case in the Chapter 3. Let $q(k)$ be a symbol of the linear operator Q and let us add

$$g_j(k) = -\frac{q_j(k)}{\mu_j(k)} = g_j \;+\; g_j' K_j \delta^2 \;+\; g_j'' K_j^2 \delta^4 + \ldots \tag{63}$$

to the (13). Then instead of (14) one has.

$$\begin{aligned}
A_0 &= 2\ f_0(k)\ A_1 * A_{-1} + \delta^2\ [f_0(k)(A_0 * A_0\ +\ 2\ A_{-2} * A_2) + \\
&\quad 3\ g_0(k)\ (2\ A_0 * A_{-1} * A_1 + A_1 * A_{-2} * A_1 + A_{-1} * A_{-1} * A_2)] \\
\frac{\partial A_1}{\partial T} &= \frac{\mu_1(k)}{\delta^4} A_1\ +\ \frac{2\ \rho_1(k)}{\delta^4}[A_1 * A_0\ +\ A_2 * A_{-1}]\ +\ \frac{3 q_1(k)}{\delta^4} A_1 * A_1 * A_{-1} + \\
&\quad \delta^{-2}\,[2\ \rho_1\ A_3 * A_{-2} + 3\ q_1\ (A_0 * A_0 * A_1 + 2\ A_0 * A_{-1} * A_2 + \\
&\quad 2\ A_1 * A_2 * A_{-2} + A_3 * A_{-1} * A_{-1})] \\
A_2 &= f_2(k)\ A_1 * A_1 + \delta^2\,[(2\ f_2(k)\ A_0 * A_2\ +\ 2\ A_{-1} * A_3) + \\
&\quad 3\ g_2(k)\ (A_0 * A_1 * A_1 + 2 A_{-1} * A_1 * A_2)] \\
A_3 &= 2\ f_3(k)\ A_1 * A_2\ +\ g_3(k) A_1 * A_1 * A_1
\end{aligned} \tag{64}$$

And analogically to (15) expanding (65) one gets

$$\begin{aligned}
A_0 &= 2\ f_0\ A_1 * A_{-1}\ +\ \delta^2\,[2\ f_0'\ K_0\ A_1 * A_{-1} + \\
&\quad (2\ f_0\ f_2\ f_{-2} + 4\ f_0^3 + 3\ g_0\ (4\ f_0 + f_{-2} + f_2)]\ A_1 * A_1 * A_{-1} * A_{-1} \\
A_2 &= f_2\ A_1 * A_1\ +\ \delta^2\,[f_2'\ K_2\ A_1 * A_1 + \\
&\quad 4\ (f_2^2\ (f_0 + 2\ f_2^2\ f_3 + g_3) + 6\ g_2\ (f_0 + f_2)]\ A_1 * A_1 * A_1 * A_{-1}) \\
A_3 &= (2\ f_3\ f_2 + g_3)\ A_1 * A_1 * A_1
\end{aligned} \tag{65}$$

Using (65) the fact that Landau constant in this $2\rho_1(2\ f_0 + f_2) + 3\ q_1 = \check{\beta}\ \delta^2$ is small of order δ^2 and returning to the original physical space we get the following degenerate Ginzburg–Landau equation.

$$\frac{\partial U_1}{\partial T} = \check{\mu}_1 U_1 + \mu_1'' \frac{\partial^2 U_1}{\partial X^2} + \check{\beta} |U_1|^2 U_1 - 2i\rho_1 \left\{ +2 f_0' U_1 \frac{\partial |U_1|^2}{\partial X} + f_2' U_{-1} \frac{\partial U_1^2}{\partial X} \right\} + \gamma_1 |U_1|^4 U_1 \tag{66}$$

with $\gamma_1 \;=\; 2\rho_1\,[4 f_0^3 + 2 f_0 f_2 f_{-2} + 12 g_0 f_0 + 3 g_0 f_{-2} + 3 g_0 f_2 + 4 f_2^2 f_0 + 4 f_2^2 f_3 + 2 f_2 g_3 + 6 g_2 f_0 + 6 g_2 f_2 + (2 f_3 f_2 + g_3)\, f_{-2}] + 3 q_1\,[4 f_0^2 + 4 f_0 f_2 + 2 f_{-2} f_2 + 2 f_3 f_2 + g_3]$.

References

[1] P. Bollerman, *On the Theory of Validity of Amplitude Equations*, PhD thesis, Utrecht University, 1996.

[2] H. Brand, P. S. Lomdahl, and A. C. Newel, *Benjamin–Feir turbulence in convective binary fluid mixtures.*, Phys. D, 23 (1986).

[3] H. R. Brand and R. Deissler, *Interaction of Localized Solutions for Subcritical Bifurcations.*, Phys. Rev. Lett., 63 (1989).

[4] P. Chossat and G. Iooss, *The Couette–Taylor Problem*, Springer Verlag, 1994.

[5] P. Collet and J.-P. Eckmann, *The time dependent amplitude equation for the Swift–Hohenberg problem*, Comm. Math. Phys., 132 (1990), pp. 139–153.

[6] M. R. Dhanak, *On certain aspects of tree–dimentional instability in parallel flows.*, Proc. R. Soc. London A, 385 (1983), pp. 53–84.

[7] A. Doelman, *On the Nonlinear Evolution of Patterns, Modulation Equations and their Solutions*, PhD thesis, University of Utrecht, 1990.

[8] A. Doelman and W. Eckhaus, *Periodic and quasi–periodic solutions of degenerate modulation equations.*, Phys. D, 53 (1991), pp. 249–266.

[9] P. Drazin and W. Reid, *Hydrodynamic Stability*, Cambridge University Press, 1981.

[10] J. Duan and P. Holmes, *On the Cauchy problem for A Generalized Ginzburg-Landau Equation.*, Nonlinear Analysis., 22 (1994), pp. 1033–1040.

[11] J. Duan, P. Holmes, and E. S. Titi, *Global Existence Theory for A Generalized Ginzburg-Landau Equation.*, Nonlinearity., 5 (1992), pp. 1303–1314.

[12] M. P. Eagles, *Supercritical flow in a diverging channel.*, J. Fluid Mech., 57 (1973), pp. 149–160.

[13] W. Eckhaus, *Studies in nonlinear stability theory.*, Springer-Verlag, 1965.

[14] W. Eckhaus and G. Iooss, *Strong selection or rejection of spatially periodic patterns in degenerate bifurcations.*, Phys. D, 39 (1989), pp. 124–146.

[15] K. Fujimura and R. E. Kelly, *Degenerate bifurcation in stably stratified plane Poiseuille flow.*, tech. report, Japan Atomic Energy Research Institute, 1995.

[16] H. Haken, ed., *Evolution of order and chaos.*, vol. 17, Springer Verlag, 1982.

[17] v. Harten, A., *On the validity of Ginzburg–Landau's equation*, J. Nonl. Sc., 1 (1991), pp. 397–422.

[18] T. Kapitula, *On the nonlinear stability of plane waves for the Ginzburg–Landau equation.*, tech. report, New York University, 1992.

[19] ——, *Singular heteroclinic orbits for degenerate modulation equations.*, Phys. D, 82 (1995), pp. 36–59.

[20] P. Kirrmann, G. Schneider, and A. Mielke, *The validity of modulation equations for extended systems with cubic nonlinearities*, Proc. R. Soc. Edinb. A, 122 (1992), pp. 85–91.

[21] E. Knobloch, *On the degenerate Hopf bifurcation with O(2) symmetry. Multiparameter bifurcation theory.* , Contemp. Math., 56 (1986), p. 193.

[22] Y. Kuramoto, *Chemical oscillations, Waves and Turbulence*, Springer Verlag, 1984.

[23] P. Laure and Y. Demay, *Symbolic computation and equation on the center manifold. Applications to the Couette–Taylor problem.*, Computer Fluids, 16 (1988), p. 229.

[24] B. Malomed and A. Nepomnyashchy, *Kinks and solitons in the generalized Ginzburg-Landau equation.*, Phys. Rev. A., 42 (1990).

[25] A. Newell and J. Whitehead, *Finite bandwidth, finite amplitude convection*, J. Fluid. Mech, 38 (1969), pp. 279–303.

[26] C. L. Pekeris and B. Shkoller, *Stability of plane Poiseuille flow to periodic disturbances of finite amplitude in the vicinity of the neutral curve .*, J. Fluid Mech., 29 (1967), pp. 31–38.

[27] W. Saarloos and P. Hohenberg, *Fronts, pulses, sources and sinks in degeneralized complex Ginzburg-Landau equation.*, Phys. D, 56 (1992), pp. 303–367.

[28] G. Schneider, *A new estimate for the Ginzburg–Landau approximation on the real axis*, J. Nonl. Sc., 4 (1994), pp. 23–34.

[29] ——, *Error estimates for the Ginzburg–Landau approximation*, J. Appl. Math. Physics (ZAMP), 45 (1994), pp. 433–457.

[30] ——, *Global existence via Ginzburg–Landau formalism and pseudo–orbits of Ginzburg–Landau approximation.*, Comm. Math. Phys., 164 (1994), pp. 157–179.

[31] ——, *Justification of Mean–field coupled Modulation Equations*, tech. report, IfAM, Universität Hannover, 1994.

[32] ——, *Validity and Limitation of the Newell–Whitehead equation*, tech. report, IfAM, Universität Hannover, 1994.

[33] L. Segel, *Distant side–walls cause slow amplitude modulation of cellular convection*, J. Fluid. Mech, 38 (1969), pp. 203–224.

[34] P. K. Sen and T. K. Vashist, *On the nonlinearity of boundary–layer flow over a flat plate.* , Proc. R. Soc. Lond. A, 424 (1989), pp. 81–92.

[35] K. Stewartson and J. Stuart, *A nonlinear instability theory for a wave system in plane Poiseuille flow*, J. Fluid. Mech, 48 (1971), pp. 529–545.

[36] H. Swinney and J. P. Gollub, *Hydrodynamic Instabilities and the Transition to Turbulence.*, Topics in Applied Physics., 45 (1981).

[37] P. Takáč, P. Bollerman, A. Doelman, A. v. Harten, and E. Titi, *Analyticity of Essentially Bounded Solutions to Semilinear Parabolic Systems and Validity of the Ginzburg-Landau Equation*, SIAM J. Math. An., 27 (1996), pp. 424–448.