Nuclear Physics B237 (1984) 274–284


# HOT TWISTS FOR THE SINGLE-POINT MODEL OF LARGE-$N$ QCD

Frans R. KLINKHAMER‡

*Leiden Observatory, PO Box 9513, 2300 RA Leiden, The Netherlands*

Pierre van BAAL

*Institute for Theoretical Physics, Princetonplein 5, PO Box 80.006, 3508 TA Utrecht, The Netherlands*

Received 20 May 1983
(Revised 10 October 1983)

We construct two types of twists for the SU($N \to \infty$) twisted-Eguchi-Kawai model, which mimic a periodic boundary condition in the temporal direction only and over an arbitrary extent $N_0$. In this way we introduce finite temperature ($T = N_0^{-1}$ in lattice units) in the single-point model. In weak coupling one gets the correct planar expansion.

## 1. Introduction

Recently much attention has been given to the reduction [1,2] of the Wilson SU($N \to \infty$) lattice gauge theory [3] to a single-point model. A model with twisted boundary conditions looks very promising for numerical [5] and analytic [6] calculations. Also the problem of how to incorporate finite temperature in (partially) reduced models has been considered [7–9]. In this paper we present general twists, which achieve this for a single-point model and which we will call "hot twists." These can be used in Monte Carlo simulations to study the large-$N$ deconfinement transition at the critical temperature $T_c$. Preliminary results indicate that $T_c \sim 3\sqrt{\sigma}$, where $\sigma$ is the string tension. We extracted from [5] $\Lambda_L \sim 3 \cdot 10^{-3}\sqrt{\sigma}$ and from [8] $T_c \sim 10^3 \Lambda_L$*. Remember that for SU(2) and SU(3) $T_c \sim 0.5\sqrt{\sigma}$ [10]. The model with our hot twist (4.7) may be used to determine, better than in [8], the value $T_c/\Lambda_L$. It seems feasible to observe renormalization group scaling while keeping the spatial finite-size effects small.

In ref. [11] it is argued that the confinement mechanism for $N \to \infty$ should be different, since factorization tells us that the string tension for adjoint quarks is twice that for fundamental quarks. Studying deconfinement for infinite $N$ might shed some light on this issue.

‡ Present address: Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA.

* We *assumed* equal $\Lambda_L$ for the twisted [5] and the quenched [8] Eguchi-Kawai model.

The program of this paper is as follows. In sect. 2 we briefly discuss the reduction procedure at finite temperature. In sect. 3 we show that non-perturbatively, i.e. at the level of the loop equations, there is agreement for $N \to \infty$ between the one-point model with hot twists and the Wilson theory at finite temperature. In sect. 4 we construct the appropriate hot twists. Sect. 5 deals with the weak-coupling limit, in which one retrieves the planar expansion at finite temperature. Finally in sect. 6 we give some further discussion of our hot twists, notably their potential value for Monte Carlo simulations.

## 2. Reduction

We start from the pure SU($N$) gauge theory on a hypercubic lattice (spacing $a = 1$) in euclidean space-time with dimension 4, where indices are denoted by greek symbols ($\mu, \nu, \ldots$). If necessary we distinguish the time (space) direction by $\mu = 0$ ($i, j, k \ldots = 1,2,3$). On a lattice of size $\Lambda = N_0 \times \Lambda_s$ with periodic boundary conditions in the time direction the partition function $Z_W$ and the action $S_W$ [3] used are*:

$$Z_W = \int \prod_{\mu, x \in \Lambda} dU_\mu(x) \exp(-\beta S_W), \tag{2.1}$$

$$S_W = \sum_{\mu \neq \nu, x \in \Lambda} \operatorname{Tr}\left(1 - U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu(x+\hat{\nu})^\dagger U_\nu(x)^\dagger\right). \tag{2.2}$$

This gives the gauge theory at equilibrium temperature $T = N_0^{-1}$, as long as the extent of the spatial directions is much larger than $N_0$. Eguchi and Kawai [1] showed that for $N \to \infty$ the theory (2.1) is equivalent to the reduced single-point model under the reduction:

$$\mathrm{R}: U_\mu(x) \to U_\mu. \tag{2.3}$$

For weak coupling ($\beta \to \infty$), modification of (2.3) proved to be necessary, first done with a quenching procedure, see [2] and references therein. Later Gonzalez-Arroyo and Okawa [4,5] proposed the more elegant approach of introducing appropriate twist $Z_{\mu\nu} \in \mathrm{Z}_N$ (the discrete centre of SU($N$)).

This twisted-Eguchi-Kawai (TEK) model is defined by:

$$Z_{\mathrm{TEK}} = \int \prod_\mu dU_\mu \exp(-\beta S_{\mathrm{TEK}}), \tag{2.4}$$

$$S_{\mathrm{TEK}} = \sum_{\mu \neq \nu} \operatorname{Tr}\left(1 - Z_{\mu\nu} U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right). \tag{2.5}$$

* Here and in the following $\beta$ is the inverse coupling constant squared and *not* the inverse temperature!

The twist $Z_{\mu\nu}$ will be labelled by the twist tensor $n_{\mu\nu} = -n_{\nu\mu}$ through:

$$Z_{\mu\nu} = \exp\left(\frac{2\pi i n_{\nu\mu}}{N}\right). \tag{2.6}$$

To guarantee in weak-coupling agreement of the internal energy, we demand the twist to be orthogonal:

$$\kappa(n) = \tfrac{1}{8} n_{\mu\nu} n_{\alpha\beta} \varepsilon_{\mu\nu\alpha\beta} = \sigma N, \tag{2.7}$$

with $\sigma$ integer. Furthermore the twist must be such that the corresponding minimum-action solution is unique up to a gauge transformation (modulo multiplication by elements of $Z_N$), which is equivalent to the statement $i(n) = 1$, with:

$$i(n) = \text{g.c.d.}\big(n_{\mu\nu}, N, \kappa(n)/N\big). \tag{2.8}$$

(g.c.d. = greatest common divisor, see [6] for details.)

In order to establish further correspondence we follow ref. [5] in identifying Wilson-loop operators. One makes the change of variables:

$$U_\mu(x) \to Z(x, \mu) U_\mu(x), \tag{2.9}$$

with $Z(x, \mu) \in Z_N$ such that

$$Z_{\mu\nu} = Z(x, \mu) Z(x + \hat{\mu}, \nu) Z(x + \hat{\nu}, \mu)^{-1} Z(x, \nu)^{-1}, \tag{2.10}$$

is independent of $x$. After the reduction, R, one retrieves the action of the TEK model. In order to see that this change of variables is possible we have to show that we can choose the $Z(x, \mu)$ such that we still respect the boundary condition:

$$Z(x, \mu) = Z(x + N_0 \hat{0}, \mu), \tag{2.11}$$

for all $x$ and $\mu$. We first construct $Z(x, \mu)$ satisfying (2.10) on the same lattice but neglecting the condition (2.11), which is easily done. We shall show that with a suitable $Z_N$-gauge transformation (2.11) will be satisfied. Each time layer represents a configuration on an infinite 3-dimensional $Z_N$ lattice. They have the same plaquette values $Z_{ij}$ and so by an appropriate $Z_N$-gauge transformation we can choose:

$$Z(x, i) = Z(x + \hat{0}, i), \tag{2.12}$$

for all $x$ and $i$. Now consider the ratio of the $Z_N$ factors picked up by two neighbouring straight timelike lines: $\prod_{k=0}^{N_0-1} Z(x + k\hat{0}, 0) \prod_{l=0}^{N_0-1} Z(x + \hat{\imath} + l\hat{0}, 0)^{-1}$.

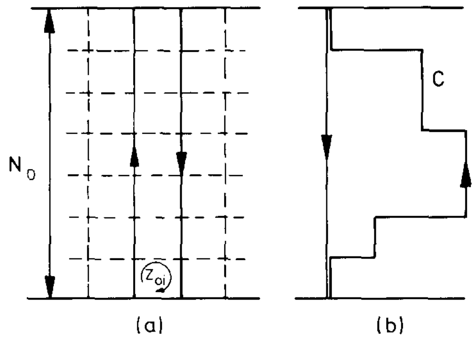


Fig. 1. (a) The timelike lines considered in the construction of the change of variables $U_\mu(x) \to Z(x,\mu)U_\mu(x)$. (b) The "closing" of a single temporal loop in order to define the appropriate surface in (2.15).

Using (2.12) it represents the $Z_N$ Wilson factor for a closed loop (see fig. 1a) and with the definition (2.10) this ratio becomes $Z_{0i}^{N_0}$. It will be shown later that we must, and can, choose the twist such that $N_0$ is the smallest positive integer with $Z_{0i}^{N_0} = 1$ for all $i$. Therefore the $Z_N$ factor:

$$Z_t \equiv \prod_{k=0}^{N_0-1} Z(x + k\hat{0}, 0), \tag{2.13}$$

is independent of $x_i$. This is necessary to have a consistent reduction of temporal loops. Unlike for $Z(x,i)$ we have $Z(x,0)$ dependent on $x_0$, but we can consistently choose $Z(x,0)$ to be periodic:

$$Z(x,0) = Z(x + N_0\hat{0}, 0), \tag{2.14}$$

because with (2.12) eq. (2.10) remains valid. We can even choose a spatially independent $Z_N$-gauge transformation which makes $Z_t \equiv 1$, which will be assumed below.

As in the TEK model one has the following correspondence for the Wilson loop C:

$$\frac{1}{N}\mathrm{Tr}\Big(\prod_{x\in C} U_\mu(x)\Big) \leftrightarrow \frac{1}{N}\prod_{x\in S} Z_{\mu\nu}\mathrm{Tr}\Big(\mathrm{R}\prod_{x\in C} U_\mu(x)\Big), \tag{2.15}$$

where S is any surface with boundary C ($C = \partial S$). For single temporal "loops" closing by the boundary conditions, S is defined by "closing" C with a straight timelike line (see fig. 1b).

## 3. Loop equations

We will briefly sketch how the loop equations in the presence of temperature can be the same for both the Wilson theory and TEK model. This imposes conditions on the construction of the twists in sect. 4. We have to show that spurious source terms [1] arising from the identification of different link variables ($U_\mu(y)$ and $U_\mu(z) \to U_\mu$) vanish. These terms contain

$$\left\langle N^{-1} \mathrm{Tr}\, \mathrm{R} \prod_{\mathrm{C}_{yz}} U_\mu(x) \right\rangle_{\mathrm{TEK}} , \tag{3.1}$$

with $\mathrm{C}_{yz}$ the part of the loop C running from $y$ to $z$. For $\beta \to \infty$ the twist eating configurations [5] $\Omega_\mu$ minimize $S_{\mathrm{TEK}}$ and the source term vanishes if $\mathrm{Tr}(\prod_{\mathrm{C}_{yz}} \Omega_\mu) = 0$. It is assumed that the twist effects are strong enough to keep (3.1) zero not only at $\beta = \infty$ but down to $\beta_{\mathrm{EK}} \sim 0.15N$; here the strong-coupling region sets in and the unbroken [12] U(1) symmetry $U_\mu \to e^{i\varphi_\mu} U_\mu$ forces (3.1) to zero. Monte Carlo simulations appear to confirm this conjecture [5].

In the presence of temperature the loop equations in the Wilson theory have source terms due to the periodicity, which should be maintained after reduction [9]. This implies that $N_0$ must be the smallest positive integer with

$$\mathrm{Tr}\left(\Omega_0^{N_0}\right) \neq 0 . \tag{3.2}$$

The spurious source terms vanish if:

$$\mathrm{Tr}\left(\Omega_0^{k_0} \Omega_1^{k_1} \Omega_2^{k_2} \Omega_3^{k_3}\right) = 0 , \tag{3.3}$$

for $k_\mu$ inside a box of dimensions $N_0 \times N_1 \times N_2 \times N_3$ with $N_i \to \infty$ for $N \to \infty$, but $N_0$ *fixed*. To illustrate this consider the loop equations for the path C of fig. 2a. On the link L the change of variables [1] $U \to (1 + i\,\epsilon\, \mathrm{T}^j)U$ is performed. In the reduced

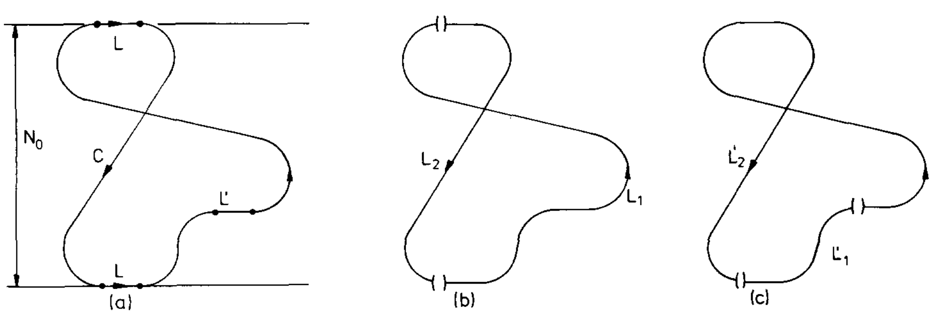


Fig. 2. The loop equation for $\langle N^{-1} \mathrm{Tr} \prod_{x \in \mathrm{C}} U_\mu(x) \rangle_{\mathrm{W}}$ contains a source term $\langle N^{-1} \mathrm{Tr} \prod_{\mathrm{L}_1} U_\mu(x) \rangle_{\mathrm{W}}$ $\langle N^{-1} \mathrm{Tr} \prod_{\mathrm{L}_2} U_\mu(x) \rangle_{\mathrm{W}}$ with $\mathrm{L}_1$ and $\mathrm{L}_2$ as in (b). These terms should survive in the reduced model, but not those (c) coming from the identification of the links L and L′ in (a). The twist makes these spurious source terms vanish.

model there are the genuine (temperature) source terms (3.1) with $C_{yz} = L_{1,2}$ (fig. 2b) and the spurious ones $L'_{1,2}$ from the identification of the links L and L′. The source terms from $L'_{1,2}$ are forced to zero by the twist, but not those from $L_{1,2}$ as follows from (3.2) and the cancelling of the $\Omega_i$. Note that if the link L′ in fig. 1a is shifted upwards to the "boundary" it gives a spurious source term, which also vanishes because $\mathrm{Tr}(\Omega_0^{N_0}\Omega_1^{k_1}\Omega_2^{k_2}\Omega_3^{k_3}) = 0$ for $k_i \neq 0 \bmod N_i$. This is necessary if we are to mimic finite temperature.

## 4. Construction of hot twists

We will first translate the constraints (3.2) and (3.3) into constraints on $n_{\mu\nu}$, which is related to $\Omega_\mu$ by:

$$\Omega_\mu \Omega_\nu \Omega_\mu^\dagger \Omega_\nu^\dagger = \exp\left(\frac{2\pi i n_{\mu\nu}}{N}\right). \tag{4.1}$$

As in [5] we find that condition (3.3) is satisfied if and only if $k_\mu$ is not an element of the sublattice $\mathrm{A} \subset \mathbb{Z}^4$:

$$\mathrm{A} = \left\{ k \in \mathbb{Z}^4 | \sigma k_\mu = \tilde{n}_{\mu\nu} q_\nu,\ q \in \mathbb{Z}^4 \right\}, \tag{4.2}$$

where $\tilde{n}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta} n_{\alpha\beta}$ and $\sigma$ is defined by (2.7). Also for all $\mu$ we have

$$\Omega_\mu \Omega_0^k \Omega_\mu^\dagger = Z_{0\mu}^k \Omega_0^k \tag{4.3}$$

and as long as $Z_{0i}^k \neq 1$ we have $\mathrm{Tr}(\Omega_0^k) = 0$. So in order to have $\mathrm{Tr}(\Omega_0^{N_0}) \neq 0$ we must have that $N$ is a multiple of $N_0$, and $n_{0i}$ is a multiple of $N/N_0$. However for at least one value of $i$, $n_{0i}$ should equal $N/N_0$. For practical purposes we take A to be a rectangular lattice, so $\mathrm{A} = N_0\mathbb{Z} \times N_1\mathbb{Z} \times N_2\mathbb{Z} \times N_3\mathbb{Z}$. We also wish $N_1$, $N_2$ and $N_3$ to be of the same order for $N \to \infty$. Furthermore, anticipating that we want to reproduce the planar expansion we require the algebra generated by

$$A(k) = \Omega_0^{k_0}\Omega_1^{k_1}\Omega_2^{k_2}\Omega_3^{k_3}, \qquad k \in \mathbb{Z}^4/\mathrm{A}, \qquad k \neq 0, \tag{4.4}$$

to be SU($N$). Obviously $A(k)\Omega_\mu = Z_\mu(k)\Omega_\mu A(k)$ with $Z_\mu(k) \in Z_N$ and $Z_\mu(k) = 1$ for all $\mu$ iff $k \in \mathrm{A}$. This guarantees that they are linearly independent. Namely suppose that $A(k^{(i)})$ for $i = 1$ to $n$ are independent and $\Sigma_{i=1}^{n+1}\alpha_i A(k^{(i)}) = 0$. Conjugation with $\Omega_\mu$ yields $\Sigma_{i=1}^{n+1}\alpha_i Z_\mu(k^{(i)})A(k^{(i)})$ and so $\Sigma_{i=1}^{n}\alpha_i \Sigma_\mu(1 - Z_\mu(k^{(i)} - k^{(n+1)}))A(k^{(i)}) = 0$. This implies $\alpha_i = 0$ for all $i$ up to $n+1$ iff for all $i$, $k^{(n+1)} \neq k^{(i)} \bmod \mathrm{A}$. In conclusion the volume $\Pi_\mu N_\mu$ of $\mathbb{Z}^4/\mathrm{A}$ has to equal $N^2$ $(= \dim(\mathrm{SU}(N)) + 1)$.

In ref. [6] it was shown that one can easily construct $\Omega_\mu$ once $n_{\mu\nu}$ is given. So we shall concentrate on finding $n_{\mu\nu}$.

Let us first take $\sigma = 1$ in (2.7). Then $i(n) = 1$ (see (2.8)) gives no extra constraint on $n_{\mu\nu}$. The condition on A is that its 4 generators $k_\mu^{(\nu)} = \tilde{n}_{\mu\nu}$ are related by a SL(4,$\mathbb{Z}$) transformation to the four basis vectors $N_\mu e^{(\mu)}$, therefore $N_\mu^{-1}\tilde{n}_{\mu\nu}$ is an integer (in the last two expressions no summation over $\mu$). The appropriate SL(4,$\mathbb{Z}$) transformation can be written as $Y = -(L^{-1}\tilde{n})^t$ where $L = \mathrm{diag}(N_0, N_1, N_2, N_3)$. One can check that all conditions on $n_{\mu\nu}$ are met by the choice:

$$n_{\mu\nu} = N_0 \begin{pmatrix} 0 & K(4K^2-1) & K(4K^2-1) & K(4K^2-1) \\ & 0 & K(2K-1) & 4K^2-1 \\ & * & 0 & K(2K+1) \\ & & & 0 \end{pmatrix}, \tag{4.5}$$

$$N = N_0^2 K(4K^2-1), \quad N_1 = N_0 K(2K+1), \quad N_2 = N_0(4K^2-1), \quad N_3 = N_0 K(2K-1)$$

Using the methods of ref. [6] we find:

$$\begin{aligned} \Omega_0 &= P_1 \otimes 1_2, \\ \Omega_1 &= Q_1 \otimes P_2^{(K+1)(2K-1)} Q_2^{1-4K^2}, \\ \Omega_2 &= Q_1 \otimes P_2^{K(2K+1)} Q_2^{-4K^2}, \\ \Omega_3 &= Q_1 \otimes P_2^{K(2K+1)} Q_2^{1-4K^2}. \end{aligned} \tag{4.6}$$

Here $P_i, Q_i$ are elements of SU($M_i$) satisfying the basic commutation relations $P_i Q_i P_i^\dagger Q_i^\dagger = \exp(2\pi i/M_i)$. For eq. (4.6) $M_1 = N_0$ and $M_2 = N_0 K(4K^2-1)$ and $\otimes$ is the tensor product of SU($M_1$) and SU($M_2$) which lies in SU($M_1 M_2 = N$).

One always has that $N_i$ is proportional to $N_0$. Here we constructed $n_{\mu\nu}$ such that the $N_i$ are as close together as possible. For $N \to \infty$ we find $N_1 : N_2 : N_3 = 1:2:1$. We would rather have that for $N \to \infty$ all $N_i$ become equal, although it certainly is not necessary. For this we had to compromise a little by allowing for $\sigma \neq 1$, supplied with a constraint on $N_0$. The twist we found is also more economic in Monte Carlo simulations (in sect. 6 we will elaborate on this). This other hot twist is given by

$$n_{\mu\nu} = N_0 \begin{pmatrix} 0 & 2K(4K^2-1) & 4K(4K^2-1) & 2K(4K^2-1) \\ & 0 & 2K(2K+1) & 4K^2-1 \\ & * & 0 & 2K(2K-1) \\ & & & 0 \end{pmatrix}, \tag{4.7}$$

$$N = 2N_0^2 K(4K^2-1), \qquad N_1 = 2N_0 K(2K-1), \qquad N_2 = N_0(4K^2-1),$$
$$N_3 = 2N_0 K(2K+1), \qquad N_0 = \text{odd},$$

and the twist-eating configuration is found to be:

$$\Omega_0 = Q_1^{-2} \otimes P_2^{2K(2K+1)(4K^2-1)} Q_2^{4K(1-4K^2)},$$

$$\Omega_1 = P_1^{K+1} \otimes P_2^{2K(2K+1)(K+1)} Q_2^{-(2K+1)^2},$$

$$\Omega_2 = P_1 \otimes P_2^{2K(2K+1)} Q_2^{-4K},$$

$$\Omega_3 = P_1^{1-K} \otimes P_2^{(1-2K^2)(2K-1)} Q_2^{(2K-1)^2}. \tag{4.8}$$

Here $M_1 = N_0$, $M_2 = 2N_0K(4K^2-1)$ define $P_i, Q_i$ as above. This twist has $\sigma = 2$. One must be careful in constructing the sublattice Λ. Here it will be generated by $k_\mu^{(0)} = \tilde{n}_{\mu 0}$, $k_\mu^{(1)} = \frac{1}{2}\tilde{n}_{\mu 1}$, $k_\mu^{(2)} = \tilde{n}_{\mu 2}$ and $k_\mu^{(3)} = \frac{1}{2}\tilde{n}_{\mu 3}$. In (4.2) $q$ is of the form $(2\mathbb{Z}, \mathbb{Z}, 2\mathbb{Z}, \mathbb{Z})$ in order to keep $k$ integer. To guarantee a unique minimum action configuration (up to a gauge) we need $i(n) = 1$, which reduces here to g.c.d. $(\sigma, N_0) = 1$ or $N_0 = \text{odd}$.

## 5. Planar graphs

For $N \to \infty$ we want to reproduce in weak coupling the planar expansion in the presence of finite temperature. The only influence of finite temperature is that the integrals over temporal momenta $p_0$ get replaced by a Fourier sum $\sum_{n=0}^{N_0-1}$ with $p_0(n) = 2\pi n T$. There is no interference with planarity because this is a property of the SU($N$) index structure only [13], unlike on the lattice where space-time momenta come from the color degrees of freedom. In establishing, for zero temperature, the correspondence between the reduced models and the continuum, one ignores the infrared problems which still plague the theory. We will do the same for finite temperature where the $n = 0$ sector behaves as a 3-dimensional gauge theory and thus gives rise to even more severe infrared problems. These infrared problems have a direct analogue in the one-point lattice where corresponding divergencies come from "long distance" behavior in *group space*. The longest wavelength in each direction is clearly $N_\mu$, which goes to infinity (for finite temperature only in the spatial directions) if $N$ tends to infinity. So finite $N$ acts as an infrared cutoff. Indeed finite $N$ corresponds to finite boundary conditions, which also in the continuum serve as an infrared cutoff. More serious might be the quartic mode problem (zero modes which are not gauge modes). When we have finite boundary conditions (a finite box) quartic modes have their influence both in a lattice [15] and in the continuum [16]. In the infinite-volume limit their influence on the lattice disappears [15], and we expect the same in the continuum*. This is one of the reasons why we carefully choose our twist such that there are only gauge-zero modes [6].

* We thank G. 't Hooft for a discussion on this point.

We can closely follow ref. [5] in establishing the correspondence. Conjugation with $\Omega_\mu$ corresponds to translation, from which one identifies the momenta $q$:

$$\Omega_\mu A(k)\Omega_\mu^\dagger = \exp\left(\frac{2\pi i}{N} n_{\mu\nu} k_\nu\right) A(k). \tag{5.1}$$

For convenience we suppose $\sigma = 1$ (generalization is straightforward) and we label $k \in \mathbb{Z}^4/\Lambda$ through

$$k_\nu = \sum_\mu \frac{\tilde{n}_{\mu\nu} q_\mu}{N_\mu}, \tag{5.2}$$

then $q_\mu = (N_\mu/N) n_{\mu\nu} k_\nu$ and $1 \leqslant q_\mu \leqslant N_\mu$. The propagator is that of a lattice with periodic boundary conditions and size $\Pi_\mu N_\mu$. For $N \to \infty$ the spatial momenta become continuous but the temporal momenta retain their correct discrete character, so that the temperature propagator in the continuum limit is reproduced ($a$ = lattice spacing; $aN_0 = T^{-1}$):

$$\lim_{a\to 0} 2a^{-2}\left(1 - \cos\left(\frac{2\pi a q_0}{aN_0}\right)\right) = \left(\frac{2\pi q_0}{aN_0}\right)^2 = p_0(q_0)^2. \tag{5.3}$$

Finally a non-planar graph will acquire a phase factor $\exp(-(2\pi i/N)$ $\Sigma_{i<j}\Sigma_{\mu>\nu} k_\mu^{(i)} n_{\mu\nu} k_\nu^{(j)}) \neq 1$, where $q_\mu^{(i)}$ are the loop momenta. Let us explicitly evaluate the phase factor for the twist (4.5) and $q^{(i)} = (0, q_1^{(i)}, 0, 0)$, for which $k_\mu^{(i)} = (-1, 0, 2K-1, 1-2K) q_1^{(i)}$. We find: $\exp([2\pi i/N]\Sigma_{i<j} k_2^{(i)} n_{23} k_3^{(j)}) = \exp([-2\pi i(2K-1)/N_0]\Sigma_{i<j} q_1^{(i)} q_1^{(j)})$. For $K \to \infty$ $\phi_1 = 2\pi q_1/N_0 K(2K+1)$ becomes continuous and the phase factor is:

$$\exp\left(\frac{-i}{2\pi}\frac{NK(2K+1)}{N_0}\sum_{i<j}\phi_1^{(i)}\phi_1^{(j)}\right)$$

which rapidly oscillates. Integration for $N \to \infty$ over $\phi_1^{(i)}$ will yield zero. So all non-planar graphs vanish for $N \to \infty$.

## 6. Discussion

We constructed the hot twists (4.5) and (4.7) for the SU($N$) single-point model, which guarantees equivalence for $N \to \infty$ with the Wilson lattice gauge theory at temperature $T$, both perturbatively and non-perturbatively (loop equations). This is rigorous in the weak-coupling limit ($\beta \to \infty$) and in the strong-coupling region, but probably holds for all $\beta$. This simple model might be useful for analytical calculations. As in [6] one can easily construct surviving fluxons (with an action of $O(1/N)$ such that $e^{-\beta S(\mathrm{fluxon})}$ remains finite for $N \to \infty$). The spectrum is the same ($\Delta S = 8\pi^2/N$ is the spacing in the action), but the "occupation numbers" are different. However one can argue that they do not contribute to the string tension for $N \to \infty$ (in weak coupling) [17].

From a practical point of view, e.g. Monte Carlo simulations, one has to keep $N$ finite. Just as for the symmetric twist of [5] we have that for finite $N$ the twist mimics a SU($N$) lattice gauge theory with periodic boundary conditions and size $N_0 \times N_1 \times N_2 \times N_3$. From [14] we know that finite-size effects are small for min $N_i \geqslant 2N_0$, and we assume this to be true for the one-point model also. For the twist (4.5) $N_\mu = N_0(1, K(2K+1), 4K^2-1, K(2K-1))$ and $N = N_0^2 K(4K^2-1)$, so that we must have $K \geqslant 2$ and therefore $N \geqslant 30N_0^2$. We now see why the twist (4.7) is more economic: $N = 2N_0^2 K(4K^2-1)$ can be as low as $6N_0^2$, because $N_\mu = N_0(1, 2K(2K-1), 4K^2-1, 2K(2K+1))$ allows for $K=1$, which gives $N_i \geqslant 2N_0$.

To be honest we have to compare this number of degrees of freedom $4(N^2-1) = 144N_0^4 - 4$ of the one-point lattice with $96N_0^4$ (or $256N_0^4$) of a $N_0 \times 2N_0 \times 2N_0 \times 2N_0$ lattice with gauge group SU(2) (or SU(3)). There is only something to be gained* for gauge group SU($M$) with $M > 3$. Of course from a numerical point of view one has to compare the one-point lattice with the lattice theory for $M = N$ and *then* the reduction is enormous. However compared to the torus model of [8], where one reduces only in the spatial directions, our model is much easier to handle. The Monte Carlo calculations are just as easy as for the symmetric twist [5].

One would like to determine $\beta_c(N_0)$, above which the global $Z_N$ symmetry is broken and deconfinement sets in, by calculating the expectation value of a single-temperature loop. In order to establish the correct scaling behaviour of $\beta_c(N_0)$ and extract $T_c$ from it, one would like to perform the calculations for quite large $N_0$ values. For the twist (4.7) we have to work with at least SU(6), SU(54) and SU(150) for $N_0 = 1$, 3 and 5 respectively. This may seem somewhat large but only in this way are we guaranteed of reasonably small boundary effects. For given $N$ and $N_0$ our model is better in this respect ($N^2 = N_0 \Pi_i N_i$) than the single-point (torus) model of ref. [9] with chosen space-time (space) quenched momenta, where $N = N_0 \Pi_i N_i$ ($N = \Pi_i N_i$).

We thank Gerard 't Hooft for a critical reading of the manuscript. For one of us (P.v.B.) this work is part of the research programme of the "Stichting voor Fundamenteel Onderzoek der Materie (FOM)", which is financially supported by the "Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek (ZWO)".

**Note added in proof:**

Having finished the present article another twist was found by one of us (P.v.B.), that generalizes the one of (4.7) to *all* $N_0$:

$$n_{\mu\nu} = N_0 \begin{pmatrix} 0 & -2K^2(4K^2-1) & 2K(4K^2-1) & 2K^2(4K^2-1) \\ & 0 & 2K(2K+1) & 4K^2-1 \\ & & 0 & 2K(2K-1) \\ & & & 0 \end{pmatrix}. \tag{4.9}$$

* But the efficiency of the updating may be somewhat lower for TEK. We thank K. Fabricius and O. Haan for bringing this to our attention.

This twist, which now has $\sigma = 1$, gives the same $N$, $N_\mu$ and $\Omega_1$, $\Omega_2$, $\Omega_3$ as in (4.7) and (4.8), whereas

$$\Omega_0 = Q_1^{-1} \otimes P_2^{2K(2K+1)(4K^2-1)} Q_2^{4K(1-4K^2)}. \tag{4.10}$$

## References


[1] T. Eguchi and H. Kawai, Phys. Rev. Lett. 48 (1982) 1063
[2] D.J. Gross and Y. Kitazawa, Nucl. Phys. B206 (1982) 440
[3] K. Wilson, Phys. Rev. D10 (1975) 2445
[4] A. Gonzalez-Arroyo and M. Okawa, Phys. Lett. 120B (1983) 174
[5] A. Gonzalez-Arroyo and M. Okawa, Phys. Rev. D27 (1983) 2397; Phys. Lett. 133B (1983) 415
[6] P. van Baal, Surviving extrema for the action on the twisted SU($\infty$) one-point lattice, Utrecht preprint (February 1983); Comm. Math. Phys. 92 (1983) 1
[7] F.R. Klinkhamer, Nucl. Phys. B218 (1983) 32
[8] H. Neuberger, Nucl. Phys. B220[FS8] (1983) 237
[9] F.R. Klinkhamer, Nucl. Phys. B228 (1983) 65
[10] J. Engels, F. Karsch, H. Satz and I. Montvay, Nucl. Phys. B205[FS5] (1982) 545; J. Engels, F. Karsch and H. Satz, Phys. Lett. 113B (1982) 398
[11] J. Greensite and M.B. Halpern, Phys. Rev. D27 (1983) 2545
[12] M. Okawa, Phys. Rev. Lett. 49 (1982) 353
[13] G. 't Hooft, Nucl. Phys. B72 (1974) 461
[14] J. Engels, F. Karsch and H. Satz, Nucl. Phys. B205[FS5] (1982) 239
[15] A. Gonzalez-Arroyo, J. Jurkiewicz and C.P. Korthals-Altes, Proc. 1981 Freiburg Nato Summer Institute, New York, (Plenum, 1982)
[16] M. Lüscher, Nucl. Phys. B219 (1983) 233; M. Lüscher and G. Münster, Nucl. Phys. B232 (1984) 445
[17] P. van Baal, Instantons versus factorization in large-$N$ field theories, Utrecht preprint (January 1984)