# RAY DISPERSION IN RANDOM ISOTROPIC MEDIA

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We give a systematic derivation of the Fokker–Planck equation for the joint probability density of the position and the direction vector of rays propagating in a random isotropic medium. Earlier results in the literature can be obtained from this by making additional approximations.

*1. Introduction.* Chernov [1] treated the problem of rays scattering in a medium with random inhomogeneities by means of the Fokker–Planck equation (FPE). He regarded the propagation of a ray as a continuous Markov process, in which the role of time is played by the path length traversed by the ray. He obtained a FPE for the angular distribution function. The method permits one to determine not only the angular but also the linear displacement of the ray from its initial direction. Moreover, in agreement with his suggestion this could be done without requiring the displacement to be small, which was an advantage of this method over the method of small perturbations.

Chernov's approach has been criticized by Keller [2], who proposed the name "dishonest" for all methods in which the average of a product is replaced by a product of averages. But later this approximation was applied to more complicated problems of ray propagation in isotropic media [3], in gyrotropic media [4], in the presence of refraction [5,6], etc. The first attempt to give a more systematic derivation was made by Klyatskin and Tatarski [7,8] for the case of vertical ray propagation in a medium without refraction. The crucial point of their proof was the identification of the width of the layer traversed by the ray with the path length. They came to the conclusion that the FPE for ray diffusion can be justified only in the small-angle approximation and has no preference over other methods.

We shall give a more general FPE, from which the results of Chernov, Yeh and Liu, Klyatskin and Tatarski can be derived, and it will appear that Chernov's equation is not confined to small angles.

*2. Derivation of the FPE for the ray dispersion.* Taking as independent variable the path length $\sigma$ of the ray, we consider the Hamilton equations of geometrical optics [7–9] (summation over repeated indices implied)

$$\frac{\mathrm{d}r_i(\sigma)}{\mathrm{d}\sigma} = N_i\,, \quad \frac{\mathrm{d}N_i(\sigma)}{\mathrm{d}\sigma} = (\delta_{ij} - N_iN_j)\,\frac{\partial\mu(\boldsymbol{r})}{\partial r_j}\,,$$
$$r_i(\sigma=0) = r_i^0\,, \quad N_i(\sigma=0) = N_i^0\,,$$
$$(i, j = 1, 2, 3;\ |\boldsymbol{N}|^2 = 1)\,, \qquad (1)$$

where $\boldsymbol{r} = \{r_i\}$ defines the position of the ray, $\boldsymbol{N} = \{N_i\}$ is the unit normal vector (in isotropic media $\boldsymbol{N}$ coincides with the ray vector), $\mu = \ln n$, and $n = n(\boldsymbol{r})$ is the refractive index. Eqs. (1) are valid under the conditions

$$\lambda \ll a\,, \quad \sqrt{\lambda L} \ll a\,,$$

i.e. the scale of the inhomogeneities $a$ is large compared to the wavelength $\lambda$ and to the size of the first Fresnel zone for the distance $L$ concerned. Moreover, we assume that the transit time of the ray is small compared to the characteristic time scale of changes

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of the inhomogeneities in time.

In this section we restrict ourselves to the case

$$n(\boldsymbol{r}) = n_0 + n_1(\boldsymbol{r}) , \quad \langle n_1(\boldsymbol{r})\rangle = 0 ,$$

with constant $n_0$ and zero average of the random inhomogeneities. Let us introduce the small dimensionless parameter $\alpha r_c \ll 1$. Here $\alpha = \nabla n_1/n_0$ is the parameter determining the size of fluctuations in question, and $r_c$ is the spatial correlation radius of the inhomogeneities, which is of the same order as their scale $a$.

If we consider the six-vector $u = \{\boldsymbol{r}, \boldsymbol{N}\}$ and divide the r.h.s. of (1) into the sure part $F_0$ and the fluctuating part $F_1$

$$F = F_0 + F_1 , \quad F_0 = \{\boldsymbol{N}, 0\} , \quad F_1 = \{0, \boldsymbol{\beta}_1\} , \qquad (2)$$

where $\beta_{1i} = (\delta_{ij} - N_i N_j) n^{-1}\, \partial n_1/\partial r_j$, then the system (1) can be rewritten in the universal form of stochastic nonlinear differential equations. In order to derive the corresponding FPE we use the method developed in ref. [10]. Applying a simple device for reducing the nonlinear problem to the linear case, one can obtain the equation for the probability density $P(u, \sigma) = P(\boldsymbol{r}, \boldsymbol{N}, \sigma)$ that, after having propagated over a distance $\sigma$, the ray has arrived in a point $\boldsymbol{r}$ with a direction $\boldsymbol{N}$:

$$\frac{\partial P(u,\sigma)}{\partial \sigma} = -\frac{\partial}{\partial u_\nu} F_\nu(u) P(u,\sigma) , \quad (\nu = 1, 2, ..., 6). \qquad (3)$$

Following [10] we expand the r.h.s. of (3) in successive powers of $\alpha r_c$.

Disregarding in the expansion terms of order $\alpha^3$ it is a rather straightforward matter to obtain according to ref. [10] the FPE

$$\frac{\partial P(u,\sigma)}{\partial \sigma} = -\frac{\partial}{\partial u_\nu} \{[F_{0\nu}(u) + C_\nu(u)] P(u,\sigma)\} + \frac{1}{2} \frac{\partial^2}{\partial u_\nu \partial u_\mu} [C_{\nu\mu}(u) P(u,\sigma)] ,$$

$$P(u, \sigma = 0) = \delta(u - u^0) . \qquad (4)$$

The coefficients in the diffusion and convection terms are

$$C_{\nu\mu}(u) = 2 \int_0^\infty \langle F_{1\nu}(u) \widetilde{F}_{1\mu}(u^{-\sigma})\rangle \,\mathrm{d}\sigma ,$$

$$C_\nu(u) = \int_0^\infty \langle [\partial F_{1\nu}(u)/\partial u_\mu] \widetilde{F}_{1\mu}(u^{-\sigma})\rangle \,\mathrm{d}\sigma , \qquad (5)$$

where the components of the six-vectors $F_0$ and $F_1$ are defined by (2) and $\widetilde{F}_1 = \{\sigma \boldsymbol{\beta}_1(u^{-\sigma}), \boldsymbol{\beta}_1(u^{-\sigma})\}$.

The six-vector $u^{-\sigma}$ can be defined for fixed $\sigma$ by means of a mapping from the initial $u(0)$ into $u(\sigma)$, i.e. $u \to u^\sigma$ with inverse $(u^\sigma)^{-\sigma} = u$. Using for determination of this mapping the solution of the unperturbed equations (1) in a medium without inhomogeneities, we shall obtain $u^{-\sigma} = \{\boldsymbol{r} - \sigma\boldsymbol{N}, \boldsymbol{N}\}$. But it should be noticed that unlike the method in refs. [7,8] the different directions of $\boldsymbol{N}$ inside the inhomogeneity define the different unperturbed trajectories. So we have no restriction with respect to the angle of scattering.

The basic assumption of our definition of the FPE was

$$\langle F_{1\nu}(u) \widetilde{F}_{1\mu}(u^{-\sigma})\rangle \approx 0 , \quad \text{for} \quad \sigma > r_c , \qquad (6)$$

and similarly for higher cumulants.

The explicit form of the coefficients (5) depends on the statistical properties of the inhomogeneities in a medium. But they are independent of $\boldsymbol{r}$ in the case considered, provided that the random field $n_1(\boldsymbol{r})$ is homogeneous.

For use of discussion we rewrite the main result (4) in the original representation, taking into account (2),

$$(\partial/\partial\sigma + \boldsymbol{N}\nabla_{\boldsymbol{r}}) P(\boldsymbol{r}, \boldsymbol{N}, \sigma) = -\frac{\partial}{\partial N_i} C_{i+3} P + \frac{1}{2} \frac{\partial^2}{\partial r_i \partial N_j} C_{i+3,j} P + \frac{1}{2} \frac{\partial^2}{\partial N_i \partial N_j} C_{i+3,j+3} P ,$$

$$P(\boldsymbol{r}, \boldsymbol{N}, \sigma = 0) = \delta(\boldsymbol{r} - \boldsymbol{r}^0)\delta(\boldsymbol{N} - \boldsymbol{N}^0) . \qquad (7)$$

Integrating (7) over $\boldsymbol{r}$ one can reduce this equation to Chernov's FPE for the angular distribution function, without the restriction in refs. [7,8].

*3. Moments of the distribution.* Here we consider only the simple case that $n_1(\boldsymbol{r})$ is a gaussian homogeneous isomeric random field, viz. $\langle n_1(\boldsymbol{r}) n_1(\boldsymbol{r}')\rangle = \langle n_1^2\rangle \exp(-|\boldsymbol{r} - \boldsymbol{r}'|^2/a^2)$. Then the FPE (7) allows us to obtain closed equations for the successive moments of the joint probability density $P(\boldsymbol{r}, \boldsymbol{N}, \sigma)$ of the position and the direction vector of the rays, which can easily be solved. The solution of this set of equations gives full information about the process in question.

When the ray propagates from the initial point $\boldsymbol{r}^0 = \{0, 0, 0\}$ in the $z$ direction $\boldsymbol{N}^0 = \{0, 0, 1\}$, our results contain some additional terms compared with the analogous expressions in refs. [1–3]. In particular for the mean square rectilinear distance from the initial point of a ray to the point at which it arrives after traversing a complicated path $\sigma$ in a medium we obtain

$$\langle r^2\rangle = \sigma/D_0 - (1/2D_0^2)[1 - \exp(-2D_0\sigma)] + \langle n_1^2\rangle\{4\sigma/D_0 - (2/D_0^2)[1 - \exp(-2D_0\sigma)]\} . \quad (8)$$

Here $D_0 = \sqrt{\pi}\langle n_1^2\rangle/n_0^2 a$, which arises when $C_\nu$ and $C_{\nu\mu}$ (5) are calculated, plays the role of a ray diffusion coefficient [1]. The third term in the r.h.s. of (8) is new, but in the case considered it can be neglected and our result coincides with refs. [1–3].

It is necessary to stress that the set of the closed moment equations can be obtained only for a medium without refraction, i.e. $n_0$ = const, and for the special choice of a random field $n_1(\boldsymbol{r})$.

*4. The FPE in the presence of refraction.* The constant $n_0$ applies, e.g., to the troposphere, but for the ionosphere we have to take $n_0 = n_0(z)$. In that case the exact solution of the unperturbed equations (1) is possible only for some special functions $n_0(z)$. Moreover, such solutions are very complicated and have no use for our purposes. However, it is possible to reduce this case to the former one, if we introduce the new additional small parameter

$$(\mathrm{d}\mu_0/\mathrm{d}z)r_\mathrm{c} \ll 1 . \quad (9)$$

Physically this means that the change of $\mathrm{d}\mu_0/\mathrm{d}z$ over the correlation distance $r_\mathrm{c}$ can be neglected, which is true in many applications.

Our main result (4) holds good with the substitution $F_0 = \{N, \boldsymbol{\beta}_0\}$ instead of (2) under the condition (9). But now the coefficients (5) have the spatial dependence even in the case of a homogeneous field $n_1(\boldsymbol{r})$. In order to reduce (4) to the FPE for the angular distribution alone, we have to assume the angle of ray scattering to be small. To put it differently, we assume that the small fluctuations of $\boldsymbol{N}$ inside inhomogeneities do not change the position of the ray, which is defined only by the unperturbed trajectory $\sigma_0$ with the given initial conditions. This permits us to consider the sure part of the refractive index $n_0$ as a function of $\sigma_0$ and to integrate eq. (4) over $\boldsymbol{r}$. This is a generalization of the Klyatskin and Tatarski approximation for the case of the stratified medium.

As mentioned above in the case considered in this section the equations for the moments of the distribution are not closed. Hence for obtaining the information of the ray scattering process it is necessary to use other methods for solving the FPE.

*5. Discussion.* First of all we discuss the conditions for applicability of the FPE (4). We expanded the r.h.s. of (3) in $\alpha r_\mathrm{c}$. Here $\alpha^{-1} = \Delta L$ is the scale on which $u$ varies and $r_\mathrm{c}$ is the scale on which the random nature of the function $n_1(\boldsymbol{r})$ becomes appreciable, i.e. relation (5) is valid. If $\alpha r_\mathrm{c}$ is small ($\alpha r_\mathrm{c} \ll 1$ and as a consequence $|n_1| \ll n_0$) it is possible to subdivide the path length in intervals $\Delta\sigma$, such that $\Delta\sigma \gg r_\mathrm{c}$, and yet $\alpha\Delta\sigma \ll 1$. That is, $u$ does not vary much during an interval $\Delta\sigma$ in which $n_1$ has forgotten its past. Combining these inequalities

$$\Delta L \gg \Delta\sigma \gg r_\mathrm{c} , \quad (10)$$

we can conclude that on the coarse-grained level determined by $\Delta\sigma$ the process is (approximately) markovian [10,11]. Hence the application of the FPE (4) in a medium without refraction is justified under the condition (10) only. In the presence of refraction one must take into account the condition (9).

We are now in a position to compare briefly our results with previous work. The FPE (4) describing the process of ray scattering in a random isotropic medium includes all equations considered earlier, both for a medium without refraction [1–3,7,8] and in a stratified medium [5,6]. In the former case with a gaussian homogeneous random field $n_1(\boldsymbol{r})$ we obtained the full set of the moments, which cover the expressions in refs. [1–3] for the linear displacement of the ray (8) and coincides with the results due to Klyatskin and Tatarski [7,8] in the small-angle limit ($D_0\sigma \ll 1$).

The second term in the r.h.s. in (7) is new; it was lost in ref. [3] and could not been obtained in refs. [7,8] as a consequence of their approximation. It reflects that for different values of $\boldsymbol{N}$ one has to take into account different rays when computing the correlations that enter into (5). To put it otherwise, we do not ignore the deviation of the actual ray from the unperturbed one. This term has the same order with regards to the small parameter of our expansion as other terms in the r.h.s. of (7). Hence it has to be taken into consideration in a systematic derivation of the FPE in question.

A more detailed discussion of the present approach will be given elsewhere, whereas a report on its application to the much more involved case of propagation of the ray in random gyrotropic media is currently in preparation.

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