A Combinatory Algebra for Sequential Functionals of Finite Type

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Abstract

It is shown that the type structure of finite-type functionals associated to a combinatory algebra of partial functions from $\mathbb{N}$ to $\mathbb{N}$ (in the same way as the type structure of the countable functionals is associated to the partial combinatory algebra of total functions from $\mathbb{N}$ to $\mathbb{N}$), is isomorphic to the type structure generated by object N (the flat domain on the natural numbers) in Ehrhard's category of "dI-domains with coherence", or his "hypercoherences".

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Introduction

PCF, "Gödel's T with unlimited recursion", was defined in Plotkin's paper [16]. It is a simply typed λ-calculus with a type o for integers and constants for basic arithmetical operations, definition by cases and fixed point recursion. More importantly, there is a special reduction relation attached to it which ensures (by Plotkin's "Activity Lemma") that all PCF-definable higher-type functionals have a sequential, i.e. non-parallel evaluation strategy. In view of this, the obvious model of Scott domains is not faithful, since it contains parallel functions. A search began for "fully abstract" domain-theoretic models for PCF.

A proliferation of ever more complicated theories of domains saw the light, inducing the father of domain theory, Dana Scott, to lament that "there are *too many* proposed categories of domains and [...] their study has become too arcane" ([17]), a judgement with which it is hard to disagree.

Although most interest in the semantics of PCF was shown by computer scientists, it became clear that there is an important overlap with higher-type recursion theory as it was recognized (I believe, initially by Robin Gandy, whose insights were transmitted by Martin Hyland and partially laid down in the paper [9]) that Kleene's late attempts ([10, 11, 12, 13]) to formalize the notion of a recursive functional of higher type, had much in common with the "full abstraction problem" for PCF. As far as I am aware however, the *exact* relationship between Kleene's work and the work on PCF still remains to be clarified.

An important model of Kleene's axioms is provided by the so-called "continuous (or countable) functionals" (see, e.g., [15]). They arise, in a standard way, as the type structure coming from the partial combinatory algebra of "Kleene's function realizability" (introduced in [14]). This is a partial combinatory algebra structure on the set of functions from $\mathbb{N}$ to $\mathbb{N}$.

A surprising result of this paper is, that a natural generalization of function realizability to *partial* functions from $\mathbb{N}$ to $\mathbb{N}$ (yielding a *total* combinatory algebra), gives a type structure of higher-type functionals which coincides with the relevant part of Ehrhard and Bucciarelli's "dI-domains with coherence" ([3, 4, 2]).

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This could be interesting for a number of reasons. First, it provides another handle on Ehrhard's work, which is complicated and rather heavily loaded with definitions; however, the fact that dI-domains with coherence have a completely independent generation process (which process is well known in logic), seems to me to enhance their naturalness as a mathematical structure. Of course, the result in this paper calls for comparison with the result in [3], viz. that dI-domains with coherence are the extensional collapse of another domain-theoretic structure, sequential structures and sequential algorithms. My result is essentially different in that it relates the dI-domains with coherence to something which is defined independently of any domain theory. But it might be conjectured that the sequential algorithms, or the part of it that is relevant to PCF, can be obtained as a kind of intensional type structure on the combinatory algebra considered here.

Secondly, it shows that Ehrhard's "strongly stable" model of PCF lives inside a realizability topos where its domain structure is *intrinsic*. This should be of interest to Synthetic Domain Theory ([7]).

Thirdly it raises the question whether maybe more models of PCF (including the fully abstract game models of [9] and [1])can be induced in this way by combinatory algebras.

Finally I should admit to an oversight: one of the stumbling blocks for me in carrying out the analysis reported in this paper, was my initial failure to recognize the importance of the stable order (an error which is almost incomprehensible in view of the fact that the game-theoretic flavour of it was directly inspired by [9], who mention that their structure is dI-domain-enriched). Now I am convinced that no *useful* attempt at unifying domain theory (as proposed in [17]) can leave stable domains out of consideration.

1 Sequential Functions

We are interested in the following game between partial functions α, $\beta : \mathbb{N} \to \mathbb{N}$. α asks, successively, values of β at given arguments; the game has no outcome if β is undefined at one of these numbers, or if α has no further move; but α may also decide, at some point, that now it has sufficient information about β, and outputs not a question, but an answer.

Formally, we define:

Definition 1.1 *A sequence $u = \langle u_0, \ldots, u_{n-1}\rangle$ (coded as a natural number) is called a* dialogue *between α and β if for all i with $0 \leq i < n \Leftrightarrow 1$, writing $u^{<i}$ for $\langle u_0, \ldots, u_{i-1}\rangle$, there is j such that*

$$\alpha(u^{<i}) = 2j \text{ and } \beta(j) = u_i$$

We say that the application $\alpha|\beta$ is defined with value n, or $\alpha|\beta = n$, if there is a dialogue u between α and β such that

$$\alpha(u) = 2n + 1$$

Of course, we read $u^{<0}$ as the empty sequence. Note, that dialogues are unique: given α and β, there is a unique (finite or infinite) dialogue between α and β.

Let $\mathcal{B}$ be the set of all partial functions from $\mathbb{N}$ to $\mathbb{N}$; then every $\alpha \in \mathcal{B}$ determines a partial function F_α from $\mathcal{B}$ to $\mathbb{N}$ by

$$F_\alpha(\beta) = n \text{ iff } \alpha|\beta = n$$

Giving $\mathcal{B}$ the topology with as subbase the collection of all

$$\mathcal{U}_p = \{\alpha \,|\, p \subseteq \alpha\}$$

for p *finite*, and $N_\perp = \mathbb{N} \cup \{\perp\}$ the topology which is discrete on $\mathbb{N}$ and has $N_\perp$ as only neighborhood of $\perp$, every F_α, considered as total function $\mathcal{B} \to N_\perp$, is continuous; but clearly, not every continuous $F : \mathcal{B} \to N_\perp$ is of the form F_α.

Examples The functions:

i) $F(\alpha) = \begin{cases} 0 & \text{if } \alpha \neq \emptyset \\ \perp & \text{else} \end{cases}$

ii) $F(\alpha) = \begin{cases} 0 & \text{if } \sum_{x \in \mathrm{dom}(\alpha), x \leq 1} \alpha(x) \geq 1 \\ \perp & \text{else} \end{cases}$

iii) $F(\alpha) = \begin{cases} 1 & \text{if } \sum_{x \in \mathrm{dom}(\alpha), x \leq 1} \alpha(x) \geq 1 \\ 0 & \text{if } \alpha(0) = \alpha(1) = 0 \\ \perp & \text{else} \end{cases}$

are all continuous, but not given as F_α.

In order to study the set of functions $\mathcal{B} \to N_\perp$ that are given by some $\alpha \in \mathcal{B}$, it is useful to consider two partial orders on this set: the *pointwise* order is defined by: $F \leq_{\mathrm{pw}} G$ iff for all $\alpha \in \mathcal{B}$ and all $n \in \mathbb{N}$: if $F(\alpha) = n$ then $G(\alpha) = n$.

The *stable* order is defined by: $F \leq_s G$ iff for all $\alpha, \beta \in \mathcal{B}$: if $\alpha \subset \beta$ then $F(\alpha) = n$ if and only if $F(\beta) = G(\alpha) = n$. Clearly, since F and G are continuous, $F \leq_s G$ implies $F \leq_{\mathrm{pw}} G$.

Every continuous $F : \mathcal{B} \to N_\perp$ has a unique *base*, that is a minimal set B of finite functions such that for all α and all n: $F(\alpha) = n$ iff there is a $p \in B$ such that $p \subset \alpha$.

Definition 1.2 *A* sequential tree *is a tree T of finite functions (ordered by $\subseteq$, so the root is the empty function) such that for each $p \in T$ there is $n \in \mathbb{N}$ such that all immediate successors q of p in T have* $\mathrm{dom}(q) = \mathrm{dom}(p) \cup \{n\}$.

Clearly:

Proposition 1.3 *A function $F : \mathcal{B} \to N_\perp$ is of the form F_α for some $\alpha \in \mathcal{B}$, iff its base is the set of leaves of a sequential tree.*

With any F_α therefore we can associate a sequential tree with set of leaves B, and a function $v : B \to \mathbb{N}$. Conversely every such pair (B, v) defines a function F which is given as F_α for some (non-unique) α. We call the pair (B, v) the *trace* of F. This is in harmony with usage in the literature of this term, cf. [5].

When is a base B of a continuous function the set of leaves of a sequential tree? Answer:

Proposition 1.4 *B is the set of leaves of a sequential tree if and only if for each nonempty finite subset B' of B we have: if $p \subseteq \bigcap B'$ then either $B' = \{p\}$ or $\bigcap_{q \in B'} \mathrm{dom}(q \setminus p) \neq \emptyset$.*

This proposition doesn't seem very informative, but yields at once:

Corollary 1.5 *Let $(B_i \mid i \in I)$ be a directed system of sets of finite functions such that each B_i is the set of leaves of a sequential tree. Then $\bigcup_{i \in I} B_i$ is the set of leaves of a sequential tree.*

From now on, we shall call functions $F : \mathcal{B} \to N_\perp$ which are given as F_α, *sequential* functions.

Let F, G be two sequential functions, with traces (B_F, v_F) and (B_G, v_G).

Proposition 1.6 *i) $F \leq_{\mathrm{pw}} G$ iff for every $p \in B_F$ there is $q \in B_G$ with $q \subseteq p$ and $v_G(q) = v_F(p)$;*

ii) $F \leq_s G$ iff $B_F \subseteq B_G$ and v_F and v_G coincide on B_F;

iii) $\alpha \subseteq \beta$ implies $F_\alpha \leq_s F_\beta$

Definition 1.7 *In a partially ordered set $(D, \leq)$ we say that d is the* least upper bound *(or* lub*, or* join*, or* supremum*) of $A \subseteq D$ if d is least such that $\forall a \in A. a \leq d$. Write $d = \bigvee A$.*

We say A is bounded *if it has an upper bound in D.*

We say that $d \in D$ is compact *if for every directed $I \subseteq D$, if $d \leq \bigvee I$ then $\exists i \in I. d \leq i$.*

*D is called ω-*algebraic *if the set of compact elements of D is countable and for each $d \in D$, the set $\{k \in D \mid k$ compact $\wedge\, k \leq d\}$ is directed and has d as least upper bound.*

D has the I-property *if for every compact* $k \in D$, $\{d \in D | \, d \leq k\}$ *is finite.*

D *is* distributive *if* $(x \vee x') \wedge y = (x \wedge y) \vee (x' \wedge y)$ *whenever these lubs exist* ($x \vee x' = \bigvee\{x, x'\}$ *and* $x \wedge x' = \bigvee\{y | \, y \leq x$ *and* $y \leq x'\}$*).*

D *is a* dI-domain *if it has least upper bounds of directed subsets and of bounded subsets, is distributive and* ω*-algebraic, and has the I-property.*

The conclusion is:

Proposition 1.8 *The set of sequential functions* $\mathcal{B} \to N_\perp$*, with the stable order, is a dI-domain.*

Moreover, the set of sequential functions $\mathcal{B} \to N_\perp$ is *atomic*, which means that every element is the supremum of the atoms below it (an atom is a non-bottom element which has no non-bottom elements stricly below it). Atomic dI-domains are known in the literature as *qualitative domains*([5]).

2 A type structure of sequential functionals

In this section we restrict ourselves to the following types: o is a type; if σ is a type, then $\sigma \to o$ is a type.

To every such type we assign a set $\mathcal{O}_\sigma$ of *sequential functionals* of type σ, and to any $f \in \mathcal{O}_\sigma$ a nonempty set $\mathrm{Ass}(f) \subset \mathcal{B}$, the set of *associates* of f. The definition is:

$\mathcal{O}_o = N_\perp$; $\mathrm{Ass}(n) = \{\alpha | \, \alpha(0) = n\}$ and $\mathrm{Ass}(\perp) = \{\alpha | \, 0 \notin \mathrm{dom}(\alpha)\}$. $\mathcal{O}_{\sigma \to o}$ consists of those functions $f : \mathcal{O} \to N_\perp$ such that there is a $\beta \in \mathcal{B}$ such that for all $x \in \mathcal{O}_\sigma$ and all $\alpha \in \mathrm{Ass}(x)$, $\beta | \alpha = n$ if and only if $f(x) = n$, and $\mathrm{Ass}(f)$ is the set of β satisfying this condition. We write $\mathrm{Ass}(\sigma)$ for the set of associates of elements of $\mathcal{O}_\sigma$.

By an easy induction on σ, if $\alpha \in \mathrm{Ass}(\sigma)$ there is a unique $f \in \mathrm{O}_\sigma$ with $\alpha \in \mathrm{Ass}(f)$; this f is denoted by $[\alpha]$ (strictly speaking, $[\alpha]$ depends on the type, but there will never be ambiguity). We write $\alpha \sim \alpha'$ if $[\alpha] = [\alpha']$, for $\alpha, \alpha' \in \mathrm{Ass}(\sigma)$.

Again, on $\mathcal{O}_\sigma$ one can define the pointwise and stable orders: on $N_\perp$, we have $\perp \leq n$ for all n, and that is all for both orders; on $\mathcal{O}_{\sigma \to o}$, $f \leq_{\mathrm{pw}} g$ iff for all $x \in \mathcal{O}_\sigma$, $fx = n$ implies $gx = n$; and $f \leq_s g$ iff for all $x \leq_s y \in \mathcal{O}_\sigma$, $fx = n \Leftrightarrow gx = fy = n$.

We extend the notation $\leq_s$ to associates and say: $\alpha \leq_s \beta$ if $[\alpha] \leq_s [\beta]$. In general on associates, $\leq_s$ can at most be a preorder. Whether $\leq_s$ is reflexive or not is equivalent to whether elements of $\mathcal{O}_{\sigma \to o}$ are monotone w.r.t. $\leq_s$. Associates are, of course, also ordered by inclusion, so the question arises what the relation between these orders is. These matters will be resolved by the following theorem.

Some more terminology: elements x, y of $\mathcal{O}_\sigma$ are called *compatible* if they have a common upper bound w.r.t. $\leq_s$. Similar for associates.

Theorem 2.1 *i) If* $\sigma = \tau \to o$ *and* $f \in \mathcal{O}_\sigma$ *then* f *is monotone w.r.t. the stable order on* $\mathcal{O}_\tau$*. In particular,* $\leq_s$ *is a partial order on* $\mathcal{O}_\sigma$*;*

ii) If $\gamma \in \mathrm{Ass}(\sigma)$ *and* $q \subset \gamma$ *is finite, there is an element* $\mu_\sigma(q)$ *of* $\mathrm{Ass}(\sigma)$ *with the properties:*

- *a)* $q \subset \mu_\sigma(q)$*;*
- *b)* $\mu_\sigma(q) \leq_s \gamma'$ *whenever* $\gamma' \in \mathrm{Ass}(\sigma)$ *and* $q \subset \gamma'$*;*
- *c) if* $\mu_\sigma(q) \leq_s \gamma'$ *for* $\gamma' \in \mathrm{Ass}(\sigma)$*, there is a finite* $q' \subset \gamma'$ *such that* $\mu_\sigma(q) \leq_s \mu_\sigma(q')$*;*
- *d)* $[\mu_\sigma(q)]$ *is a compact element of* $\mathcal{O}_\sigma$*.*

iii) For $\alpha, \beta \in \mathrm{Ass}(\sigma)$*,* $\alpha \subseteq \beta$ *implies* $\alpha \leq_s \beta$*, and* $f \leq_s g$ *in* $\mathcal{O}_\sigma$ *implies that for every* $\beta \in \mathrm{Ass}(g)$ *there is* $\alpha \in \mathrm{Ass}(f)$ *with* $\alpha \subseteq \beta$*;*

iv) If $x, y \in \mathcal{O}_\sigma$ *are compatible then their meet* $x \wedge y$ *exists;*

v) *If $\sigma = \tau \to o$, $f \in \mathcal{O}_\sigma$, then f preserves meets of compatible elements: for $x, y \in \mathcal{O}_\tau$ compatible, $f(x \wedge y) = n$ iff $fx = fy = n$.*

Proof. The proof is somewhat involved; it is a simultaneous induction on the type σ.

For $\sigma = o$, the first part of i) is vacuous and it's clear that $\leq_s$ is a partial order on $\mathcal{O}_o$; ii) take $\mu_o(q) = q$; the rest is left to the reader; iii) and iv) are obvious, and v) is vacuous.

Now let $\sigma = \tau \to o$.

i). If $f \in \mathcal{O}_\sigma$ and $x \leq_s y$ in $\mathcal{O}_\tau$ then since f has an associate γ and, by the induction hypothesis of iii), every associate β of y contains an associate α of x, $fx = n \Leftrightarrow \gamma|\alpha = n \Rightarrow \gamma|\beta = n \Leftrightarrow fy = n$. So, f is monotone and $f \leq_s f$; clearly then, $\leq_s$ is a partial order on $\mathcal{O}_\sigma$.

ii). Let q be finite such that $q \subset \gamma$ for some $\gamma \in \mathrm{Ass}(\sigma)$. There is a finite set E of finite functions p such that $p \subset \beta$ for some $\beta \in \mathrm{Ass}(\tau)$, and $q|p$ is defined. If $E = \emptyset$, we simply put $\mu_\sigma(q) = q$; in that case it's clear that q itself is an associate of the function $\lambda x.\bot$. Assume now that $E \neq \emptyset$; let $E' = \{p_1, \ldots, p_n\} \subseteq E$ be such that for each $p \in E$ there is a unique $p_i \in E'$ with $\mu_\tau(p_i) \leq_s \mu_\tau(p)$. For a finite function r we put

$$\Delta_r = \{\delta \in \mathrm{Ass}(\tau) |\, r \subset \delta \text{ and for some } i, \mu_\tau(p_i) \leq_s \delta\}$$

Define $, = \mu_\sigma(q)$ as follows:

$$,(u) = \begin{cases} q(u) & \text{if } u \in \mathrm{dom}(q) \\ \text{undefined} & \text{if } u \text{ is not a dialogue between } , \text{ and} \\ & \text{some finite function } r, \text{ or if } \Delta_r = \emptyset \\ 2k+1 & \text{if there is a } \gamma \in \mathrm{Ass}(\sigma) \text{ with } q \subset \gamma,\ \gamma|r = k \text{ and} \\ & \text{there is } r' \supseteq r \text{ and } i \text{ with } \mu_\tau(p_i) \leq_s \mu_\tau(r') \\ 2l & \text{for } l = \min(\bigcap_{\delta \in \Delta_r} \mathrm{dom}(\delta \setminus q)) \text{ else, if} \\ & \bigcap_{\delta \in \Delta_r} \mathrm{dom}(\delta \setminus q) \neq \emptyset \\ \text{undefined} & \text{else} \end{cases}$$

First, let us remark that the case $\Delta_r \neq \emptyset, \bigcap_{\delta \in \Delta_r} \mathrm{dom}(\delta \setminus r) = \emptyset$ can only apply if $\Delta_r = \{r\}$ since if $q \subset \gamma \in \mathrm{Ass}(\sigma)$, $\gamma|\delta$ is defined for all $\delta \in \Delta_r$ (by induction hypothesis i), since $\gamma|p_i$ is defined, hence $\gamma|\mu_\tau(p_i)$ is defined).

A second remark is, that if $\gamma|r = k$ for $\gamma \in \mathrm{Ass}(\sigma)$ with $q \subset \gamma$, and $r' \supseteq r$ is such that $\mu_\tau(p_i) \leq_s \mu_\tau(r')$, we must already have that $\mu_\tau(p_i) \leq_s \mu_\tau(r)$; because in that case, by induction hypothesis ii), $\mu_\tau(p_i)$ and $\mu_\tau(r)$ are compatible and $\gamma|\mu_\tau(p_i) = \gamma|\mu_\tau(r) = k$; by induction hypothesis there is an associate ε of their meet, with $\varepsilon \subseteq \mu_\tau(p_i)$, and $\gamma|\varepsilon = k$; but then, $p_i \subseteq \varepsilon$ (because $q|p_i = k$, $q \subseteq \gamma$ and $p_i \subseteq \mu_\tau(p_i)$); hence $\mu_\tau(p_i) \leq_s \varepsilon$ and so, $\mu_\tau(p_i) \leq_s \mu_\tau(r)$.

Therefore, $,|\delta = k$ if and only if there is $r \subseteq \delta$, r finite, and i with $q|p_i = k$ and $\mu_\tau(p_i) \leq_s \mu_\tau(r)$ (note, that in a dialogue u between γ and r, if $\gamma(u) = 2l$ and $l \notin \mathrm{dom}(r)$, $l \in \bigcap_{\delta \in \Delta_r} \mathrm{dom}(\delta \setminus r)$ will always hold, so in a continuing dialogue between $,$ and δ, these questions will ultimately be posed). Now if $,|\delta = k$ and $\delta \sim \delta'$, there is $r \subseteq \delta$ with $\mu_\tau(p_i) \leq_s \mu_\tau(r) \leq_s \delta'$ hence by induction hypothesis there is $r' \subseteq \delta'$ with $\mu_\tau(p_i) \leq_s (r')$; so $,|\delta' = k$. This proves that $, \in \mathrm{Ass}(\sigma)$.

Suppose $q \subseteq \gamma$, $\gamma \in \mathrm{Ass}(\sigma)$. Let $x \leq_s y \in \mathcal{O}_\tau$. By induction hypothesis iii) there are associates $\theta \subseteq \zeta$ of x, y respectively. If $,|\zeta = \gamma|\theta = k$ let $q_1 \subseteq \zeta$, $q_2 \subseteq \theta$ finite with $,|q_1 = \gamma|q_2 = k$. Since $\mu_\tau(p_i) \leq_s \mu_\tau(q_1 \cup q_2)$ for some i, we have that $,|\theta = k$; so, $, \leq_s \gamma$.

We still have to prove that if $, \leq_s \gamma \in \mathrm{Ass}(\sigma)$, there is a finite part r of γ such that $, \leq_s \mu_\sigma(r)$. For this, it is sufficient to note that for $\gamma \in \mathrm{Ass}(\sigma)$ the following two conditions are equivalent:

a) $, \leq_s \gamma$

b) for all i, $\gamma|\mu_\tau(p_i) = q|p_i$, and if p is such that $\mu_\tau(p)$ is compatible with $\mu_\tau(p_i)$ and $\gamma|p = \gamma|p_i$, then $\mu_\tau(p_i) \leq_s \mu_\tau(p)$.

For a)$\Rightarrow$b), that $\gamma|\mu_\tau(p_i) = q|p_i$ is clear, and if $\mu_\tau(p)$ is compatible with $\mu_\tau(p_i)$, and x their meet in $\mathcal{O}_\tau$, and $\mu_\tau(p_i) \not\leq_s \mu_\tau(p)$, then x is strictly below $[\mu_\tau(p_i)]$, and $[,](x)$ is undefined but $[\gamma](x)$ is defined (induction hypothesis!); contradiction with $, \leq_s \gamma$.

For b)$\Rightarrow$a), if (using induction hypothesis) $\theta \subseteq \zeta$ are in $\mathrm{Ass}(\tau)$ and , $|\zeta = \gamma|\theta = k$, there are finite $r_1 \subseteq \zeta$, $r_2 \subseteq \theta$ with , $|r_1 = \gamma|r_2$. Then there is i with $\mu_\tau(p_i) \leq_s \mu_\tau(r_1)$, and $\mu_\tau(r_1)$ and $\mu_\tau(r_2)$ are compatible since $\theta \subseteq \zeta$. By b), $\mu_\tau(p_i) \leq_s \mu_\tau(r_2)$, so , $|\theta = k$. The other implication is left to the reader.

Now if , $\leq_s \gamma \in \mathrm{Ass}(\sigma)$ there is a finite $r \subseteq \gamma$ such that for all i, $r|\mu_\tau(p_i) = q|p_i$. Since the second condition of b) clearly remains true if we replace γ by something $\leq_s \gamma$, we have , $\leq_s \mu_\sigma(r)$.

By a similar argument, left to the reader, $[\mu_\sigma(q)]$ is compact in $\mathcal{O}_\sigma$.

iii). If $\alpha, \beta \in \mathrm{Ass}(\sigma)$ with $\alpha \subseteq \beta$, and $x \leq_s y \in \mathcal{O}_\tau$ then by induction hypothesis every associate ζ of y contains an associate θ of x, hence, $[\alpha](x) = n$ iff $\alpha|\theta = n$ iff $\alpha|\zeta = \beta|\theta = n$ iff $[\alpha](y) = [\beta](x) = n$; so $\alpha \leq_s \beta$.

Now let $f \leq_s g \in \mathcal{O}_\sigma$ and $\beta \in \mathrm{Ass}(g)$. We define $\alpha \subseteq \beta$ by stipulating that $u \in \mathrm{dom}(\alpha)$ iff the following hold:

a) u is a dialogue between β and some finite function q;

b) there is $\gamma \in \mathrm{Ass}(\tau)$ with $q \subseteq \gamma$ and $f([\gamma]) \neq \bot$;

c) if $\beta(u) = 2k+1$ then we must have: for all $\gamma \in \mathrm{Ass}(\tau)$, if $q \subseteq \gamma$ then $f([\gamma]) = k$.

Then for $\gamma \in \mathrm{Ass}(\tau)$, the implication $\alpha|\gamma = k \Rightarrow f([\gamma]) = k$ clearly follows. For the converse, if $f([\gamma]) = k$ then certainly $\beta|\gamma = k$ so $\beta|q = k$ for finite $q \subseteq \gamma$. The only way that $\alpha|\gamma = k$ can fail to hold is that there is another $\gamma' \in \mathrm{Ass}(\tau)$ with $q \subseteq \gamma'$, and $f([\gamma']) \neq k$. Then $f([\gamma']) = \bot$ since $f \leq_s g$. For $\mu_\tau(q)$ from induction hypothesis ii) however, we have $\beta|\mu_\tau(q) = k$ because $q \subseteq \mu_\tau(q)$, and $f([\mu_\tau(q)]) = \bot$ by i), since $\mu_\tau(q) \leq_s \gamma'$. But also $\mu_\tau(q) \leq_s \gamma$, and we obtain a contradiction with $f \leq_s g$.

iv) and **v)** are now easy: if x, y are compatible with upper bound z, let $\gamma \in \mathrm{Ass}(z)$ and pick associates α, β for x, y with $\alpha, \beta \subseteq \gamma$. Then $\alpha \cap \beta$ is an associate of $x \wedge y$, and such meets are clearly respected by any $f \in \mathcal{O}_{\sigma \to o}$. ∎

From this theorem we shall obtain a series of corollaries, which culminates in the theorem that every $\mathcal{O}_\sigma$ is a qualitative domain, and that every $f \in \mathcal{O}_{\sigma \to o}$ is a so-called *stable* function (theorem 2.8).

Corollary 2.2 *Let $\sigma = \tau \to o$. Then to any $f \in \mathcal{O}_\sigma$ an associate β_f can be assigned in such a way, that $f \leq_s g$ if and only if $F_{\beta_f} \leq_s F_{\beta_g}$ as sequential functions: $\mathcal{B} \to N_\bot$.*

Proof. Define β_f by:

$$\beta_f(u) = \begin{cases} \text{undefined} & \text{if } u \text{ is not a dialogue between } \beta_f \text{ and some} \\ & \text{finite function } q, \text{ or if there is no } \gamma \in \mathrm{Ass}(\tau) \cap \mathcal{U}_q \\ & \text{for which } f([\gamma]) \neq \bot; \\ 2k+1 & \text{if } f([\gamma]) = k \text{ for all } \gamma \in \mathrm{Ass}(\tau) \cap \mathcal{U}_q; \\ 2k & \text{for } k = \min(\bigcap\{\mathrm{dom}(\gamma \setminus q) \,|\, \gamma \in \mathrm{Ass}(\tau) \cap \mathcal{U}_q,\ f([\gamma]) \neq \bot\}), \text{ else} \end{cases}$$

If $\delta \in \mathrm{Ass}(f)$, u a dialogue between δ and q, and p a finite part of some $\gamma \in \mathrm{Ass}(\tau) \cap \mathcal{U}_q$ such that $f([\gamma]) \neq \bot$, then if $\delta(u) = 2l$, either $l \in \mathrm{dom}(p)$ or

$$l \in \bigcap\{\mathrm{dom}(\gamma' \setminus p) \,|\, \gamma' \in \mathrm{Ass}(\tau) \cap \mathcal{U}_p, f([\gamma']) \neq \bot\}$$

Therefore, if $f([\gamma]) = k$ then $\beta_f|\gamma = k$. The converse is obvious, so $\beta_f \in \mathrm{Ass}(f)$.

Now suppose $f \leq_s g$ in $\mathcal{O}_\sigma$ and $\theta \subseteq \zeta$ in $\mathcal{B}$. Definitely if $\beta_f|\theta = k$ then $\beta_g|\theta = \beta_f|\zeta = k$; conversely if $\beta_g|\theta = \beta_f|\zeta = k$ there are q_1, q_2 finite, $q_1 \subseteq \zeta$, $q_2 \subseteq \theta$ such that f is constant k on $\mathrm{Ass}(\tau) \cap \mathcal{U}_{q_1}$ and g is constant k on $\mathrm{Ass}(\tau) \cap \mathcal{U}_{q_2}$ (and the sets $\mathrm{Ass}(\tau) \cap \mathcal{U}_{q_i}$ are nonempty).

Then $f([\mu_\tau(q_1 \cup q_2)]) = g([\mu_\tau(q_2)]) = k$ hence by $f \leq_s g$, $f([\mu_\tau(q_2)]) = k$ hence f is constant on $\mathrm{Ass}(\tau) \cap \mathcal{U}_{q_2}$, hence $\beta_f|\theta = k$.

This shows that $f \leq_s g$ implies $F_{\beta_f} \leq_s F_{\beta_g}$ as sequential functions; conversely, since for $x \leq_s y$ in $\mathcal{O}_\tau$ there are associates $\theta \subseteq \zeta$ of x, y respectively, we have

$$fy = gx = k \Leftrightarrow F_{\beta_f}|\zeta = F_{\beta_g}|\theta = k \Leftrightarrow F_{\beta_f}|\theta = k \Leftrightarrow fx = k$$

∎

Corollary 2.3 *$\mathcal{O}_\sigma$ has directed joins for all σ, and they are preserved by any $f \in \mathcal{O}_{\sigma\to o}$.*

Proof. The first statement is a straightforward combination of the previous corollary and the theorem that the set of sequential functions has directed joins. To see that they are respected by $f \in \mathcal{O}_{\sigma\to o}$, suppose $I \subseteq \mathcal{O}_\sigma$ directed, $\gamma \in \mathrm{Ass}(\bigvee I)$, $f(\bigvee I) = k$. There is $q \subseteq \gamma$ finite such that $f([\mu_\sigma(q)]) = k$; since $[\mu_\sigma(q)]$ is compact there is $i \in I$ with $f(i) = k$. ∎

Corollary 2.4 *$\mathcal{O}_\sigma$ has joins of bounded subsets.*

Proof. Let $A \subseteq \mathcal{O}_\sigma$ have upper bound z with associate γ; pick for each $a \in A$ an associate $\gamma_a \subseteq \gamma$. We may assume that each γ_a is only defined on dialogues. Then $\bigcup(\{\gamma_a | \, a \in A\}) \in \mathrm{Ass}(\sigma)$ and is an associate of the pointwise join of A. ∎

Corollary 2.5 *If $A \subseteq \mathcal{O}_\sigma$ is nonempty and bounded, then $\bigwedge A$ exists and is preserved by any $f \in \mathcal{O}_{\sigma\to o}$.*

Proof. That $\bigwedge A$ exists follows from the previous corollary. But, in the notation of that proof, $\bigcap\{\gamma_a | \, a \in A\}$ is an associate of $\bigwedge A$. If $\beta|\gamma_a = k$ for all a, then $\beta|\bigcap_{a\in A}\gamma_a = k$. So $f(\bigwedge A) = k$ iff for all $a \in A$, $f(a) = k$. ∎

Corollary 2.6 *$\mathcal{O}_\sigma$ is distributive.*

Proof. To show that $(x \vee x') \wedge y \leq (x \wedge y) \vee (x' \wedge y)$ (the other inequality always holds), we may assume that $x \vee x'$ and y are compatible (otherwise we replace y by $(x \vee x') \wedge y$); then the statement follows from ordinary distributivity of $\cap$ over $\cup$. ∎

Corollary 2.7 *$\mathcal{O}_\sigma$ has the I-property.*

Proof. For $\sigma = o$ this is trivial, and for $\sigma = \tau{\to}o$, first since for every $x \in \mathcal{O}_\sigma$, $x = \bigvee\{[\mu_\sigma(q)] | \, q \subseteq \gamma\}$ for any $\gamma \in \mathrm{Ass}(x)$ and this join is directed, every compact element c of $\mathcal{O}_\sigma$ is less than some $\mu_\sigma(q)$. For $\mu_\sigma(q)$ there is a finite set $\{p_1, \ldots p_n\}$ such that if $\mu_\sigma(q)|\delta = k$ then $\mu_\tau(p_i) \leq_s \delta$ for some i. So if $c([\delta]) = k$ then by $c \leq_s [\mu_\sigma(q)]$ we have $c([\mu_\tau(p_i)]) = k$. So c determines a subset of $\{p_1, \ldots, p_n\}$ on which it is defined. Hence $c = \mu_\sigma(q')$ for some q', and there are only finitely many elements of $\mathcal{O}_\sigma$ below $[\mu_\sigma(q)]$. ∎

Let us summarize:

Theorem 2.8 *Every $\mathcal{O}_\sigma$ is a dI-domain, and every $f \in \mathcal{O}_{\sigma\to o}$ is a* stable *function, meaning that it preserves directed joins and meets of nonempty bounded subsets.*

Moreover, from the proof of corollary 2.7 it follows that every $\mathcal{O}_\sigma$ is atomic, hence a qualitative domain. Note, that this gives another proof of corollary 2.5, since a stable function between qualitative domains automatically preserves meets of nonempty bounded subsets.

Now we want to characterize the structure $\{\mathcal{O}_\sigma | \, \sigma \text{ a type}\}$ as subcategory of the category of qualitative domains and stable functions. Clearly, not every stable function from $\mathcal{O}_\sigma$ to $N_\perp$ is an element of $\mathcal{O}_{\sigma\to o}$.

Example. Let f_1, f_2, f_3 be the partial functions:

$$\begin{array}{lll} f_1(0) = 0 & f_1(1) = 0 & \\ & f_2(1) = 1 & f_2(2) = 0 \\ f_3(0) = 1 & & f_3(2) = 1 \end{array}$$

Note that f_1, f_2, f_3 are pairwise incompatible but $\{f_1, f_2, f_3\}$ is not the set of leaves of a sequential tree. Considering f_1, f_2, f_3 as elements of $\mathcal{O}_{o\to o}$, there is a stable function $\phi : \mathcal{O}_{o\to o} \to \mathcal{O}_o$ with basis $\{f_1, f_2, f_3\}$, but ϕ is not an element of $\mathcal{O}_{(o\to o)\to o}$.

In the next section we shall see that the sequential functionals as defined here, are part of a structure known in the literature, namely Ehrhard's *strongly stable model*([3, 4]).

3 Sequential Functionals and Strong Stability

The definitions of dI-domains with coherence, strongly stable functions etc. below, are all due to Thomas Ehrhard ([3]).

Definition 3.1 *A* dI-domain with coherence *is a pair $\mathcal{D} = (D, \mathcal{C}(D))$ where D is a dI-domain and $\mathcal{C}(D)$ a set of finite, nonempty subsets of D (called the* coherent *subsets), satisfying:*

i) for every $d \in D$, $\{d\} \in \mathcal{C}(D)$;

ii) if $A \in \mathcal{C}(D)$, B finite with $\forall b \in B \exists a \in A\, b \leq a$ and $\forall a \in A \exists b \in B\, b \leq a$, then $B \in \mathcal{C}(D)$;

iii) if $E_1 \subseteq D, \ldots, E_n \subseteq D$ are directed and such that for all $x_1 \in E_1, \ldots, x_n \in E_n$, $\{x_1, \ldots, x_n\} \in \mathcal{C}(D)$ then $\{\bigvee E_1, \ldots, \bigvee E_n\} \in \mathcal{C}(D)$.

From this definition it follows immediately that the set $\mathcal{C}(D)$ is determined by the set of all coherent sets of compact elements. Note also that every finite set that has an upper bound in D, is coherent.

Definition 3.2 *Let $\mathcal{D} = (D, \mathcal{C}(D))$ and $\mathcal{E} = (E, \mathcal{C}(E))$ be two dI-domains with coherence. A continuous function $f : D \to E$ is called* strongly stable *if for every $A \in \mathcal{C}(D)$, $f[A] \in \mathcal{C}(E)$ and $f(\bigwedge A) = \bigwedge f[A]$.*

Note, that every strongly stable function is stable. Evidently there is a category **dIC** of dI-domains with coherence and strongly stable functions, and it is a subcategory of the category of dI-domains and stable functions. Ehrhard shows that **dIC** is cartesian closed: the product $\mathcal{D} \times \mathcal{E}$ is $(D \times E), \mathcal{C}(D \times E))$ where $D \times E$ is the product of dI-domains, and $A \subseteq D \times E$ is coherent iff both its projections are coherent. The function space $\mathcal{D} \Rightarrow \mathcal{E}$ is $(D \Rightarrow E, \mathcal{C}(D \Rightarrow C))$ where $D \Rightarrow E$ is the set of strongly stable functions from D to E (which is a dI-domain, with the stable order), and $\{f_1, \ldots, f_n\}$ is coherent iff for every coherent $\{d_1, \ldots d_m\} \subseteq D$ and every $K \subseteq \{1, \ldots, n\} \times \{1, \ldots, m\}$ such that K projects surjectively onto $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$, one has that $\{f_i(d_j) \mid (i, j) \in K\}$ is coherent in E and

$$\bigwedge \{f_i(d_j) \mid (i, j) \in K\} = (\bigwedge_{i=1}^{n} f_i)(\bigwedge_{j=1}^{m} d_j)$$

In **dIC** we have the object $N = (N_\perp, \mathcal{C}(N))$ where $A \subseteq N_\perp$ is coherent if either $\perp \in A$ or $A = \{n\}$ for some $n \in \mathbb{N}$. Using this object N and the cartesian closedness of **dIC** we have an obvious interpretation of the types of section 2. We shall show that this interpretation yields exactly the type structure of sequential functionals from section 2.

To begin with, we have noticed in the previous section that each $\mathcal{O}_\sigma$ is a qualitative domain, so a word about continuous functions between qualitative domains is in order; here I restrict to functions $\mathcal{O}_\sigma \to N_\perp$. Every such function f has (and is conversely determined by) its *trace* (B, v) where $B \subseteq \mathcal{O}_\sigma$ a set of compact elements b which are minimal w.r.t. the property that $f(b) \neq \perp$, and $v : B \to \mathbb{N}$ a function. We call B the *base* of f. In order to check stability-like properties for f it is the base B that matters (v being redundant), e.g.:

- f is stable if and only if $b \neq b'$ implies that b and b' are incompatible, for $b, b' \in B$;
- f is strongly stable iff every coherent subset of B is a singleton;
- $f : \mathcal{O}_\sigma \to N_\perp$ is sequential iff the set

  $$\{p \text{ finite} \mid [\mu_\sigma(p)] \in B\}$$

  is the set of leaves of a sequential tree.

I leave the verification of these facts to the reader.

We now turn to the dI-domains $\mathcal{O}_\sigma$. Even without knowing that the elements of $\mathcal{O}_{\sigma\to o}$ are strongly stable functions we can still define a coherence on them as if they were, i.e.:

- $A \subset N_\perp$ is coherent iff $\perp \in A$ or $A = \{n\}$ for some $n \in \mathbb{N}$;
- $A \subset \mathcal{O}_{\sigma\to o}$ is coherent iff for each coherent $B \subset \mathcal{O}_\sigma$ and each $E \subseteq A \times B$ such that $\pi_1(E) = A$ and $\pi_2(E) = B$, we have that $\{f(b) \,|\, (f,b) \in E\}$ is coherent, and, for each $n \in \mathbb{N}$, if for all $(f,b) \in E$, $f(b) = n$ then $(\bigwedge A)(\bigwedge B) = n$.

Note that this definition is equivalent to Ehrhard's for function spaces. We shall prove that in fact, $\mathcal{O}_{\sigma\to o}$ is the dIC of strongly stable functions from $\mathcal{O}$ to $N_\perp$.

First some simple remarks about associates of type $(\tau\to o)\to o$.

Lemma 3.3 *Let $\gamma \in \mathrm{Ass}((\tau\to o)\to o)$. Then there is $\gamma' \sim \gamma$ with the property that for every $\beta \in \mathrm{Ass}(\tau\to o)$ such that $\gamma'|\beta$ is defined and every dialogue u between γ' and β:*

> *$\gamma'(u^{<i})$ is of the form $2v$ where $v = \langle v_0, \ldots, v_{n-1}\rangle$ is a dialogue between β and some $\delta \in \mathrm{Ass}(\tau)$ with $\beta|\delta$ defined, and for every $m < n$ there is $j < i$ with $\gamma'(u^{<j}) = 2v^{<m}$*

Proof. Since we may assume that β is only defined on dialogues between β and some $\delta \in \mathrm{Ass}(\tau)$ with $\beta|\delta$ defined, $\gamma(\langle\rangle)$ must be $2v$ where v is such a dialogue; let γ' first question β on all subdialogues of v, etc. ∎

So basically, what $\gamma \in \mathrm{Ass}((\tau\to o)\to o)$ can do when confronted with a hypothetical $\beta \in \mathrm{Ass}(\tau\to o)$ is: feed it some δ, and see. But since γ a priori knows nothing about β except for the arguments at which β wants to know δ, the following lemma (which formalizes this idea) should be clear:

Lemma 3.4 *Let $\gamma \in \mathrm{Ass}(f)$, $f \in \mathcal{O}_{(\tau\to o)\to o}$ and suppose $\{c_1, \ldots, c_n\} \subset \mathrm{Base}(f)$. Then one of the following three possibilities occurs:*

i) $n = 1$;

ii) *there is an $x \in \mathcal{O}_\tau$ such that $c_i(x) \neq \perp$ for all $i \leq n$, but there are $i, j \leq n$ with $c_i(x) \neq c_j(x)$;*

iii) *there are finite functions $p_1, \ldots, p_n$, $p_i \subseteq \delta_i \in \mathrm{Ass}(\tau)$, and associates β_i of c_i, with $\beta_i|p_i$ defined but no β_i defined on a proper subfunction of p_i, and a $q \subseteq \bigcap_{i=1}^n p_i$ such that $\bigcap\{\mathrm{dom}(p_i \setminus q) |\, i \leq n\} = \emptyset$.*

Proof. Suppose $n > 1$ and assume γ satisfies lemma 3.3. Take any $\beta_1 \in \mathrm{Ass}(c_1)$ and let u be the dialogue between γ and β_1. There must be a least index i such that for some $j \neq 1$, for no $\beta_1' \sim \beta_1$ and $\beta_j \in \mathrm{Ass}(c_j)$, $\langle u_0, \ldots, u_i\rangle$ is both a dialogue between γ and β_1', and γ and β_j. Now pick for each $j > 1$ an associate β_j such that the dialogue between γ and β_j starts with $u^{<i}$. $u^{<i}$ may contain already several finished dialogues between the β's and some finite functions p, but at point i we have $\gamma(u^{<i}) = 2v$ where v is a dialogue between some p and *all* β_j's. Pick for each j, now $1 \leq j \leq n$, a p_j such that $\beta_j|p_j$ is defined, $p_j \subseteq \delta_j \in \mathrm{Ass}(\tau)$, and v is a dialogue between β_j and p_j. Then $p \subseteq \bigcap_{j=1}^n p_j$.

If $\beta_1(v) = 2l$ then for some j, l cannot be in $\mathrm{dom}(p_j)$ since otherwise there would be an associate $\beta_j' \sim \beta_j$ which also asks l at this point. So then (iii) holds. If $\beta_1(v) = 2l+1$ and (iii) does not hold, then for all j, $\beta_j|p$ is defined but the values must be different hence (ii) holds. ∎

Theorem 3.5 *For every type σ we have:*

i) *A set $\{c_1, \ldots, c_n\}$ of compact elements of $\mathcal{O}_\sigma$ is coherent $\iff$ $n = 1$ or there are $p_1, \ldots, p_n$ finite with $c_i = [\mu_\sigma(p_i)]$, and for some $q \subseteq \bigcap_{i=1}^n p_i$, $\bigcap\{\mathrm{dom}(p_i \setminus q) \,|\, 1 \leq i \leq n\} = \emptyset$;*

ii) *For a continuous function $f : \mathcal{O}_\sigma \to N_\perp$ we have: $f \in \mathcal{O}_{\sigma\to o} \iff f$ is strongly stable.*

Proof. Induction on σ. For $\sigma = o$ the facts are obvious; so let $\sigma = \tau \rightarrow o$. We prove i)$\Leftarrow$, ii)$\Leftarrow$, ii)$\Rightarrow$, i)$\Rightarrow$.

i)$\Leftarrow$. If $n = 1$ then $\{c_1, \ldots, c_n\}$ is coherent by the first axiom of coherence. If $n > 1$, $p_1, \ldots, p_n$ finite with $c_i = [\mu_\sigma(p_i)]$ and $q \subseteq \bigcap_{i=1}^n p_i$ with $\bigcap\{\mathrm{dom}(p_i \setminus q) \mid 1 \leq i \leq n\} = \emptyset$, let $\{x_1, \ldots, x_m\}$ be a coherent set of compact elements of $\mathcal{O}_\tau$ and $K \subseteq \{1, \ldots, n\} \times \{1, \ldots, m\}$ which projects surjectively onto $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$. By induction hypothesis ii) we may assume that all elements of $\mathcal{O}_\sigma$ are strongly stable functions from $\mathcal{O}_\tau$ to $N_\perp$; we apply induction hypothesis i) to $\{x_1, \ldots, x_m\}$. If $m = 1$ then we must have that either some $c_i(x_1) = \perp$ or $\{c_1(x_1), \ldots, c_n(x_1)\} = \{[\mu_\sigma(q)](x_1)\}$ by the assumption on $\{c_1, \ldots, c_n\}$. So this is coherent but also if $r_1, \ldots, r_m$ finite with $x_i = [\mu_\tau(r_i)]$, and $s \subseteq \bigcap_{j=1}^m r_j$ with $\bigcap\{\mathrm{dom}(r_j \setminus s) \mid 1 \leq j \leq m\} = \emptyset$ and $p_i | r_j$ is defined for $(i, j) \in K$, we must have that $q|s$ is defined. This proves that $\{c_1, \ldots, c_n\}$ is coherent.

ii)$\Leftarrow$ follows from i)$\Leftarrow$ just proved: if $f : \mathcal{O}_\sigma \to N_\perp$ is strongly stable then no nonempty, finite, non-singleton subset of $\mathrm{Base}(f)$ can be coherent hence $\{p \text{ finite} \mid [\mu_\sigma(p)] \in \mathrm{Base}(f)\}$ is the set of leaves of a sequential tree, so $f \in \mathcal{O}_{\sigma \to o}$.

ii)$\Rightarrow$ Let $f \in \mathcal{O}_{\sigma \to o}$ and $\{c_1, \ldots, c_n\} \subseteq \mathrm{Base}(f)$. Suppose $n > 1$. Apply lemma 3.4: if case ii) holds then clearly $\{c_1, \ldots, c_n\}$ cannot be coherent. So suppose case iii) holds, i.e. there are finite functions $p_1, \ldots, p_n$, $p_i \subseteq \delta_i \in \mathrm{Ass}(\tau)$ and $\beta_i \in \mathrm{Ass}(c_i)$ with $\beta_i | p_i$ defined, and $q \subseteq \bigcap_{i=1}^n p_i$ with $\bigcap\{\mathrm{dom}(p_i \setminus q) \mid 1 \leq i \leq n\} = \emptyset$. By induction hypothesis i) we have that the set $\{[\mu_\tau(p_i)] \mid 1 \leq i \leq n\}$ is coherent; hence if $\{c_1, \ldots, c_n\}$ were coherent we would already have that $\beta_i | q$ defined (verify that $[\mu_\tau(q)] = \bigwedge\{[\mu_\tau(p_i)] \mid 1 \leq i \leq n\}$), which contradicts the choice of the p_i. So $\{c_1, \ldots, c_n\}$ is not coherent, hence f is strongly stable.

i)$\Rightarrow$ follows from ii)$\Rightarrow$ just proved: if the conclusion of i)$\Rightarrow$ does not hold for $\{c_1, \ldots, c_n\}$ then $\{c_1, \ldots, c_n\} \subseteq \mathrm{Base}(f)$ for some $f \in \mathcal{O}_{\sigma \to o}$ which by ii)$\Rightarrow$ contradicts coherence of $\{c_1, \ldots, c_n\}$. ∎

There is a full sub-ccc of the ccc **dIC** on objects which are qualitative domains and whose coherence is generated by coherence on atoms. Ehrhard (l.c.) gives a presentation of this category in the style of Girard's qualitative domains. He calls the objects *hypercoherences*. Since $N_\perp$ is a hypercoherence, it turns out that in fact our whole type structure lands in the category of hypercoherences.

4 $\mathcal{B}$ as a combinatory algebra

For $\alpha \in \mathcal{B}$ and $x \in \mathbb{N}$ let α_x denote the partial function which sends $y \in \mathbb{N}$ to $\alpha(\langle x, y\rangle)$ (if this is defined), now $\langle \cdot, \cdot \rangle$ referring to some (recursive) bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

Definition 4.1 *Given $\alpha, \beta \in \mathcal{B}$ let $\alpha \bullet \beta$ denote the partial function*

$$\lambda x.\alpha_x | \beta$$

Theorem 4.2 *With $(\cdot)\bullet(\cdot)$ as defined in 4.1, $\mathcal{B}$ is a combinatory algebra.*

I record this fact without proof. My own proof was a laborious calculation of the combinators **k** and **s**, which is not very illuminating. Another proof could consist in showing that every recursive operator $\{e\}^{F_1, \ldots, F_n}$ in n partial oracles is in fact of the form:

$$\{e\}^{F_1, \ldots, F_n}(x) = y \Leftrightarrow (\cdots(\alpha \bullet F_1) \cdots \bullet F_n)(x) = y$$

for some α. A third approach would establish a characterization of those functions $F : \mathcal{B}^n \to \mathcal{B}$ which are of form $F(\beta_1, \ldots, \beta_n) = (\cdots(\alpha \bullet \beta_1) \cdots \bullet \beta_n)$ for some α. This involves some combinatorics with sequential trees.

Let me just make clear in what way the type structure $\{\mathcal{O}_\sigma \mid \sigma \text{ type}\}$ of section 2, and hence the corresponding part of Ehrhard's Hypercoherences, fits into the realizability topos generated by the combinatory algebra $\mathcal{B}$ (for realizability toposes consult [8, 6]). Let us call it $\mathcal{E}ff(\mathcal{B})$. An important subcategory (the subcategory of $\neg\neg$-separated objects) of $\mathcal{E}ff(\mathcal{B})$ can be described as follows:

Let $\mathcal{B}$-Set be the category with objects pairs (X, E_X) where X is a set and $E_X : X \to \mathcal{P}(\mathcal{B})$ a function. A function $f : X \to Y$ is a morphism from (X, E_X) to (Y, E_Y) if for some $\alpha \in \mathcal{B}$:

$$\forall x \in X \forall \beta \in E_X(x)\, \alpha \bullet \beta \in E_Y(f(x))$$

α is said to *track* f.

The category $\mathcal{B}$-Set is cartesian closed: the function space $(Y, E_Y)^{(X,E_X)}$ may be rendered as $(Y^X, E_{X\Rightarrow Y})$ where $\alpha \in E_{X\Rightarrow Y}(f)$ iff α tracks f.

For each type σ, $(\mathcal{O}_\sigma, \mathrm{Ass})$ is an object of $\mathcal{B}$-Set and it is an easy exercise to verify that $(\mathcal{O}_{\sigma\to o}, \mathrm{Ass})$ is isomorphic in $\mathcal{B}$-Set to $(N_\perp, \mathrm{Ass})^{(\mathcal{O}_\sigma, \mathrm{Ass})}$.

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