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A Criterion for Sets of V-Interpolation

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1. INTRODUCTION

Let $\Gamma_1, \Gamma_2, ..., \Gamma_n$, $n \geqslant 2$, be discrete spaces, and $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ their product. We consider the tensor algebra $V(\Gamma) = C(\Gamma_1) \hat{\otimes} \cdots \hat{\otimes} C(\Gamma_n)$, where $C(\Delta)$, Δ a discrete space, denotes the Banach algebra of bounded functions on Δ with the supremum norm. A set $E \subset \Gamma$ is called a *set of V-interpolation* if $V(E) = C(E)$. Here $V(E) = V(\Gamma)/I(E)$, where

$$I(E) = \{\varphi \in V(\Gamma); \varphi^{-1}(0) \supset E\},$$

and $V(E)$ is provided with the quotient norm.

We introduce the following notations. The indices α, β, γ will always be elements of $\{1, 2, ..., n\}$. $\pi_\alpha : \Gamma \to \Gamma_\alpha$ is the natural projection; $\pi_{\alpha\alpha} = \pi_\alpha$; if $\alpha \neq \beta$ then $\pi_{\alpha\beta} : \Gamma \to \Gamma_\alpha \times \Gamma_\beta$ is the natural projection; $\Gamma_\alpha \times \Gamma_\beta$ and $\Gamma_\beta \times \Gamma_\alpha$ are not considered as different spaces, thus $\pi_{\alpha\beta} = \pi_{\beta\alpha}$.

We shall prove in this paper the following criterion for sets of V-interpolation:

THEOREM 1. *A set $E \subset \Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ is a set of V-interpolation if and only if there exists a decomposition*

$$E = \bigcup_{i=1}^{k} \bigcup_{\alpha=1}^{n} E_{i\alpha}$$

of E into a finite number of disjoint sets $E_{i\alpha}$ (possibly empty) such that the following two conditions are satisfied:

(i) $\pi_\alpha : E_{i\alpha} \to \Gamma_\alpha$ *is injective for all* $1 \leqslant i \leqslant k$, $1 \leqslant \alpha \leqslant n$,

(ii) *For all* $E_{i\alpha} \neq E_{j\beta}$ *either* (1) *or* (2) *below is fulfilled*:

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(1) $\pi_{\alpha\beta} E_{i\alpha} \cap \pi_{\alpha\beta} E_{j\beta} = \varnothing$,

(2) *There is a* γ, $1 \leqslant \gamma \leqslant n$, *such that* $\pi_\gamma E_{i\alpha} \cap \pi_\gamma E_{j\beta} = \varnothing$.

Condition (i) just says that $E_{i\alpha}$ is an α-section, i.e., the graph of a function $\Gamma_\alpha \to \Gamma_1 \times \cdots \times \check{\Gamma}_\alpha \times \cdots \times \Gamma_n$, whereas condition (ii) restricts the "intertwining" of these graphs.

The case $n = 2$ deserves special attention, because it has been studied more intensively than the case $n \geqslant 3$ (cf. [1]) and because it brings out most clearly the idea of our criterion. If, in the case $n = 2$, $\alpha \neq \beta$, then (1) is automatically satisfied because $\pi_{\alpha\beta}$ is the identity mapping. If $\alpha = \beta$, then either (1) is not satisfied or $E_{i\alpha}$ and $E_{j\beta}$ can be considered as just one α-section. It follows therefore that the theorem for the case $n = 2$ can be reformulated in the following way:

THEOREM 2. *A set $E \subset \Gamma_1 \times \Gamma_2$ is a set of V-interpolation if and only if E is the union of a finite number of bisections with disjoint ranges.*

We have to explain the terminology. A set $E \subset \Gamma_1 \times \Gamma_2$ is a *bisection* if E can be decomposed, $E = E_1 \cup E_2$, such that $\pi_i : E_i \to \Gamma_i$ is injective, $i = 1, 2$ (see [1]). The *range* of E is the disjoint union of the sets $\pi_2 E_1$ and $\pi_1 E_2$.

The two parts of Theorem 1 will be proved in Sections 4 and 5, respectively.

2. COLORATION

Let $F \subset E \subset \Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ and let $1 \leqslant N < +\infty$. A mapping $X : F \to \{1, 2, \ldots, N\}$ is a *coloration of F with respect to E* if for all sets $A = \{a_1, a_2, \ldots, a_n\} \subset F$ with $2 \leqslant \operatorname{card} A \leqslant n$ and for all $b \in E \backslash A$ the following holds:

(3) $\pi_\alpha b = \pi_\alpha a_\alpha$ for all $1 \leqslant \alpha \leqslant n \Rightarrow \operatorname{card} X(A) \geqslant 2$.

One can express this in the following way: It is not allowed that any point b of E can "see" points of F all of the same colour in each of the n hyperplanes of codimension 1 containing b.

If $n = 2$ and $F = E$, this is exactly the definition of coloration in [1]. Notice that card A may be strictly less than n, i.e., some of the points of A might lie in the intersection of several hyperplanes through b. It is of no use, however, to consider card $A = 1$.

A coloration of E with respect to E itself will be called simply a coloration of E. A coloration of F with respect to $E \supset F$ is in particular a coloration of F.

The following two lemmas will enable us to construct colorations.

LEMMA 2.1. *If $F \subset E \subset \Gamma$, if $E_1, \ldots, E_k$ are subsets of E containing F and with union E, and if for every j, $1 \leqslant j \leqslant k$, there exists a coloration X_j of F with respect to E_j with $N_j < +\infty$ colours, then there is a coloration X of F with respect to E with $\prod_{j=1}^{k} N_j$ colours.*

Proof. Denote $Z_j = \{1, 2, \ldots, N_j\}$; we have then $X_j : F \to Z_j$. Define $X : F \to Z_1 \times Z_2 \times \cdots \times Z_k \cong \{1, 2, \ldots, \prod_{j=1}^{k} N_j\}$ by $X(a) = (X_1(a), \ldots, X_k(a))$ for all $a \in F$. Then it is evident that X is a coloration of F with respect to E.

LEMMA 2.2. *If a set $E \subset \Gamma$ can be decomposed into a finite number of subsets $E = F_1 \cup F_2 \cup \cdots \cup F_k$ such that for each of the F_i, $1 \leqslant i \leqslant k$, there exists a coloration X_i of F_i with respect to E, with $N_i < +\infty$ colours, then there exists a coloration X of E with $\sum_{i=1}^{k} N_i$ colours.*

Proof. We can consider the F_i to be disjoint. The mapping $X : E \to \{(i, n);\ 1 \leqslant i \leqslant k,\ 1 \leqslant n \leqslant N_i\} \cong \{1, 2, \ldots, \sum_{i=1}^{k} N_i\}$ defined by $X(a) = (i, X_i(a))$ if $a \in F_i$, evidently is a coloration of E.

The next lemma will be used in Section 5.

LEMMA 2.3. *If $F \subset \Gamma$ is monochromatic, i.e., if the mapping $X : F \to \{1\}$ is a coloration of F, then F can be decomposed into n disjoint subsets, $F = F^{(1)} \cup \cdots \cup F^{(n)}$ such that*

(i) $\pi_\alpha : F^{(\alpha)} \to \Gamma_\alpha$ *is injective,*

(ii) $\pi_\alpha F^{(\alpha)} \cap \pi_\alpha(F \backslash F^{(\alpha)}) = \varnothing$,

(iii) $\pi_\alpha F^{(\alpha)} \cap \pi_\alpha F^{(\beta)} = \varnothing \ (\alpha \neq \beta)$,

(iv) $\pi_{\alpha\beta} F^{(\alpha)} \cap \pi_{\alpha\beta} F^{(\beta)} = \varnothing \ (\alpha \neq \beta)$.

Proof. Put $F_\alpha = \{b \in F;\ a \in F,\ a \neq b \Rightarrow \pi_\alpha a \neq \pi_\alpha b\}$, $\alpha = 1, \ldots, n$. Then $\bigcup_{\alpha=1}^{n} F_\alpha = F$. Indeed if $b \in F$ and $b \notin F_\alpha$ for all α, then there is a set $A = \{a_1, a_2, \ldots, a_n\} \subset F$, $2 \leqslant \operatorname{card} A \leqslant n$, such that $b \notin A$ and $\pi_\alpha a_\alpha = \pi_\alpha b$, in contradiction with the fact that F is monochromatic. The F_α need not be disjoint, but one can of course choose disjoint subsets $F^{(\alpha)} \subset F_\alpha$ such that the union is still F. (i) and (ii) now follow at once, (iii) is an immediate consequence of (ii), and (iv) follows directly from (iii).

3. THE FUNDAMENTAL LEMMA

The relation between sets of V-interpolation and coloration is described in the following fundamental lemma, which is just the analog for arbitrary n of the corresponding result in [1] where only the case $n = 2$ is considered.

LEMMA 3.1. *A set $E \subset \Gamma$ is a set of V-interpolation if and only if there exists a finite coloration of E.*

The proof, entirely analogous to that in [1], is based on the following two facts:

(A) A set $E \subset \Gamma$ is a set of V-interpolation if and only if for every subset $F \subset E$ there is a finite number of cubes $\Delta_p = \Delta_p^{(1)} \times \cdots \times \Delta_p^{(n)}$, $\Delta_p^{(\alpha)} \subset \Gamma_\alpha$, $1 \leqslant p \leqslant q$, $q = q(F) < +\infty$, such that $(\Delta_1 \cup \cdots \cup \Delta_q) \cap E = F$.

(B) If $E \subset \Gamma$ is a set of V-interpolation then E can be decomposed into a finite number of sections:

$$E = \bigcup_{i=1}^{k} \bigcup_{\alpha=1}^{n} E_{i\alpha}, \qquad \pi_\alpha : E_{i\alpha} \to \Gamma_\alpha \quad \text{injective.}$$

The proof of (A) rests upon the possibility to approximate functions in $C(\Gamma_\alpha)$ and $C(\Gamma)$ uniformly by linear combinations of characteristic functions such that the absolute sum of the coefficients is bounded by 4 (or even $2\sqrt{3}$) times the norm of the function. To prove (B) one uses that a set of V-interpolation is a V-Sidon set, and all V-Sidon sets have such a decomposition. For details we refer to [1].

Proof of the lemma. Let $X : E \to \{1, 2,..., N\}$ be a coloration of E. Take any $F \subset E$. Put $F_i = \{a \in F; X(a) = i\}$, $i = 1,..., N$, and put $\Delta_i = \pi_1 F_i \times \cdots \times \pi_n F_i$ the cube generated by F_i. Then $\Delta_i \cap E = F_i$. For suppose there is an element $b \in (\Delta_i \cap E)\backslash F_i$, then there exists a set $A = \{a_1 ,..., a_n\} \subset F_i$ with $2 \leqslant \operatorname{card} A \leqslant n$ such that $\pi_\alpha a_\alpha = \pi_\alpha b$, which contradicts the fact that $\operatorname{card} X(A) = 1$, X being a coloration. We conclude that $(\Delta_1 \cup \cdots \cup \Delta_N) \cap E = F$, which by (A) above shows that E is a set of V-interpolation.

Conversely, let $E \subset \Gamma$ be a set of V-interpolation. By (B) we have $E = \bigcup_{i=1}^k \bigcup_{\alpha=1}^n E_{i\alpha}$ with $\pi_\alpha : E_{i\alpha} \to \Gamma_\alpha$ injective. By (A) we can find cubes $\Delta_{i\alpha p}$, $1 \leqslant p \leqslant q$, $q = q(i, \alpha)$, such that $(\Delta_{i\alpha 1} \cup \cdots \cup \Delta_{i\alpha q}) \cap E = E_{i\alpha}$, $1 \leqslant i \leqslant k$, $1 \leqslant \alpha \leqslant n$. We can even assume, without loss of generality, that the cubes $\Delta_{i\alpha p}$, $1 \leqslant p \leqslant q(i, \alpha)$, are disjoint, because a finite number of cubes can always be replaced by a still finite number of disjoint cubes such that both collections of cubes have the same union. Now define

$$\begin{aligned} X : E &\to \{(i, \alpha, p); 1 \leqslant i \leqslant k, 1 \leqslant \alpha \leqslant n, 1 \leqslant p \leqslant Q = \max q(i, \alpha)\} \\ &\simeq \{1, 2,..., knQ\} \end{aligned}$$

by $X(a) = (i, \alpha, p)$ if $a \in E \cap \Delta_{i\alpha p} \cdot X$ is well defined and is a coloration, as is easily seen.

4. Proof of the Sufficiency

We need the following easy technical lemma:

Lemma 4.1. *If S is any set, $D \subset S$ a subset of S, and $d : D \to S$ a mapping such that $ds \neq s$ for all $s \in D$, then there exists a mapping $Y : S \to \{1, 2, 3\}$ such that $Y(ds) \neq Y(s)$ for all $s \in D$.*

Proof. We define $D_0 = S \backslash D$ and inductively for all $n \geqslant 1$, $D_n = \{s \in D; ds \in D_{n-1}\}$. We denote $D_\infty = D \backslash \bigcup_{n \geqslant 1} D_n$ and remark that $dD_\infty \subset D_\infty$, $d \bigcup_{n \geqslant 1} D_n \subset \bigcup_{n \geqslant 0} D_n$, so that Y can be defined independently on D_∞ and $\bigcup_{n \geqslant 0} D_n$, respectively. If we put $Y(s) = 1$ if $s \in D_n$ and n is odd, $Y(s) = 2$ if $s \in D_n$ and n is even, then clearly $Y(ds) \neq Y(s)$ for all $s \in \bigcup_{n \geqslant 1} D_n$. To define Y on D_∞ we proceed as follows: d defines an equivalence relation in $D_\infty : a \sim b$ if and only if there exist $n, m \geqslant 0$ such that $d^n a = d^m b$ (d^0 is the identity on D_∞, and $d^{n+1} = d \circ d^n$, $n \geqslant 0$). We can define Y independently on each equivalence class. Let $T \subset D_\infty$ be such a class. We distinguish two possible cases:

Case 1. $d^n a \neq a$ for all $n \geqslant 1$ and all $a \in T$. Fix an element $b \in T$. For any $a \in T$ take n and m such that $d^n a = d^m b$. Now define $Y(a) = 1$ if $n - m$ is odd, $Y(a) = 2$ if $n - m$ is even. This definition does not depend on the choice of n and m: if also $d^{n'} a = d^{m'} b$, then $d^{m+n'} a = d^{m+m'} b = d^{n+m'} a$, hence $m + n' = n + m'$ (Case 1), thus $n - m = n' - m'$. It is clear, furthermore, that always $Y(da) \neq Y(a)$.

Case 2. There is an element $c \in T$ and $n \geqslant 2$ such that $d^n c = c$, $d^i c \neq c$, $1 \leqslant i < n$ (remark that $dc \neq c$ by assumption). Put $c_i = d^i c$, $i = 1, \ldots, n$; then $d^n c_i = c_i$ for all $i = 1, \ldots, n$. For all $a \in T \backslash \{c_1, \ldots, c_n\}$ there are uniquely determined integers k and i, $k \geqslant 1$, $1 \leqslant i \leqslant n$, such that $d^k a = c_i$, $d^{k-1} a \neq c_{i-1}$ (we put $c_0 = c_n$). Define $Y(c_i) = 1$ if i is odd and $1 \leqslant i < n$, $Y(c_i) = 2$ if i is even, $Y(c_n) = 3$ if n is odd, $Y(a) = 1$ if $k + i$ is odd, $Y(a) = 2$ if $k + i$ is even (a, k, i as above). It is again clear that $Y(da) \neq Y(a)$ for all $a \in T$. This finishes the proof of the lemma.

Now suppose $E \subset \Gamma$ has a decomposition $E = \bigcup_{i=1}^{k} \bigcup_{\alpha=1}^{n} E_{i\alpha}$ into disjoint sets satisfying (i) and (ii) of our Theorem 1. We shall construct then for each ordered pair $E_{i\alpha}$, $E_{j\beta}$, $(i, \alpha) \neq (j, \beta)$, a coloration of $E_{i\alpha}$

with respect to $E_{i\alpha} \cup E_{j\beta}$. By Lemma 2.1, this will give colorations of the sets $E_{i\alpha}$ with respect to E, and this in turn by Lemma 2.2 will result in a coloration of E. An application of Lemma 3.1 then will finish the proof.

We distinguish two cases:

Case 1. $E_{i\alpha}$ and $E_{j\beta}$ satisfy (2) of condition (ii). Then $X : E_{i\alpha} \to \{1\}$ is already a coloration of $E_{i\alpha}$ with respect to $E_{i\alpha} \cup E_{j\beta}$. For if $a, b \in E_{i\alpha}$, $a \neq b$, then $\pi_\alpha a \neq \pi_\alpha b$, and if $a \in E_{i\alpha}$, $b \in E_{j\beta}$ then $\pi_\gamma a \neq \pi_\gamma b$ for some γ, so that never a set $A \subset E_{i\alpha}$ and an element $b \in (E_{i\alpha} \cup E_{j\beta})\backslash A$ can violate (3).

Case 2. $E_{i\alpha}$ and $E_{j\beta}$ do not satisfy (2), hence they satisfy (1), and $\alpha \neq \beta$. For any (l, γ), $1 \leqslant l \leqslant k$, $1 \leqslant \gamma \leqslant n$, we shall denote $\pi_\gamma E_{l\gamma} = \Gamma_{l\gamma} \subset \Gamma_\gamma$, and $p_{l\gamma} : \Gamma_{l\gamma} \to E_{l\gamma}$ the inverse of $\pi_{\gamma|E_{l\gamma}}$.

Consider the mapping $d = \pi_\beta p_{i\alpha} \pi_\alpha p_{j\beta} : D \to \Gamma_\beta$, where D is the natural domain of d. Take any $t \in D$, put $b = p_{j\beta} t \in E_{j\beta}$ and $a = p_{i\alpha} \pi_\alpha b \in E_{i\alpha}$ (a exists because $t \in D$). $\pi_\alpha a = \pi_\alpha b$; hence by (1), $\pi_\beta a \neq \pi_\beta b$, i.e., $dt \neq t$. We can thus apply Lemma 4.1 and obtain a mapping $Y : \Gamma_\beta \to \{1, 2, 3\}$ such that $Y(dt) \neq Y(t)$ for all $t \in D$. We define $X = Y\pi_\beta : E_{i\alpha} \to \{1, 2, 3\}$ and claim that X is a coloration as desired. Indeed, take $A = \{a_1, \dots, a_n\} \subset E_{i\alpha}$ with $2 \leqslant \operatorname{card} A \leqslant n$, and $b \in (E_{i\alpha} \cup E_{j\beta})\backslash A$ such that $\pi_\gamma b = \pi_\gamma a_\gamma$ for all γ. Taking $\gamma = \alpha$ we see that necessarily $b \in E_{j\beta}$, and we have, in particular, $\pi_\alpha b = \pi_\alpha a_\alpha$, $\pi_\beta b = \pi_\beta a_\beta$. Writing $\pi_\beta b = t \in D$ we derive

$$X(a_\alpha) = Y\pi_\beta p_{i\alpha} \pi_\alpha a_\alpha = Y\pi_\beta p_{i\alpha} \pi_\alpha p_{j\beta} t = Y(dt),$$

$$X(a_\beta) = Y\pi_\beta a_\beta = Y\pi_\beta p_{j\beta} t = Y(t).$$

Hence $X(a_\alpha) \neq X(a_\beta)$, thus $\operatorname{card} X(a) \geqslant 2$. We conclude that X is a coloration.

5. Proof of the Necessity

Suppose $E \subset \Gamma$ is a set of V-interpolation. Let $X : E \to Z = \{1, 2, \dots, N\}$ be a coloration of E with $N < +\infty$ colours. Such a coloration exists by Lemma 3.1. Actually, the mapping X constructed there has the property that every monochromatic subset is a section, and the reasoning below could be slightly shortened if we used this fact. However, to make clear that we don't really need it at this point, we prefer to start from an arbitrary coloration. Put $E_i = X^{-1}(i)$ and $\pi_\alpha(E_i) = \Delta_i^{(\alpha)} \subset \Gamma_\alpha$, $1 \leqslant \alpha \leqslant n$, $i \in Z$.

For every $m \geqslant 1$ we will denote by H_m the set of all m-tuples $J = (I_1, ..., I_m)$ of subsets of Z. There are natural mappings $\varphi_\alpha : H_{n-1} \times H_1 \to H_n$, defined by

$$\varphi_\alpha((I_1, ..., \check{I}_\alpha, ..., I_n), I_\alpha) = (I_1, ..., I_\alpha, ..., I_n),$$

where $(I_1, ..., \check{I}_\alpha, ..., I_n)$ of course means $(I_1, ..., I_{\alpha-1}, I_{\alpha+1}, ..., I_n)$.

For all $I \in H_1$ and $1 \leqslant \alpha \leqslant n$ we put

$$\Delta_I^{(\alpha)} = \bigcap_{i \in Z} \{t \in \Gamma_\alpha ; t \in \Delta_i^{(\alpha)} \Leftrightarrow i \in I\}.$$

$\Gamma_\alpha = \bigcup_{I \in H_1} \Delta_I^{(\alpha)}$, and the sets $\Delta_I^{(\alpha)}$ are disjoint. For all

$$J = (I_1, ..., I_n) \in H_n$$

we denote $D_J = \Delta_{I_1}^{(1)} \times \cdots \times \Delta_{I_n}^{(n)} \subset \Gamma$, and $E_J = D_J \cap E$. The cubes D_J, $J \in H_n$, are disjoint and cover Γ, hence $E = \bigcup_{J \in H_n} E_J$ is a decomposition of E into disjoint subsets. Let us denote for every $J = (I_1, ..., I_n) \in H_n$, $c(J) = \operatorname{card}(I_1 \cap I_2 \cap \cdots \cap I_n)$. We claim then that $c(J) \neq 1$ implies $E_J = \varnothing$. Indeed, if $x \in E_J$ and $X(x) = i \in Z$, then $i \in I_1 \cap \cdots \cap I_n$, hence $c(J) \geqslant 1$. If also $j \in I_1 \cap \cdots \cap I_n$, $j \neq i$, then there is a subset $A = \{y_1, ..., y_n\} \subset E$, $x \notin A$, $2 \leqslant \operatorname{card} A \leqslant n$, with $X(A) = \{j\}$ and $\pi_\alpha y_\alpha = \pi_\alpha x$, $\alpha = 1, ..., n$. This contradicts the fact that X is a coloration of E. It follows that all sets E_J are monochromatic (we don't need the fact that often $E_J = \varnothing$). By Lemma 2.3, there is a decomposition $E_J = E_J^{(1)} \cup \cdots \cup E_J^{(n)}$ into disjoint subsets with the properties (i) to (iv) of that lemma.

For every $1 \leqslant \alpha \leqslant n$ and any $K = (I_1, ..., \check{I}_\alpha, ..., I_n) \in H_{n-1}$ we shall denote $E_K^{(\alpha)} = \bigcup_{I \in H_1} E_{\varphi_\alpha(K,I)}^{(\alpha)}$. It is evident that the sets $E_K^{(\alpha)}$ are disjoint and that $\bigcup_{K \in H_{n-1}} \bigcup_{\alpha=1}^{n} E_K^{(\alpha)} = \bigcup_{J \in H_n} E_J = E$. We shall show that the sets $E_K^{(\alpha)}$, $K \in H_{n-1}$, $1 \leqslant \alpha \leqslant n$, satisfy the conditions (i) and (ii) of Theorem 1.

The following inclusions are evident:

$$\pi_\alpha E_{\varphi_\alpha(K,I)}^{(\alpha)} \subset \Delta_I^{(\alpha)},$$

$$\pi_\beta E_{\varphi_\alpha(K,I)}^{(\alpha)} \subset \Delta_{I_\beta}^{(\beta)} \qquad (\beta \neq \alpha).$$

$\pi_\alpha : E_K^{(\alpha)} \to \Gamma_\alpha$ is injective because $\pi_\alpha : E_{\varphi_\alpha(K,I)}^{(\alpha)} \to \Gamma_\alpha$ is injective for all $I \in H_1$ (property (i) of Lemma 2.3), and the sets $\pi_\alpha E_{\varphi_\alpha(K,I)}^{(\alpha)} \subset \Delta_I^{(\alpha)}$, $I \in H_1$, are disjoint. So condition (i) is satisfied.

To prove that condition (ii) is fulfilled, we consider two different pairs (K, α), (K', β), $1 \leqslant \alpha \leqslant n$, $1 \leqslant \beta \leqslant n$,

$$K = (I_1, \dots, \check{I}_\alpha, \dots, I_n) \in H_{n-1}, \qquad K' = (I_1', \dots, \check{I}_\beta', \dots, I_n') \in H_{n-1}.$$

If there exists a $\gamma \notin \{\alpha, \beta\}$ such that $I_\gamma \neq I_\gamma'$, then

$$\pi_\gamma E_K^{(\alpha)} \cap \pi_\gamma E_{K'}^{(\beta)} \subset \Delta_{I_\gamma}^{(\gamma)} \cap \Delta_{I_\gamma'}^{(\gamma)} = \varnothing,$$

so that (2) of condition (ii) is satisfied. Now suppose $I_\gamma = I_\gamma'$ for all $\gamma \notin \{\alpha, \beta\}$. Then necessarily $\alpha \neq \beta$ because $(K, \alpha) \neq (K', \beta)$. We shall show that in this case (1) of condition (ii) is satisfied. We have the following inclusions:

$$\pi_{\alpha\beta} E_{\varphi_\alpha(K, I)}^{(\alpha)} \subset \Delta_I^{(\alpha)} \times \Delta_{I_\beta}^{(\beta)},$$

$$\pi_{\alpha\beta} E_{\varphi_\beta(K', I')}^{(\beta)} \subset \Delta_{I_\alpha'}^{(\alpha)} \times \Delta_{I'}^{(\beta)}.$$

The only rectangle occuring twice here is the rectangle $\Delta_{I_\alpha'}^{(\alpha)} \times \Delta_{I_\beta}^{(\beta)}$, all others having empty intersection. It follows therefore that

$$\pi_{\alpha\beta} E_K^{(\alpha)} \cap \pi_{\alpha\beta} E_{K'}^{(\beta)} = \pi_{\alpha\beta} E_{\varphi_\alpha(K, I_\alpha')}^{(\alpha)} \cap \pi_{\alpha\beta} E_{\varphi_\beta(K', I_\beta)}^{(\beta)} = \pi_{\alpha\beta} E_J^{(\alpha)} \cap \pi_{\alpha\beta} E_J^{(\beta)},$$

where $J = (I_1'', \dots, I_n'')$ with $I_\alpha'' = I_\alpha'$, $I_\beta'' = I_\beta$, $I_\gamma'' = I_\gamma = I_\gamma'$ if $\gamma \notin \{\alpha, \beta\}$. Now (iv) of Lemma 2.3 shows that (1) is indeed satisfied.

This finishes the proof.

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