INSTANTONS VERSUS FACTORIZATION IN LARGE-N FIELD THEORIES

Pierre van BAAL
Institute for Theoretical Physics, Princetonplein 5, P.O. Box 80.006, 3508 TA Utrecht, The Netherlands

Received 26 January 1984
Revised manuscript received 27 February 1984

We discuss the contribution of surviving extrema for the action in $N \to \infty$ Yang–Mills theories in weak coupling and their relevance for factorization. In particular we discuss the role of fluxons in the twisted Eguchi–Kawai model.

Recently there has been much interest in large-N gauge theories [1], because the reduction [2] of a large-N lattice gauge theory to a one-point lattice, makes numerical and hopefully analytic calculations feasible. Factorization is crucial, both for simplifying the original large-N theory (only planar Feynman diagrams contribute) and for the reduction (the Schwinger–Dyson equations for the Wilson loop are at $N \to \infty$ a closed set of equations [3]). What one wishes to understand is confinement and hopefully $N \to \infty$ is not too crude an approximation to obtain information for finite N.

Tomboulis [4] claims to have proved confinement for all coupling β and $d \leqslant 4$ in SU(2), however *the mechanism* for confinement is still unclear. One believes that Z_N-vortices are responsible for confinement [5,6]. It is therefore natural to look for Z_N-type configurations which might survive for $N \to \infty$, either in the continuum [7] or on the lattice. In the latter case they were studied for the twisted Eguchi–Kawai [8] (TEK) model in ref. [9]. These configurations are instanton-like and survive for $N \to \infty$, because their action is proportional to $1/N$. Coleman's [10] argument tells us that ordinary instantons cannot contribute to the Wilson loop, this is however no longer valid for the instantons on the torus with nontrivial "twist" [11]. So there is still a possibility that configurations with nontrivial Z_N structure contribute to confinement. It is the purpose of this letter to show that at least for TEK, fluxons [9] do not change perturbative results.

As was pointed out by Greensite and Halpern [12] it is very unlikely that Z_N-type configurations play an essential role for $N \to \infty$ confinement. Their argument is based on the fact that factorization implies:

$$\langle \mathrm{tr}_A U(C) \rangle = |\langle \mathrm{tr}_F U(C) \rangle|^2 \,, \tag{1}$$

where $\mathrm{tr}_{A(F)} U(C)$ is the Wilson loop in the adjoint (fundamental) representation. This means that the string tension in the adjoint representation is twice that in the fundamental representation. The adjoint representation is insensitive for Z_N and indeed for finite N adjoint quarks are not confined. So if confinement persists it is probably of a different nature for $N \to \infty$. This makes it very unlikely that the above mentioned surviving extrema will be responsible for $N = \infty$ confinement.

We will now show that they have no influence for $N \to \infty$ if we insist on factorization for the non-perturbative sector. The argument is valid at least for fluxons in TEK. Let the extrema be numerated by k, a positive integer, with action S_k. k represents the ordering of the extrema: $S_k \geqslant S_l$ for $k > l$. Different extrema (disconnected in configuration space) with the same action will of course have different k-values. Extrema are said to survive if:

$$\lim_{N\to\infty} N S_k < \infty \,, \tag{2}$$

since then the factor $\exp(-\beta S_k)$ has a non-vanishing $N \to \infty$ limit, where β/N is kept fixed. Denote by $\langle O \rangle_k$ the expectation value of the operator O, expanding around the kth extremum:

$$\langle O\rangle_k = \frac{\int DA^{(k)} O(A) \exp\{-\beta[S(A) - S_k]\}}{\int DA^{(k)} \exp\{-\beta[S(A) - S_k]\}}\,, \tag{3}$$

where $A^{(k)}$ is the expansion of the appropriate field around the kth extremum. Furthermore the weight of each configuration is given by:

$$W_k = \left(\frac{\int DA^{(k)} \exp\{-\beta[S(A) - S_k]\}}{\int DA^{(0)} \exp[-\beta S(A)]}\right) \exp(-\beta S_k)\,. \tag{4}$$

The factor between parentheses is a perturbative quantity (of order 1 if the number of zero modes for the kth instanton equals that of the ground state). By factorization we will mean the following: There is a set of operators, such that for any pair O_1 and O_2 in this set $\langle O_1 O_2\rangle = \langle O_1\rangle\langle O_2\rangle$. This should of course at least be true in perturbation theory. For Yang–Mills theories, O will be a Wilson loop operator. A necessary condition for factorization is that the variance of an operator O is zero:

$$\langle O^2\rangle - \langle O\rangle^2 = 0\,. \tag{5}$$

If we work out the expectation values in the steepest descent approximation [12] we find [‡1]:

$$\langle O^2\rangle - \langle O\rangle^2 = \sum W_k \langle (O - \langle O\rangle_k)^2\rangle_k \Big/ \sum W_k + \sum_{k>l} W_k W_l (\langle O\rangle_k - \langle O\rangle_l)^2 \Big/ \Big(\sum W_k\Big)^2\,. \tag{6}$$

Since the integration measure in (3) is positive definite, we have for any positive definite operator O, i.e. $O(A) \geqslant 0$ for all A, that $\langle O\rangle \geqslant 0$. So W_k, $\langle (O - \langle O\rangle_k)^2\rangle_k$, $W_k W_l$ and $(\langle O\rangle_k - \langle O\rangle_l)^2$ are all positive definite and (5) implies for all k with $W_k \neq 0$ ($W_k \equiv 0$ extrema do not contribute anyhow):

$$\langle O\rangle_k = \langle O\rangle_0\,. \tag{7}$$

And this implies that the extrema have no influence (in weak coupling) since from (7) we deduce:

$$\langle O\rangle = \langle O\rangle_0\,. \tag{8}$$

$\langle O\rangle_0$ is the purely perturbative expectation value and thus all operators satisfying factorization perturbatively also factorize in the non-perturbative sector as soon as (5) is valid.

‡1 $\langle O\rangle = \sum_k W_k \langle O\rangle_k / \sum W_k$.

Our findings are therefore consistent with those of Greensite and Halpern [12]. Also we see once again an illustration of the fact that a single masterfield in the sense of Witten [13] does not exist. The $N \to \infty$ limit is more like a thermodynamic limit [14], which allows for different equilibrium configurations, with the same value in each configuration for macroscopic variables (the Wilson loops). It is also a simple exercise to show that "microscopic observables" of the kind considered by Haan [14] do not factorize due to the instantons: $\langle O\rangle_k/\langle O\rangle \neq 1 + O(1/N)$. Since for reduced models factorization for all couplings is an essential ingredient eq. (7) is a severe constraint on these models and this is the practical use of our observations.

Let us discuss the situation for the TEK model in somewhat more detail [9]. The action is given by:

$$S_{\rm TEK} = \sum_{\mu\neq\nu=1}^{4} \mathrm{tr}(1 - Z_{\mu\nu} U_\mu U_\nu U_\mu^+ U_\nu^+)\,, \tag{9}$$

where $U_\mu \in SU(N)$ are the link variables and

$$Z_{\mu\nu} = \exp(-2\pi i n_{\mu\nu}/N)\,, \tag{10}$$

with $n_{\mu\nu}$ the antisymmetric twist tensor with integer entries (mod N). The for $N \to \infty$ surviving extrema (fluxons) are given by the solutions to the equation:

$$U_\mu U_\nu U_\mu^+ U_\nu^+ = \exp(2\pi i m_{\mu\nu}/N)\,, \tag{11}$$

and they survive for $N \to \infty$ if:

$$\frac{1}{4}\sum_{\mu\neq\nu} (n_{\mu\nu} - m_{\mu\nu})^2 = k\,, \tag{12}$$

with k an integer. In order that in perturbation theory, the results of the unreduced model are retrieved, one demands that the pfaffian of n satisfies:

$$\mathrm{Pf}(n) = \tfrac{1}{8}\epsilon_{\mu\nu\alpha\beta} n_{\mu\nu} n_{\alpha\beta} = \pm N\,, \tag{13}$$

which also guarantees a zero action solution of eq. (11) with $m_{\mu\nu} = n_{\mu\nu}$. Eqs. (11) and (12) also in general enforce $\mathrm{Pf}(m) = \pm N$. Then all solutions to (11) for a given $m_{\mu\nu}$ are unique: that is up to a gauge ($U_\mu \to \Omega U_\mu \Omega^+$) and multiplication with an element of the center $Z_N (U_\mu \to Z_\mu U_\mu)$. There is a subgroup H of Z_n^4 containing N^2 elements, all equivalent to 1. To be precise for $Z_\mu \in H$ there exists an $\Omega \in SU(N)$ such that $Z_\mu U_\mu = \Omega U_\mu \Omega^+$ for all μ (this is independent of the choice of U_μ). This result is not in ref. [9], but will be published elsewhere. The solution manifold consists therefore of N^2 disconnected gauge orbits, labelled by elements of Z_N^4/H. The closed Wilson loops do not feel the degeneracy. Since it is the same for each fluxon we can ignore this N^2-fold degeneracy. In the gaussian approximation (using eq. (16) of ref. [9]) one then finds:

$$\langle O\rangle_{n-m}^n = \langle O\rangle_0^m [1 + O(1/N^2)] , \quad (14)$$

where now the subscript $n - m$ labels the different fluxons ($S_{n-m} = 8\pi^2 k/N$, for N large, see (12)) and the superscript denotes the twist tensor of the model (see (10)). Perturbatively the $1/N$ corrections correspond to the finite size corrections of the effective lattice spanned by the four four-vectors $k_\mu^{(\nu)} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} n_{\alpha\beta}$ [8]. For the fluxons these effective lattices deviate typically $O(1/N^{1/2})$ from the ground state effective lattice, which guarantees for large N:

$$\langle O\rangle_0^m = \langle O\rangle_0^n . \quad (15)$$

Together with (14) this implies (7) and thus factorization. To give an explicit example, consider the symmetric twist [8] : $n_{\mu\nu} = L, \nu < \mu, N = L^2$. The effective lattice is square with sides of length L. For the first fluxon in table 1, ref. [9] :

$$m_{\mu\nu} - n_{\mu\nu} = \delta_{\mu 1}(\delta_{\nu 2} + \delta_{\nu 3}) - \delta_{\nu 1}(\delta_{\mu 2} + \delta_{\mu 3}) ,$$

with $S_1 = 8\pi^2/N$, the effective lattice is spanned by:

$$\{(L, 1, -1, 0), (0, L+1, -1, 1),$$
$$(0, 1, L-1, 1), (0, 0, 0, L)\} ,$$

which is indeed close to a square L^4 lattice. (Note that for the fluxons $W_k = \mu_k \exp(-8\pi^2 k\beta/N)$, with μ_k the number of solutions to eq. (12)).

Let us discuss in how far the above results can be generalized to any other $N \to \infty$ theory. First we have to stress that the result is only valid for weak coupling, or equivalently for small Wilson loops. Increasing the size of the Wilson loop corresponds to an increasing effective coupling. Our analysis makes one *suspect* that surviving extrema will not have a significant contribution. The weak coupling computation is however of practical importance as a test for factorization, which is an essential feature for reduction to hold. It is crucial to realize that W_k in general also includes the contribution of approximate extrema ("multiple tunneling").

There are two ways out if one wants to avoid our results. It was implicitly assumed that W_k and $\Sigma_{k=0}^\infty W_k$ have a smooth and finite limit for $N \to \infty$. It is in principle however possible that especially ΣW_k has no finite limit. In that case one replaces W_k by $\hat{W}_k = W_k/\Sigma_{l=0}^\infty W_l$ which certainly is finite for $N \to \infty$ with $\Sigma_{k=0}^\infty \hat{W}_k = 1$, but $\hat{W}_0 = 0$. So we have instead of eq. (8): $\langle O\rangle = \langle O\rangle_k$, with $k \neq 0$ and $\hat{W}_k \neq 0$ but otherwise k is arbitrary. Again the expectation value is purely perturbative but now in a nontrivial background. The second assumption made was that the set of S_k does not have a limit point. It is beyond the scope of this letter to discuss what conditions factorization impose in the case that such an assumption is not made.

There is an alternative for the TEK, namely the quenched Eguchi–Kawai [15] (QEK) model, for which surviving extrema were found by Neuberger [16] and investigated in more detail by Parsons [17]. He claims that the extrema can potentially contribute to confinement, in particular by the first way out of eq. (8), as described above [‡2]. The extrema seem to have no connection whatsoever with those of the TEK model. This favours the assumption, also put forward in ref. [17], that some or all extrema in both reduced models are artefacts of the reduction. (To reassure the reader: The apparent discrepancy between TEK and QEK (compare eq. (4.3) of ref. [17]) only occurs for large Wilson loops, where weak coupling is not a good approximation.)

So we conclude that the confinement mechanism is probably different for $N \to \infty$. Monte Carlo data [18] seem to indicate that the string tension tends to a constant for $N \to \infty$, but this is by no means conclusive. If we want to save factorization it might just as well be possible that confinement does not survive for $N \to \infty$. This would be signalled by a zero string tension at $N = \infty$, which would fit more naturally to the equality of fundamental and adjoint string tensions. Finally one should keep an open mind for the possibility of violation of factorization especially in the cross-over region from weak to strong coupling. Checking factorization [19] is therefore important in the light of our investigations.

We thank Mark Gross, Frans Klinkhamer, Gerard 't Hooft, Jan Smit and Sander Bais for discus-

‡2 We thank the referee for pointing out that this is what was meant in ref. [17].

sion. This work is part of the research program of FOM, financially supported by ZWO.

References

[1] G. 't Hooft, Nucl. Phys. B72 (1974) 461.
[2] T. Eguchi and H. Kawai, Phys. Rev. Lett. 48 (1982) 1063.
[3] A.A. Migdal, Ann. Phys. 126 (1980) 279;
Yu.M. Makeenko and A.A. Migdal, Phys. Lett. 88B (1979) 135.
[4] E.T. Tomboulis, Phys. Rev. Lett. 50 (1983) 885.
[5] G. 't Hooft, Nucl. Phys. B138 (1978) 1.
[6] G. Mack, Phys. Rev. Lett. 45 (1980) 1378.
[7] G. 't Hooft, Commun. Math. Phys. 81 (1981) 267.
[8] A. Gonzalez-Arroyo and A. Okawa, Phys. Rev. D27 (1983) 2397.
[9] P. van Baal, Commun. Math. Phys. 92 (1983) 1.
[10] S. Coleman, The uses of instantons, Erice Lectures (1977).
[11] G. 't Hooft, Nucl. Phys. B153 (1979) 141.
[12] J. Greensite and M.B. Halpern, Phys. Rev. D27 (1983) 2545.
[13] E. Witten, in: Recent developments in gauge theories, Proc. Cargèse Summer Institute (1979), eds. G. 't Hooft et al. (Plenum, New York, 1980), p. 217.
[14] O. Haan, Phys. Lett. 106B (1981) 207.
[15] G. Bhanot, U.M. Heller and H. Neuberger, Phys. Lett. 113B (1982) 47.
[16] H. Neuberger, Phys. Lett. 119B (1982) 179.
[17] N.H. Parsons, A discussion of some nonperturbative effects and their relevance to quark confinement in the quenched Eguchi–Kawai model, University of Oxford preprint (July 1983).
[18] A. Gonzalez-Arroyo and M. Okawa, Phys. Lett. 133B (1983) 415.
[19] J. Carlson, J. Greensite, M.B. Halpern and T. Sterling, Nucl. Phys. B217 (1983) 461;
K. Fabricius and O. Haan, Phys. Lett. B131 (1983) 399.