RAY DISPERSION IN RANDOM GYROTROPIC MEDIA

S.M. GOLYNSKI [1]

Institute for Theoretical Physics of the University at Utrecht, Princetonplein 5, P.O. Box 80.006, 3508 TA Utrecht, the Netherlands

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We give a systematic derivation of the Fokker–Planck equation for the joint probability density of the position and the ray vector of rays propagating in a gyrotropic medium with random inhomogeneities. The results are a generalization of preceding work for the case of an isotropic medium.

In our previous publication [1] it was shown that the process of ray scattering in random media can be regarded as approximately markovian, in which the role of time is played by the path length traversed by the ray. We answered the question about the Fokker–Planck equation (FPE) describing the ray dispersion in isotropic media with random inhomogeneities.

In this letter we extend our preceding results to a random gyrotropic medium in the presence of a constant magnetic field. The picture of ray propagation in a gyrotropic medium becomes much more involved. We have to distinguish the ray vector $\boldsymbol{S}$ ($|\boldsymbol{S}| = 1$), which coincides with the direction of the time-average Poynting vector, from the normal vector $\boldsymbol{N}$ ($|\boldsymbol{N}| = 1$) perpendicular to the wave front. In the general case the directions of $\boldsymbol{S}$ and $\boldsymbol{N}$ are different [2–4]. Moreover the refractive indices $\mu = \mu(\boldsymbol{r}, \boldsymbol{S})$ and $n = n(\boldsymbol{r}, \boldsymbol{S})$ along the directions of $\boldsymbol{S}$ and $\boldsymbol{N}$ respectively are functions not only of the position $\boldsymbol{r}$ but also for the direction of the ray $\boldsymbol{S}$.

We begin our derivation of the FPE with the Hamilton equations of geometrical optics [2]

$$\mathrm{d}\boldsymbol{r}(\sigma)/\mathrm{d}\sigma = \boldsymbol{S}, \quad \mathrm{d}\boldsymbol{p}(\sigma)/\mathrm{d}\sigma = \nabla_r \mu\,, \tag{1}$$

where the independent variable σ is the path length of the ray and according to ref. [2]

$$\boldsymbol{p}(\boldsymbol{r}, \boldsymbol{S}) = n\boldsymbol{N} = \mu\boldsymbol{S} + \nabla_s\mu - \boldsymbol{S}(\boldsymbol{S}\nabla_s\mu)\,. \tag{2}$$

[1] Permanent address: Moscow Factory-Institute attached to the ZIL-plant, Avtozavodskaya ul. 16, 180968 Mowscow, USSR.

Substituting (2) into the second equation of (1) and transforming it, we rewrite the initial equations in the more convenient form

$$\mathrm{d}\boldsymbol{r}(\sigma)/\mathrm{d}\sigma = \boldsymbol{S}\,,$$

$$\mathrm{d}\boldsymbol{S}(\sigma)/\mathrm{d}\sigma = [\mu - \boldsymbol{S}\nabla_s\mu]^{-1}\{\nabla_r\mu - \boldsymbol{S}(\boldsymbol{S}\nabla_r\mu) - \boldsymbol{S}\nabla_r(\nabla_s\mu) + \boldsymbol{S}[\boldsymbol{S}\nabla_r(\boldsymbol{S}\nabla_s\mu)]\} \equiv \beta(\boldsymbol{r}, \boldsymbol{S})\,. \tag{3}$$

We restrict ourselves to the case

$$\mu(\boldsymbol{r}, \boldsymbol{S}) = \mu_0(\boldsymbol{r}, \boldsymbol{S}) + \alpha\mu_1(\boldsymbol{r}, \boldsymbol{S})\,, \quad \langle \mu_1(\boldsymbol{r}, \boldsymbol{S}) \rangle = 0\,,$$

with sure part μ_0 and zero average of the random inhomogeneities. Here α is the small dimensionless parameter determining the size of the fluctuation of the process under consideration.

In what follows we make use of the notation adopted in refs. [1,5]. For brevity we introduce the six-vector $u = \{\boldsymbol{r}, \boldsymbol{S}\}$ and divide the r.h.s. of (3) into the sure part

$$F_0 = \{\boldsymbol{S}, \beta_0\} = \{\boldsymbol{S}, (\mu_0 - \boldsymbol{S}\nabla_s\mu_0)^{-1}[\nabla_r\mu_0 - \boldsymbol{S}(\boldsymbol{S}\nabla_r\mu_0) - \boldsymbol{S}\nabla_r(\nabla_s\mu_0) + \boldsymbol{S}(\boldsymbol{S}\nabla_r(\boldsymbol{S}\nabla_s\mu_0))]\}\,, \tag{4}$$

and the fluctuating part $F_1 = \{0, \boldsymbol{\beta}_1\}$, whose explicit form will be defined below. Then the system (3) can be rewritten in the universal form of stochastic nonlinear differential equations as

$$\mathrm{d}u(\sigma)/\mathrm{d}\sigma = F_0(u) + \alpha F_1(u)\,, \quad \langle F_1(u) \rangle = 0\,.$$

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Applying a simple device for reducing the nonlinear problem to the linear case developed in ref. [5] one can obtain the stochastic Liouville equation for the probability density $P(u, \sigma) = P(\boldsymbol{r}, \boldsymbol{S}, \sigma)$ that, after having propagated over a distance σ, the ray has arrived in the point $\boldsymbol{r}$ with direction $\boldsymbol{S}$

$$\partial P(u, \sigma)/\partial\sigma = -\nabla[F_0(u) + \alpha F_1(u)]P(u, \sigma) . \quad (5)$$

Here ∇ is used for the operator that differentiates everything that comes after it with respect to u.

Following ref. [5] we expand the r.h.s. of (5) in successive powers of a new dimensionless parameter $\epsilon r_c \ll 1$, where $\epsilon = \max(\epsilon_1, \epsilon_2)$,

$$\epsilon_1 = \alpha(\mu_0 - \boldsymbol{S}\nabla_s\mu_0)^{-1}\nabla_r\mu_1 ,$$

$$\epsilon_2 = \alpha(\mu_0 - \boldsymbol{S}\nabla_s\mu_0)^{-1}\nabla_r(\boldsymbol{S}\nabla_s\mu_1) , \quad (6)$$

and r_c is the spatial correlation radius of the inhomogeneities in question. Disregarding in the expansion terms of order ϵ^3 we obtain

$$\frac{\partial P(u, \sigma)}{\partial\sigma} = \nabla\left(-F_0(u) + \alpha^2\int_0^\infty \frac{d(u^{-\sigma})}{d(u)} \times \langle F_1(u)\nabla_{-\sigma}F_1(u^{-\sigma}, \sigma)\rangle \frac{d(u)}{d(u^{-\sigma})}\,d\sigma\right)P(u, \sigma) . \quad (7)$$

The six-vector $u^{-\sigma}$ is defined for fixed σ by means of a mapping from the initial $u(0)$ into $u(\sigma)$ with inverse $(u^\sigma)^{-\sigma} = u$. The operator $\nabla_{-\sigma}$ denotes differentiation with respect to $u^{-\sigma}$. The first and the last terms in the integrand (7) are the jacobian determinants of the mapping. To determine this mapping one should solve the unperturbed equations (3), whose solution is a very complicated problem in the general case.

To proceed further we introduce the new limitations

$$r_c(\mu_0 - \boldsymbol{S}\nabla_s\mu_0)^{-1}\nabla_r\mu_0 \ll 1 ,$$

$$r_c(\mu_0 - \boldsymbol{S}\nabla_s\mu_0)^{-1}\nabla_r(\boldsymbol{S}\nabla_s\mu_0) \ll 1 . \quad (8)$$

Physically this means that we consider the unperturbed trajectories inside the inhomogeneities as straight lines. Under the conditions (8) one can obtain for $\sigma \lesssim r_c$

$$u^{-\sigma} = \{\boldsymbol{r} - \sigma\boldsymbol{S}, \boldsymbol{S}\} ,$$

$$d(u^{-\sigma})/d(u) = d(u)/d(u^{-\sigma}) = 1 ,$$

$$\partial/\partial r_i^{-\sigma} = \partial/\partial r_i , \quad \partial/\partial S_i^{-\sigma} = \partial/\partial S_i + \sigma\partial/\partial r_i \quad (i = 1, 2, 3) . \quad (9)$$

It should be stressed that the different directions of $\boldsymbol{S}$ inside the inhomogeneities define the different unperturbed paths. Hence we do not ignore the deviation of the actual ray from the unperturbed one and we have no restrictions with regard to the angle of scattering.

Bearing in mind (9) one can transform (7) into the usual form of the FPE (sum convention applied)

$$\frac{\partial P(u, \sigma)}{\partial\sigma} = -\frac{\partial}{\partial u_\nu}\{[F_{0\nu}(u) + C_\nu(u)]\}P(u, \sigma) + \frac{1}{2}\frac{\partial^2}{\partial u_\nu\partial u_\mu}\{C_{\nu\mu}(u)P(u, \sigma)\} \quad (\nu, \mu = 1, 2, ..., 6) , \quad (10)$$

where $F_{0\nu}(u)$ is given by (4) and the coefficients in the diffusion and convection terms are

$$C_{\nu\mu}(u) = 2\int_0^\infty \langle F_{1\nu}(u)F_{1\mu}(u^{-\sigma}, \sigma)\rangle\,d\sigma ,$$

$$C_\nu(u) = \int_0^\infty \langle[\partial F_{1\nu}(u)/\partial u_\mu]\widetilde{F}_{1\mu}(u^{-\sigma}, \sigma)\rangle\,d\sigma . \quad (11)$$

Here the explicit form of the six-vectors $F_1 = \{\boldsymbol{0}, \boldsymbol{\beta}_1\}$ and $\widetilde{F}_1 = \{\sigma\boldsymbol{\beta}_1, \boldsymbol{\beta}_1\}$ can be defined with the use of (3), (8) and (9) by

$$\boldsymbol{\beta}_1 = (\mu_0 - \boldsymbol{S}\nabla_s\mu_0)^{-1}\{\nabla_r\mu_1 - \boldsymbol{S}(\boldsymbol{S}\nabla_r\mu_1) - \boldsymbol{S}\nabla_r(\nabla_s\mu_1) + \boldsymbol{S}[\boldsymbol{S}\nabla_r(\boldsymbol{S}\nabla_s\mu_1)]\} . \quad (12)$$

According to ref. [5] the basic assumption of our derivation is

$$\langle F_{1\nu}(u)\widetilde{F}_{1\mu}(u^{-\sigma}, \sigma)\rangle \approx 0 \quad \text{for } \sigma > r_c ,$$

and similary for higher cumulants.

Taking into account (4), our main result (10) in the original representation is

$$\left(\frac{\partial}{\partial\sigma} + \boldsymbol{S}\nabla_r\right)P(\boldsymbol{r}, \boldsymbol{S}, \sigma) = -\frac{\partial}{\partial S_i}\{[F_{0,i+3} + C_{i+3}]P\} + \frac{1}{2}\frac{\partial^2}{\partial r_i\partial S_j}\{C_{i+3,j}P\} + \frac{1}{2}\frac{\partial^2}{\partial S_i\partial S_j}\{C_{i+3,j+3}P\} \quad (i, j = 1, 2, 3) . \quad (13)$$

In an isotropic medium, when $\boldsymbol{S} = \boldsymbol{N}$ and the refractive index depends only on the position of the ray, i.e. $\mu = n = n(\boldsymbol{r})$, the expressions (4) and (12) reduce to

$$F_0 = \{N, n_0^{-1}[\nabla_{\boldsymbol{r}} n_0 - N(N\nabla_{\boldsymbol{r}} n_0)]\},$$

$$\beta_1 = n_0^{-1}[\nabla_{\boldsymbol{r}} n_1 - N(N\nabla_{\boldsymbol{r}} n_1)] \ . \tag{14}$$

Substituting (14) into (11) we come to the conclusion that the equations (10) and (13) coincide with analogous ones obtained in ref. [1].

We evaluate the condition for the applicability of the FPE by analogy with ref. [1]. Defining the scale on which u varies by $\Delta L = \epsilon^{-1}$ (6) we subdivide the path length in intervals $\Delta\sigma$ such that

$$\Delta L \gg \Delta\sigma \gg r_c. \tag{15}$$

These inequalities permit us to consider the process of ray scattering as (approximately) markovian on the coarse-grained level determined by $\Delta\sigma$ [1,5].

The application of the FPE (10) in a random gyrotropic medium is justified under the conditions (8) and (15). If the sure part of the refractive index does not depend on the position of the ray, it is sufficient to bear in mind condition (15) alone.

In some applications of ray propagation in a gyrotropic medium one can be interested in the FPE for the probability density $P(\boldsymbol{r}, N, \sigma)$ instead of the $P(\boldsymbol{r}, \boldsymbol{S}, \sigma)$ considered. The solution of this problem will be given in ref. [6].

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