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Existence Results without Convexity Conditions for General Problems of Optimal Control with Singular Components

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This note presents a new, quick approach to existence results without convexity conditions for optimal control problems with singular components in the sense of E. J. McShane (*SIAM J. Control* **5** (1967), 438–485). Starting from the resolvent kernel representation of the solutions of a linear integral equation, a version of Fatou's lemma in several dimensions is shown to lead directly to a compactness result for the attainable set and an existence result for a Mayer problem. These results subsume those of L. W. Neustadt (*J. Math. Anal. Appl.* **7** (1963), 110–117), C. Olech (*J. Differential Equations* **2** (1966), 74–101), M. Q. Jacobs ("Mathematical Theory of Control," pp. 46–53, Academic Press, 1967), L. Cesari (*SIAM J. Control* **12** (1974), 319–331) and T. S. Angell (*J. Optim. Theory Appl.* **19** (1976), 63–79).

1. INTRODUCTION

Let $(T, \mathscr{T}, \mu)$ be a finite measure space and m a given dimension. The following version of Fatou's lemma in several dimensions will be the main tool of this note.

FATOU LEMMA (Balder [4]). *Suppose $\{f_k\} \subset \mathscr{L}_1(T; \mathbb{R}^m)$ is such that*

$$\{f_k^-\} \text{ is uniformly integrable,} \tag{1.1}$$

$$\lim_k \int_T f_k \, d\mu \text{ exists (in } \mathbb{R}^m). \tag{1.2}$$

Then there exists $f_* \in \mathscr{L}_1(T; \mathbb{R}^m)$ with

$$f_*(t) \text{ is a limit point of } \{f_k(t)\} \text{ a.e. in } T, \tag{1.3}$$

$$\int_T f_* \, d\mu \leqslant \lim_k \int_T f_k \, d\mu. \tag{1.4}$$

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Here $\mathscr{L}_1(T; \mathbb{R}^m)$ denotes the set of all integrable functions from T into $\mathbb{R}^m$ and $f_k^-(t) \equiv (\max(-f_k^1(t), 0), ..., \max(-f_k^m(t), 0))'$. The first such result is due to Schmeidler [14]. The present version subsumes his original version as well as similar ones by Cesari and Suryanarayana [8] and Artstein [3].

In Balder [4] the Fatou lemma was used in a way which is reminiscent of the popular deparametrization (or reduction) approach in optimal control theory, and it was demonstrated how existence results for allocation problems in economics follow quickly from it. The argument there only required taking elementary pointwise limits and applying Aumann's measurable selection theorem. Here this line of thought will be continued by showing the apparently fundamental significance of the above version of Fatou's lemma in several dimensions for the subject of existence without convexity conditions in optimal control theory, notably in connection with singular components in the sense of the fundamental paper by McShane [11].

Existence results without convexity conditions were first obtained by Neustadt [12]. Until now, the only paper in this area considering singular components (for the optimal control of a linear ordinary differential equation) is that by Cesari [7]. It will become apparent that the approach of this note is considerably more economical than his. Our main results subsume the compactness result for the attainable set of Jacobs [10, Theorem 1] (and a fortiori those of Neustadt [12] and Olech [13]) and the existence results for Mayer problems in Cesari [7, Theorem 2.1] and Angell [1, Theorem 5.2].

In connection with existence "with convexity conditions" for problems with singular components, Fatou's lemma in several dimensions has played a role already, as is seen by combining the results of Cesari and Suryanarayana [8] and Angell [2] (note also Remark 3.5 of the former reference). However, this is a rather different role which is much less direct in nature.

Throughout the remainder of this note T will be the unit interval [0, 1], equipped with the Lebesgue σ-algebra $\mathscr{E}$, and μ will be the Lebesgue measure.

Let us introduce some notation. For $m, n \in \mathbb{N}$, $\mathbb{R}^{m \times n}$ denotes the set of all $m \times n$-matrices; it is equipped with the usual norm obtained by identifying matrices with linear operators. The set of all measurable functions from T into $\mathbb{R}^{m \times n}$ is denoted by $\mathscr{M}(T; \mathbb{R}^{m \times n})$; also, the set of all integrable (essentially bounded) functions in $\mathscr{M}(T; \mathbb{R}^{m \times n})$ will be denoted by $\mathscr{L}_1(T; \mathbb{R}^{m \times n})$ ($\mathscr{L}_\infty(T; \mathbb{R}^{m \times n})$). Finally, for any Banach space S the set of all continuous functions from T into S is denoted by $\mathscr{C}(T; S)$.

2. Main Results

Let n, $\bar{n}$, $m \in \mathbb{N}$ be given dimensions and let a, $\bar{a}$, b, $\bar{b}$ be given kernels on $T^2 \equiv T \times T$ with $a \in \mathscr{C}(T; \mathscr{L}_1(T; \mathbb{R}^{n \times n}))$[1], $\bar{a} \in \mathscr{C}(T; \mathscr{L}_1(T; \mathbb{R}^{\bar{n} \times \bar{n}}))$, $b \in \mathscr{M}(T^2; \mathbb{R}^{n \times n})$, $\bar{b} \in \mathscr{M}(T^2; \mathbb{R}^{\bar{n} \times n})$ such that

$$a(t, \tau) = 0 \qquad \text{if} \quad t < \tau, \tag{2.1}$$

$$\bar{a}(t, \tau) = 0 \qquad \text{if} \quad t < \tau, \tag{2.2}$$

$$b(t, \tau) = 0 \qquad \text{if} \quad t < \tau, \qquad b \text{ is continuous on } \Delta, \tag{2.3}$$

$$\bar{b}(\cdot, \tau) \text{ is continuous on } \{t \in T : t < \tau\}, \qquad \bar{b} \text{ is continuous on } \Delta, \tag{2.4}$$

$$\sup_{T^2} |\bar{b}| < +\infty, \tag{2.5}$$

$$\bar{b}(t, \tau)\, \bar{P} \subset \bar{P} \qquad \text{if} \quad t \geqslant \tau. \tag{2.6}$$

Here Δ denotes the set $\{(t, \tau) \in T^2 : t \geqslant \tau\}$ and $\bar{P}$ the set of all $x \in \mathbb{R}^{\bar{n}}$ with $x \geqslant 0$. Also, let $c : T \times \mathbb{R}^m \to \mathbb{R}^n$, $\bar{c} : T \times \mathbb{R}^m \to (-\infty, +\infty]^{\bar{n}}$ be such that

$$c \text{ and } \bar{c} \text{ are } \mathscr{T} \times \mathscr{B}(\mathbb{R}^m)\text{-measurable}, \tag{2.7}$$

$$Q(t) \text{ is closed a.e. in } T, \tag{2.8}$$

where

$$Q(t) \equiv \{(\xi, \bar{\xi}) \in \mathbb{R}^{n+\bar{n}} : \xi = c(t, u), \bar{\xi} \geqslant \bar{c}(t, u), u \in \mathbb{R}^m\}.$$

Let Π be the set of all control parameters $\pi \equiv (\alpha, \beta, v, \bar{v}, u)$, $(\alpha, \beta) \in \Delta$, $v \in \mathbb{R}^n$, $\bar{v} \in \mathbb{R}^n$, $u \in \mathscr{M}([\alpha, \beta]; \mathbb{R}^m)$. The solution $(y, \bar{y}) \in \mathscr{C}([\alpha, \beta]; \mathbb{R}^{n+\bar{n}})$ of the system of linear integral equations

$$y(t) = \int_\alpha^t a(t, \tau)\, y(\tau)\, d\tau + \int_\alpha^t b(t, \tau)\, c(\tau, u(\tau))\, d\tau + b(t, \alpha)\, v, \tag{I$_1$}$$

$$\bar{y}(t) = \int_\alpha^t \bar{a}(t, \tau)\, y(\tau)\, d\tau + \int_\alpha^\beta \bar{b}(t, \tau)\, \bar{c}(\tau, u(\tau))\, d\tau + \bar{b}(t, \alpha)\, \bar{v} \tag{I$_2$}$$

is denoted by $(y_\pi, \bar{y}_\pi)$, provided it exists. Note the dynamically extraneous role of the part $\bar{y}$ of the state vector $(y, \bar{y})$. Entirely in accordance with McShane [11], this part is to be made up by the singular components (cf. Cesari [6, 7]).

An important specialization of (I_1)–(I_2) is as follows. Let $A \in$

[1] That is, $t \mapsto a(t, \cdot)$ is continuous from T into $\mathscr{L}_1(T; \mathbb{R}^{n \times n})$, etc.

$\mathscr{L}_1(T; \mathbb{R}^{n \times n})$, $\bar{A} \in \mathscr{L}_1(T; \mathbb{R}^{\bar{n} \times n})$ and let (D_1)–(D_2) be the system of linear ordinary differential equations

$$\dot{y}(t) = A(t)\, y(t) + c(t, u(t)), \tag{D_1}$$

$$\dot{\bar{y}}(t) = \bar{A}(t)\, y(t) + \bar{c}(t, u(t)) \qquad \text{a.e. in } [\alpha, \beta], \tag{D_2}$$

with initial conditions $y(\alpha) = v$, $\bar{y}(\alpha) = \bar{v}$.

LEMMA 2.1. *Suppose that for* $\pi \equiv (\alpha, \beta, v, \bar{v}, u) \in \Pi$

$$c(\cdot, u(\cdot)) \in \mathscr{L}_1([\alpha, \beta]; \mathbb{R}^n), \tag{2.9}$$

$$\bar{c}(\cdot, u(\cdot)) \in \mathscr{L}_1([\alpha, \beta]; \mathbb{R}^{\bar{n}}). \tag{2.10}$$

Then the solution $(y_\pi, \bar{y}_\pi) \in \mathscr{C}([\alpha, \beta]; \mathbb{R}^{n+\bar{n}})$ *of* (I_1)–(I_2) *exists. Moreover, there exists a continuous kernel* $Y: \Delta \to \mathbb{R}^{n \times n}$ *such that for every* $\pi \in \Pi$ *satisfying* (2.9)–(2.10) *the solution* y_π *of* (I_1) *can be expressed as*

$$y_\pi(t) = \int_\alpha^t Y(t, \tau)\, c(\tau, u(\tau))\, d\tau + Y(t, \alpha)\, v. \tag{2.11}$$

Proof. Define the operators $\mathscr{A}$ and $\bar{\mathscr{A}}$ by

$$\mathscr{A} y(t) \equiv \int_T a(t, \tau)\, y(\tau)\, d\tau,$$

and an analogous expression for $\bar{\mathscr{A}}$. Define C_π, $\bar{C}_\pi$ by

$$C_\pi(t) \equiv \int_\alpha^\beta b(t, \tau)\, c(\tau, u(\tau))\, d\tau + b(t, \alpha)\, v, \tag{2.12}$$

and an analogous expression for $\bar{C}_\pi$. From the continuity property of a and $\bar{a}$ it follows that $\mathscr{A}$ maps $\mathscr{C}(T; \mathbb{R}^n)$ into $\mathscr{C}(T; \mathbb{R}^n)$ and $\bar{\mathscr{A}}$ maps $\mathscr{C}(T; \mathbb{R}^n)$ into $\mathscr{C}(T; \mathbb{R}^{\bar{n}})$. By the dominated convergence theorem it follows easily from (2.2)–(2.5) and (2.9)–(2.10) that $C_\pi \in \mathscr{C}([\alpha, \beta]; \mathbb{R}^n)$, $\bar{C}_\pi \in \mathscr{C}([\alpha, \beta]; \mathbb{R}^{\bar{n}})$. We shall apply Warga [16, II.5.5, II.5.6]. First of all, note that although formally this result requires $\mathscr{T}$ to be the Borel σ-algebra on T, it also remains valid if we take $\mathscr{T}$ to be the Lebesgue σ-algebra on T (viz., the completion of the Borel σ-algebra). In view of the properties of the kernel a—notably (2.1)—it follows now that we may apply this result. Hence, the operator $I - \mathscr{A}$ is a homeomorphism, the operators $\mathscr{A}$ and $\mathscr{A}^* \equiv (I - \mathscr{A})^{-1} - I$ are compact and there exists $\kappa \in \mathscr{C}(T; \mathscr{L}_1(T; \mathbb{R}^{n \times n}))$ such that

$$\mathscr{A}^* z(t) = \int_T \kappa(t, \tau)\, z(\tau)\, d\tau. \tag{2.13}$$

Given $C \in \mathscr{C}([\alpha, \beta]; \mathbb{R}^n)$, define $\hat{C}^p \in \mathscr{C}(T; \mathbb{R}^n)$ to be any continuous extension of C to T such that $\hat{C}^p(t) = 0$ on $[0, \alpha - p^{-1}]$ and $[\alpha + p^{-1}, 1]$, with $\sup_T |\hat{C}^p| \leqslant \sup_{[\alpha,\beta]} |C|$, $p \in \mathbb{N}$. Let $\hat{y}^p \equiv (I - \mathscr{A})^{-1} \hat{C}^p$. By compactness of $\mathscr{A}^*$, a subsequence of $\{\hat{y}^p - \hat{C}^p\}$ converges to some $\hat{z} \in \mathscr{C}(T; \mathbb{R}^n)$. By (2.13) it follows directly that $\hat{z}(t) = \int_\alpha^\beta \kappa(t, \tau)\, C(\tau)\, d\tau$ on $[\alpha, \beta]$. Also, it is easy to check that on $[\alpha, \beta]$, $y(t) = Ay(t) + C(t)$ for $y \equiv \hat{z} + C$. Further, we claim that we may write $\hat{z}(t) = \int_\alpha^t \kappa(t, \tau)\, C(\tau)\, d\tau$, since $\int_t^\beta \kappa(t, \tau)$ $C(\tau)\, d\tau = 0$ for every $\alpha \leqslant t < \beta$. To see this, note that in the above argument every $\hat{y}^p$ has $\hat{y}^p(t) = 0$ on $[0, \alpha - p^{-1}]$. This gives $\hat{z}(t) = 0$ on $[0, \alpha]$; in particular, $\hat{z}(\alpha) = \int_\alpha^\beta \kappa(\alpha, \tau)\, C(\tau)\, d\tau = 0$. Since the restriction of C to any subinterval $[\alpha', \beta]$ of $[\alpha, \beta]$ is also continuous, we find now that $\int_{\alpha'}^\beta \kappa(\alpha', \tau)$ $C(\tau)\, d\tau = 0$ for all $\alpha' \geqslant \alpha$. This proves our claim. It is now a matter of straightforward calculation to combine the above with (2.13) and (2.3) so as to arrive at (2.11) with Y defined by

$$Y(t, \tau) \equiv \int_\tau^t \kappa(t, \sigma)\, b(\sigma, \tau)\, d\sigma + b(t, \tau).$$

To see that Y is continuous on Δ, note that for $(t, \tau), (t', \tau') \in \Delta$

$$|Y(t, \tau) - Y(t', \tau')| \leqslant \varepsilon_1 + \varepsilon_2 + \varepsilon_3,$$

where

$$\varepsilon_1 \equiv \sup_\Delta |b| \int_T |\kappa(t, \sigma) - \kappa(t', \sigma)|\, d\sigma,$$

$$\varepsilon_2 \equiv \int_{\tau'}^{t'} |\kappa(t, \sigma)|\, |b(\sigma, \tau) - b(\sigma, \tau')|\, d\sigma,$$

$$\varepsilon_3 \equiv \left| \int_\tau^t \kappa(t, \sigma)\, b(\sigma, \tau)\, d\sigma - \int_{\tau'}^{t'} \kappa(t, \sigma)\, b(\sigma, \tau)\, d\sigma \right|,$$

Since $\kappa \in \mathscr{C}(T; \mathscr{L}_1(T; \mathbb{R}^{n \times n}))$, we obviously have $\varepsilon_1 \to 0$ as $t' \to t$. Also, it follows from this and (2.3) that $\varepsilon_2 \to 0$ as $(t', \tau') \to (t, \tau)$ by an obvious application of the dominated convergence theorem. Finally, $\varepsilon_3 \to 0$ as $(t', \tau') \to (t, \tau)$ by the dominated convergence theorem, since $|\kappa(t, \cdot)\, b(\cdot, \tau)|$ is integrable. Q.E.D.

Of course, for the special case given by (D_1)–(D_2) the above lemma agrees with the variation of parameters formula for the solution of (D_1); cf. Warga [16, II.4.8]. Hence, the reader who is only interested in the optimal control of (D_1)–(D_2) can skim over the above lemma.

Our fundamental result on existence without convexity conditions for the optimal control of (I_1)–(I_2) is as follows.

THEOREM 2.2. *Suppose* $\{\pi_k\} \subset \Pi$, $\pi_k = (\alpha_k, \beta_k, v_k, \bar{v}_k, u_k)$, *satisfies*

$$\{(\alpha_k, \beta_k, v_k, \bar{v}_k)\} \text{ converges to } (\alpha_0, \beta_0, v_0, \bar{v}_0) \in \Delta \times \mathbb{R}^{n+\bar{n}}, \tag{2.14}$$

$$\{c(\cdot, u_k(\cdot))\} \text{ is uniformly integrable,}^2 \tag{2.15}$$

$$\{\bar{c}^-(\cdot, u_k(\cdot))\} \text{ is uniformly integrable,}^2 \tag{2.16}$$

$$\sup_k \int_{\alpha_k}^{\beta_k} |\bar{c}^+(t, u_k(t))|\, dt < +\infty. \tag{2.17}$$

Then there exists a subsequence $\{\pi_{k'}\}$ *of* $\{\pi_k\}$ *and* $u_* \in \mathcal{M}([\alpha_0, \beta_0]; \mathbb{R}^m)$ *such that for* $\pi_* \equiv (\alpha_0, \beta_0, v_0, \bar{v}_0, u_*)$

$$\lim_{k'} (y_{\pi_{k'}}(\alpha_{k'}), y_{\pi_{k'}}(\beta_{k'})) = (y_{\pi_*}(\alpha_0),\ y_{\pi_*}(\beta_0)), \tag{2.18}$$

$$\lim_{k'} (\bar{y}_{\pi_{k'}}(\alpha_{k'}), \bar{y}_{\pi_{k'}}(\beta_{k'})) \geqslant (\bar{y}_{\pi_*}(\alpha_0), \bar{y}_{\pi_*}(\beta_0)). \tag{2.19}$$

Proof. We shall write y_k for y_{π_k}, etc. Also, we shall only prove (2.18)–(2.19) for the terminal states $\{(y_k(\beta_k), \bar{y}_k(\beta_k))\}$, since the proof for the initial states is an exact copy. (Let us note incidentally that $y_k(\alpha_k) = b(\alpha_k, \alpha_k)\, v_k$ by our assumptions.) By Lemma 2.1 we have

$$\int_T f_k(t)\, dt = y_k(\beta_k) - Y(\beta_k, \alpha_k)\, v_k,$$

$$\int_T \bar{f}_k(t)\, dt = \bar{y}_k(\beta_k) - \bar{Y}(\beta_k, \alpha_k)\, v_k - \bar{b}(\beta_k, \alpha_k)\, \bar{v}_k,$$

with $f_k(t) \equiv Y(\beta_k, t)\, c(t, u_k(t))$ on $[\alpha_k, \beta_k]$, $\equiv 0$ elsewhere and $\bar{f}_k(t) \equiv \bar{Y}(\beta_k, t)$ $c(t, u_k(t)) + \bar{b}(\beta_k, t)\, \bar{c}(t, u_k(t))$ on $[\alpha_k, \beta_k]$, $\equiv 0$ elsewhere. Here

$$\bar{Y}(t, \tau) \equiv \int_\tau^t \bar{a}(t, \sigma)\, Y(\sigma, \tau)\, d\sigma.$$

In complete analogy to the proof of the continuity of Y on Δ in Lemma 2.1, we can prove that $\bar{Y} : \Delta \to \mathbb{R}^{\bar{n} \times n}$ is continuous. By the above and (2.15)–(2.17) it follows that the sequence $\{(y_k(\beta_k), \bar{y}_k(\beta_k))\}$ is bounded. Hence we may suppose without loss of generality that $z \equiv \lim_k y_k(\beta_k)$ and $\bar{z} \equiv \lim_k \bar{y}_k(\beta_k)$ exist. By continuity of Y, $\bar{Y}$, $\bar{b}$ on Δ we now have

$$\lim_k \int_T f_k(t)\, dt = z - Y(\beta_0, \alpha_0)\, v_0, \tag{2.20}$$

$$\lim_k \int_T \bar{f}_k(t)\, dt = \bar{z} - \bar{Y}(\beta_0, \alpha_0)\, v_0 - \bar{b}(\beta_0, \alpha_0)\, \bar{v}_0. \tag{2.21}$$

[2] Each $c(\cdot, u_k(\cdot))$, $\bar{c}(\cdot, u_k(\cdot))$ is taken to be zero outside $[\alpha_k, \beta_k]$.

Further, let us define $c_k(t) \equiv c(t, u_k(t))$ on $[\alpha_k, \beta_k]$, $\equiv 0$ elsewhere, and $\bar{c}_k(t) \equiv \bar{c}(t, u_k(t))$ on $[\alpha_k, \beta_k]$, $\equiv 0$ elsewhere. Again by (2.15)–(2.17), we may suppose without loss of generality that $\lim_k \int_T c_k(t)\, dt$ as well as $\lim_k \int_T \bar{c}_k(t)\, dt$ exist. By the Fatou lemma of Section 1, applied to

$$\{\phi_k\} \subset \mathscr{L}_1(T; \mathbb{R}^{3n+2\bar{n}}), \qquad \phi_k \equiv (f_k, -f_k, c_k, \bar{f}_k, \bar{c}_k), \tag{2.22}$$

we find that there exist $f_*, f^*, c_* \in \mathscr{L}_1(T; \mathbb{R}^n)$ and $\bar{f}_*, \bar{c}_* \in \mathscr{L}_1(T; \mathbb{R}^{\bar{n}})$ such that

$$(f_*, f^*, c_*, \bar{f}_*, \bar{c}_*)(t) \text{ is a limit point of } \{\phi_k(t)\} \quad \text{a.e.} \quad \text{in } T, \tag{2.23}$$

$$\int_T f_*(t)\, dt \leqslant \lim_k \int_T f_k(t)\, dt, \qquad \int_T f^*(t)\, dt \leqslant -\lim_k \int_T f_k(t)\, dt, \tag{2.24}$$

$$\int_T \bar{f}_*(t)\, dt \leqslant \lim_k \int_T \bar{f}_k(t)\, dt. \tag{2.25}$$

From (2.23) we conclude that $f_*(t) = -f^*(t)$ a.e. in T. Hence, the two inequalities in (2.24) can be summarized as

$$\int_T f_*(t)\, dt \equiv \lim_k \int_T f_k(t)\, dt. \tag{2.24}$$

For a.e. $t \in T$ there exists by (2.23) a subsequence $\{k_t\}$ of $\{k\}$ such that $(f_*(t), c_*(t), \bar{f}_*(t), \bar{c}_*(t)) = \lim_{k_t} \psi_{k_t}(t)$, where $\psi_k \equiv (f_k, c_k, \bar{f}_k, \bar{c}_k)$. We claim that for a.e. t there exists $u_t \in \mathbb{R}^m$ such that

$$f_*(t) = Y(\beta_0, t)\, c(t, u_t) \qquad \text{on} \quad [\alpha_0, \beta_0], \tag{2.26}$$

$$\bar{f}_*(t) \geqslant \bar{Y}(\beta_0, t)\, c(t, u_t) + \bar{b}(\beta_0, t)\, \bar{c}(t, u_t) \qquad \text{on} \quad [\alpha_0, \beta_0], \tag{2.27}$$

and $f_*(t) = 0$, $\bar{f}_*(t) = 0$ elsewhere. First, suppose that $\alpha_0 < t < \beta_0$; then eventually $\alpha_k < t < \beta_k$, so by definition of $f_k(t)$, $\bar{f}_k(t)$ and continuity of Y, $\bar{Y}$, $\bar{b}$ we have by the above

$$f_*(t) = Y(\beta_0, t)\, c_*(t),$$
$$\bar{f}_*(t) = \bar{Y}(\beta_0, t)\, c_*(t) + \bar{b}(\beta_0, t)\, \bar{c}_*(t).$$

Also, by (2.8) it follows that

$$(c_*(t), \bar{c}_*(t)) \in Q(t).$$

In view of the definition of $Q(t)$ and property (2.6) of $\bar{b}$, (2.26)–(2.27) follow. Second, if $t < \alpha_0$ then eventually $t < \alpha_k$, so by definition of $f_k(t), \bar{f}_k(t)$ we conclude that $f_*(t) = 0$, $\bar{f}_*(t) = 0$; of course, when $t > \beta_0$ a similar argument holds. Hence our claim concerning (2.26)–(2.27) has been proven.

Next, we apply Aumann's measurable selection theorem. By (2.7) the set of all $(t, u) \in [\alpha_0, \beta_0] \times \mathbb{R}^m$ such that $f_*(t) = Y(\beta_0, t)\, c(t, u), \bar{f}_*(t) \geqslant \bar{Y}(\beta_0, t)\, c(t, u) + \bar{b}(\beta_0, t)\, \bar{c}(t, u)$ is $\mathscr{E}_0 \times (\mathbb{R}^m)$-measurable, where $\mathscr{E}_0$ denotes the Lebesgue σ-algebra on $[\alpha_0, \beta_0]$. Hence, by Aumann's theorem (Himmelberg [9, Theorem 5.2]) there exists $u_* \in \mathscr{M}([\alpha_0, \beta_0]; \mathbb{R}^m)$ such that a.e. in T

$$f_*(t) = Y(\beta_0, t)\, c(t, u_*(t)),$$
$$\bar{f}_*(t) \geqslant \bar{Y}(\beta_0, t)\, c(t, u_*(t)) + \bar{b}(\beta_0, t)\, \bar{c}(t, u_*(t)).$$

Combining (2.20)–(2.27) with the explicit representation (2.11) for the terminal state $(y(\beta_0), \bar{y}(\beta_0))$, we conclude that (2.18)–(2.19) have been proven. Q.E.D.

Remark 2.3. The major part of the above proof deals with the control functions u_k via the functions c_k, $\bar{c}_k$. The only time when the topological nature of the control space plays a role is when we apply Aumann's measurable selection result. Hence, instead of working with $U = \mathbb{R}^m$, we might as well have worked with a more abstract control space U. Aumann's measurable selection result remains valid if we take U to be a metrizable Lusin (alias standard Borel) space, so this is what we could have started with.

Remark 2.4. In usual models the control parameters $\pi \in \Pi$, $\pi \equiv (\alpha, \beta, v, \bar{v}, u)$ are subject to the restriction

$$u(t) \in U(t) \qquad \text{a.e. in} \quad [\alpha, \beta]. \tag{2.28}$$

Here $t \mapsto U(t)$ is a multifunction from T into $\mathbb{R}^m$ with a graph which is $\mathscr{E} \times \mathscr{B}(\mathbb{R}^m)$-measurable. Such a restriction can be dealth with directly by Theorem 2.2, as one can see by introducing $\bar{c}_{\bar{n}+1} : T \times \mathbb{R}^m \to [0, +\infty]$, defined by

$$\bar{c}_{\bar{n}+1}(t, u) \equiv 0 \qquad \text{if} \quad u \in U(t), \qquad \equiv +\infty \qquad \text{if} \quad u \notin U(t),$$

and redefining $\bar{\bar{c}} \equiv (\bar{c}, c_{\bar{n}+1})'$. Note that (2.7), (2.16)–(2.17) remain valid for $(c, \bar{\bar{c}})$ under (2.28). Also, to require now (2.8) is to say that

$$Q'(t) = \{(\xi, \bar{\xi}) \in \mathbb{R}^{n+\bar{n}} : \xi = c(t, u), \bar{\xi} \geqslant \bar{c}(t, u), u \in U(t)\}$$

must be closed a.e. in T.

Remark 2.5. The above remark is actually a special case of the following. Suppose the control parameters are subject to the restriction

$$\int_{\alpha}^{\beta} h(t, u(t))\, dt \leqslant 1, \tag{2.29}$$

where $h : T \times \mathbb{R}^m \to (-\infty, +\infty]$ is $\mathscr{T} \times \mathscr{B}(\mathbb{R}^m)$-measurable. Then Theorem 2.2 applies to this situation, provided that

$$\{h^-(\cdot, u_k(\cdot))\} \text{ is uniformly integrable}$$

and (2.8) is replaced by the condition that

$$Q'(t) \equiv \{(\xi, \tilde{\xi}) \in \mathbb{R}^{n+\bar{n}+1} : \xi = c(t, u), \tilde{\xi} \geqslant \tilde{c}(t, u), u \in \mathbb{R}^m\}$$

is closed a.e. in T; here $\tilde{c} \equiv (\bar{c}, h)'$. Moreover, if c, $\bar{c}$ are such that for every $\varepsilon > 0$ there exists $f_\varepsilon \in \mathscr{L}_1(T)$ such that

$$\max(|c(t, u)|, |\bar{c}^-(t, u)|) \leqslant \varepsilon h(t, u) + f_\varepsilon(t) \qquad \text{on} \quad T \times \mathbb{R}^m$$

then (2.15)–(2.16) are automatically satisfied under (2.29). If in addition

$$|\bar{c}^+(t, u)| \leqslant \eta h(t, u) + f(t)$$

for certain $\eta > 0$, $f \in \mathscr{L}_1(T)$, then also (2.17) holds under (2.29).

Remark 2.6. Suppose that for every $t \in T$

$$c(t, \cdot) \text{ is continuous} \quad \text{on } \mathbb{R}^m,$$

$$\bar{c}_1(t, \cdot) \text{ is inf-compact}^3 \quad \text{on } \mathbb{R}^m,$$

$$\bar{c}_j(t, \cdot) \text{ is lower semicontinuous} \quad \text{on } \mathbb{R}^m, \qquad j = 2, \ldots, \bar{n}.$$

Then (2.8) holds automatically.

We shall now derive two corollaries from Theorem 2.2. For our conceptual convenience we shall suppose from now on that

$$\bar{b}(t, \tau) = 0 \qquad \text{if} \quad t < \tau,$$

$$b(t, t) = I, \qquad \bar{b}(t, t) = I,$$

where I denotes the respective unit matrices. As a consequence, we have in (I_1)–(I_2) that

$$(y(\alpha), \bar{y}(\alpha)) = (v, \bar{v}).$$

[3] That is, for every $\beta \in \mathbb{R}$ the set $\{u \in \mathbb{R}^m : \bar{c}_1(t, u) \leqslant \beta\}$ is compact.

Let Γ be a given closed subset of Δ and V, $\bar{V}$ given compact sets in $\mathbb{R}^n$ and $\mathbb{R}^{\bar{n}}$, respectively. Define $R(\Pi_0)$ to be the set of all attainable points $(z, \bar{z})$ for which there exists $\pi \in \Pi_0$, $\pi \equiv (\alpha, \beta, v, \bar{v}, u)$, such that

$$z = y_\pi(\beta), \qquad \bar{z} = \bar{y}_\pi(\beta).$$

Here Π_0 is the set of all $(\alpha, \beta, v, \bar{v}, u)$ such that $(\alpha, \beta) \in \Gamma$, $v \in V$, $\bar{v} \in \bar{V}$ and $u \in \mathscr{M}([\alpha, \beta]; \mathbb{R}^m)$.

COROLLARY 2.7. *Suppose that*

$$\{c(\cdot, u(\cdot)) : \pi \in \Pi_0\} \text{ is uniformly integrable,}^4 \tag{2.30}$$

$$\{\bar{c}^-(\cdot, u(\cdot)) : \pi \in \Pi_0\} \text{ is uniformly integrable,}^4 \tag{2.31}$$

$$\sup_{\pi \in \Pi_0} \int_\alpha^\beta |\bar{c}^+(t, u(t))| \, dt < +\infty. \tag{2.32}$$

Then for every sequence $\{(z_k, \bar{z}_k)\} \subset R(\Pi_0)$ there exist a subsequence $\{(z_{k'}, \bar{z}_{k'})\}$ and $(z_*, \bar{z}_*) \in R(\Pi_0)$ such that

$$\lim_{k'} z_{k'} = z_*, \qquad \lim_{k'} \bar{z}_{k'} \geqslant \bar{z}_*.$$

Remark 2.8. Constraints of the types mentioned in Remarks 2.4–2.5 can also be dealt with directly by Corollary 2.7, namely, by augmenting the singular components in the way indicated there.

Let $e : \Gamma \times V \times \mathbb{R}^n \times \bar{V} \times \mathbb{R}^{\bar{n}} \to (-\infty, +\infty]$ be lower semicontinuous in all variables and nondecreasing in its last variable.

COROLLARY 2.9. *Suppose that* (2.30)–(2.32) *hold. Then there exists* $\pi_* \in \Pi_0$ *such that*

$$J(\pi_*) = \inf_{\Pi_0} J,$$

where $J(\pi)$ *denotes the Mayer criterion*

$$J(\pi) \equiv e(\alpha, \beta, v, y_\pi(\beta), \bar{v}, \bar{y}_\pi(\beta)).$$

Proof. Let $\{\pi_k\}$ be a minimizing sequence for the problem, $\pi_k \equiv (\alpha_k, \beta_k, v_k, \bar{v}_k, u_k)$. Since Γ, V and $\bar{V}$ are compact we may suppose without loss of generality that there exist $(\alpha_0, \beta_0) \in \Gamma$, $v_0 \in V$ and $\bar{v}_0 \in \bar{V}$ such that $\{(\alpha_k, \beta_k, v_k, \bar{v}_k)\}$ converges to $(\alpha_0, \beta_0, v_0, \bar{v}_0)$. By Theorem 2.2 there exists a subsequence $\{\pi_{k'}\}$ and $u_* \in \mathscr{M}([\alpha_0, \beta_0]; \mathbb{R}^m)$ with for $\pi_* \equiv (\alpha_0, \beta_0, v_0, \bar{v}_0, u_*) \in \Pi$

[4] Cf. footnote 2.

that $\lim_{k'} y_{k'}(\beta_{k'}) = y_{\pi_*}(\beta_0)$, $\bar{z} \equiv \lim_{k'} \bar{y}_{k'}(\beta_{k'}) \geqslant \bar{y}_{\pi_*}(\beta_0)$. Note that $\pi_* \in \Pi_0$. By lower semicontinuity of e it follows that

$$\inf_{\Pi_0} J \geqslant e(\alpha_0, \beta_0, v_0, y_{\pi_*}(\beta_0), \bar{v}_0, \bar{z}).$$

Hence, the desired result follows now from the monotonicity property of e.

Q.E.D.

Remark 2.10. Let B be a closed subset of $\Gamma \times V \times \mathbb{R}^n \times \bar{V} \times \mathbb{R}^{\bar{n}}$ such that $(\alpha, \beta, v, x, \bar{v}, \bar{x}) \notin B$ implies $(\alpha, \beta, v, x, \bar{v}, \tilde{x}) \notin B$ whenever $\tilde{x} \geqslant \bar{x}$. The additional constraint

$$(\alpha, \beta, v, y_\pi(\beta), \bar{v}, \bar{y}_\pi(\beta)) \in B$$

is already dealt with by Corollary 2.9, as is seen by redefining e: we set $e' \equiv e$ on B, $e' \equiv +\infty$ on the complement of B.

Corollary 2.7 extends the compactness result of Jacobs [10, Theorem 1] to a model with singular components and with a linear integral equation as its dynamical system. Also, the result of Jacobs [10] deals only with a constant constraint set $U(t)$ (cf. Remark 2.4). In view of our remarks, Corollary 2.9 extends the existence result of Cesari [7, Theorem 2.1] to a model with a linear integral equation. (Actually, no compactness conditions are imposed upon V, $\bar{V}$ by Cesari [7], but it is easy to see that such conditions are implied by his compactness condition involving the set P (his notation).) In view of our remarks, Corollary 2.9 also extends the existence result of Angell [1, Theorem 5.2] to a model with singular components. Notice also that of the cited results in the literature none implies the other ones. Apart from this, there is a substantial number of more technical differences between this note and the above papers (e.g., our treatment of constraints by means of functions that can take the value $+\infty$ and the fact that the only topological conditions on c, $\bar{c}$ are imposed here through our condition (2.8) for the "orientor field" Q). On each of these technical counts this note turns out to have the least restrictive assumptions.

It is possible to consider a more abstract version of the dynamical system (I_1)–(I_2) with a compact metric space as the time domain. That such an extension can be treated in essentially the same way as presented here is already apparent from the role of Warga [16, II.5.5, II.5.6] in Lemma 2.1. For a model with a variable time domain, however, such an extension would seem to be rather artificial. In contrast, when the model admits a fixed time domain an even more general dynamical system can be considered, namely, a linear functional-integral equation of the type considered in Warga [16, II.5.5, II.5.6]. For such a model there is no need to demonstrate the continuity of the kernel Y, as was done in Lemma 2.1; hence the corresponding version of Lemma 2.1 requires a proof that is already contained in our present proof of that lemma. In such a model the proof of the corresponding version of Theorem 2.2 is also simpler. We shall leave it to the reader to work out these remarks.

Different control problems with fixed time domain have been studied by Banks and Jacobs [5] (optimal control of a certain functional-integral equation) and Suryanarayana [15] (optimal control of a certain partial differential equation). Although each of these problems belongs to a category not discussed above, they, as well as their extensions having singular components, can also be approached by means of the Fatou lemma. This is evident if one considers the explicit representations given in Banks and Jacobs [5, (2.7)] and Suryanarayana [15, (10)]; in both instances the set T of this note has to be taken multidimensional.

3. Epilogue

In this note a new approach has been indicated to proving existence results without convexity. We hope that it has become apparent that the employment of this approach (mainly a matter of working out pointwise limit arguments) is much less demanding than the usual approaches involving Lyapunov's theorem. The point of the matter is, of course, that Lyapunov's theorem has already been "processed" into the Fatou lemma and that the end-product (viz. the Fatou lemma) is so much easier to work with than the totality of its ingredients in complicated situations where the existence question is raised.

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