

Vortex-Lattice Melting in a One-Dimensional Optical Lattice

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We investigate quantum fluctuations of a vortex lattice in a one-dimensional optical lattice for realistic numbers of particles and vortices. Our method gives full access to all the modes of the vortex lattice and we discuss in particular the Bloch bands of the Tkachenko modes. Because of the small number of particles in the pancake Bose-Einstein condensates at every site of the optical lattice, finite-size effects become very important. Therefore, the fluctuations in the vortex positions are inhomogeneous and the melting of the lattice occurs from the outside inwards. By looking into correlations between neighboring vortices, we identify new solid and liquid phases. Tunneling between neighboring pancakes substantially reduces the inhomogeneity as well as the size of the fluctuations.

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Introduction.—Very rapidly rotating ultracold bosonic gases have been predicted to form highly correlated quantum states [1–3]. In these states, the Bose-Einstein condensate has been completely depleted by quantum fluctuations, and quantum liquids appear with excitations that can carry fractional statistics. Some of these states have been identified with (bosonic) fractional quantum Hall states [2,4,5]. It is by now a long-standing goal to observe the experimental signatures of these very interesting states in the context of ultracold quantum gases. The conditions for these states to form have been expressed in the requirement that the ratio $\nu = N/N_v$ of the number of atoms N and the number of vortices N_v should be smaller than some critical value ν_c . The ratio ν plays the role of the filling factor and estimates for the critical ν_c are typically around 8 [2,6]. These estimates are made for infinitely large systems.

However, observed filling factors are, up until now, always greater than 100, where almost perfect hexagonal lattices form and no sign of melting can be seen [7]. These experiments are carried out with Bose-Einstein condensates consisting of typically 10^5 particles, whereas the maximum number of vortices observed is around 300. Decreasing the number of particles results in loss of experimental signal, whereas the number of vortices is limited by the rotation frequency that has to be smaller than the transverse trapping frequency. Adding a quartic potential, which stabilizes the condensate also for rotation frequencies higher than the transverse trap frequency, has until now not improved this situation [8], although it has opened up the possibility of forming a giant vortex in the center of the cloud [9,10]. In recent work, however, it was realized that quantum fluctuations of the vortices can be greatly enhanced, without losing experimental signal, by using a one-dimensional optical lattice [11]. The optical lattice divides the Bose-Einstein condensate into a stack of two-dimensional pancake condensates that are weakly coupled by tunneling as schematically shown in Fig. 1. The number of particles in a single pancake is much

smaller than in a Bose-Einstein condensate in a harmonic trap, and therefore the fluctuations are much bigger. Moreover, by varying the coupling between the pancakes, it is possible to study the dimensional crossover between decoupled two-dimensional melting, and the strong-coupling limit where the three-dimensional situation is recovered. Recently, the density profiles for quantum Hall liquids in such a geometry have also been calculated [12]. Because of the small number of particles in each pancake shaped Bose-Einstein condensate, finite-size effects become very pronounced in this setup. In particular, the critical filling factor for the melting of the lattice ν_c changes compared to the homogeneous situation. Moreover, melting is not expected to occur homogeneously but starts at the outside and then gradually moves inwards as the rotation speed increases [13]. Therefore, phase coexistence is expected, where a vortex crystal is surrounded by a vortex liquid. In this Letter we study this interesting physics by investigating the quantum fluctuations of the vortex lattice for realistic numbers of particles and vortices. To decide whether or not the vortex lattice is melted we use the Lindemann criterion, which in this case has to be applied locally.

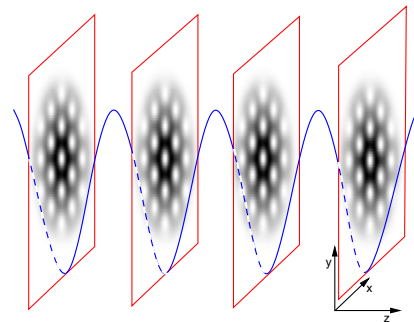


FIG. 1 (color online). Setup in which the melting of the vortex lattice is studied. An optical lattice along the z direction (indicated in blue) divides the condensate into pancake condensates which are coupled by tunneling processes.

Besides the possibility of observing highly correlated quantum states after the vortex lattice has melted, also the vortices themselves have very interesting properties. For moderately rotating gases, the vortices are observed to order in a Abrikosov lattice [14,15]. The lowest mode of the lattice is the so-called Tkachenko mode [16], which recently has been investigated experimentally [7,17] and theoretically [18–20]. Since the study of the fluctuations of the vortex lattice requires full information on all the modes of the lattice, we also can identify this mode. Because of the optical lattice, this mode actually becomes a Bloch band and we also determine the dispersion of the Tkachenko modes in the axial direction of the optical lattice.

Lattice vibrations.—The action describing the system in the rotating frame is given by $S = \int dt \int d^3\mathbf{x} \mathcal{L}(\mathbf{x}, t)$, with the Lagrange density given by

$$\mathcal{L} = \Psi^* \left[i\hbar \partial_t + \frac{\hbar^2 \nabla^2}{2m} - V(\mathbf{x}) + \Omega L_z - \frac{g}{2} |\Psi|^2 \right] \Psi, \quad (1)$$

where $\Psi(\mathbf{x}, t)$ is the Bose-Einstein condensate wave function, m is the mass of the atoms, Ω is the rotation frequency, $L_z = i\hbar(y\partial_x - x\partial_y)$ is the angular momentum operator, and $g = 4\pi\hbar^2 a/m$ is the interaction strength, with a the three-dimensional scattering length. The combination of an optical lattice in the axial direction and harmonic confinement in the radial direction gives an external potential $V(\mathbf{x}) = V_z \cos^2(2\pi z/\lambda) + \frac{1}{2}m\omega_\perp^2(x^2 + y^2)$, where λ is the wavelength of the laser producing the optical lattice. Assuming that the optical lattice is deep enough, we can perform a tight-binding approximation and write $\Psi(\mathbf{x}, t) = \sum_i \Psi(z - z_i) \Psi_i(x, y, t)$, where $\Psi(z)$ is chosen to be the lowest Wannier function of the optical lattice. Hence the Lagrange density can be written as $\mathcal{L} = t \sum_{\langle ij \rangle} \Psi_i^* \Psi_j + \sum_i \mathcal{L}_i$, with the on-site terms

$$\mathcal{L}_i = \Psi_i^* \left[i\hbar \partial_t + \frac{\hbar^2 \nabla^2}{2m} - \frac{m\omega_\perp^2 r^2}{2} + \Omega L_z - \frac{g'}{2} |\Psi_i|^2 \right] \Psi_i, \quad (2)$$

and $\langle ij \rangle$ denotes that the sum is taken over nearest neighboring sites. Moreover, we have defined the interaction coefficient $g' = 4\pi\hbar^2 a/\sqrt{2\pi}\ell_z m$ and hopping amplitude $t = 4V_z^{3/4} E_z^{1/4} \exp[-2\sqrt{V_z/E_z}]/\sqrt{\pi}$, where $E_z = 2\pi^2\hbar^2/\lambda^2 m$ is the recoil energy associated with the optical lattice and $\ell_z = \lambda/2\pi(V_z/E_z)^{1/4}$ is the harmonic length associated with the optical lattice.

Melting is only expected for a Bose-Einstein condensate that is rotating very fast. The wave function can then be taken to be part of the lowest Landau level, implying that we describe the condensate as a compressible fluid. Thus we consider wave functions of the form $\prod_n (w - w_n) \times \exp[-|w|^2/2]$, where $w = (x + iy)/\ell$, $w_n = (x_n + iy_n)/\ell$, and (x_n, y_n) is the position of the n th vortex. Here ℓ is the “magnetic length.” To increase the validity of this study, we use ℓ as a variational parameter instead of fixing it to

the radial harmonic length [21]. The associated frequency is $\omega = \hbar/m\ell^2$. From now on distances are rescaled by ℓ , frequencies are scaled by the radial trapping frequency ω_\perp , and we define a dimensionless interaction strength by means of $U = Nm g'/4\pi\hbar^2 = Na/\sqrt{2\pi}\ell_z$. In the calculations we take the scattering length of ^{87}Rb , $\lambda = 700$ nm, and $V_z/E_z = 16$, which gives $U = 25N$. The on-site energy part of the Lagrange density depends in this approximation only on the particle density and becomes $(\omega + 1/\omega - 2\Omega)r^2 n(\mathbf{r})/2 + 2\pi\omega U n^2(\mathbf{r})$. The lowest Landau level wave functions and, therefore, also the atomic density, are fully determined by the location of the vortices. Hence, we replace the functional integral over the condensate wave functions by a path integral over the vortex positions.

To determine the quadratic fluctuations around the Abrikosov lattice, we first have to find the classical ground state, which we calculated for up to 37 vortices. For small numbers of vortices, the ground state is distorted from the hexagonal lattice [22]. In general, there are also vortices far outside the condensate. The coarse-grained atomic density is well approximated by a Thomas-Fermi profile [23]. For fixed interaction U and different rotation frequencies pictures of the classical ground state are given in Fig. 2. Then we study the quadratic fluctuations by expanding the action up to second order in the fluctuations [24]. This yields an action of the form

$$S = \sum_i \mathbf{u}_i \cdot (\mathbf{T}i\partial_t - \mathbf{E}) \cdot \mathbf{u}_i - t \sum_{\langle ij \rangle} \mathbf{u}_i \cdot \mathbf{J} \cdot \mathbf{u}_j, \quad (3)$$

where $\mathbf{u}_i = (\delta x_{1,i}, \delta y_{2,i}, \delta x_{2,i}, \delta y_{2,i}, \dots) \equiv (\dots, \mathbf{u}_{ni}, \dots)$ is the total displacement vector of all the point vortices in the pancake condensate at site i , and \mathbf{T} , \mathbf{E} , and \mathbf{J} are matrices

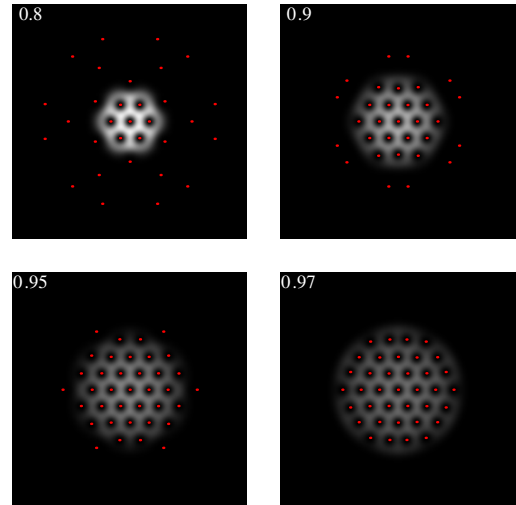


FIG. 2 (color online). Classical vortex lattice and density profile for rotation frequencies $\Omega/\omega_\perp = 0.8, 0.9, 0.95$, and 0.97 . Here $U = 10$, which corresponds to $N = 250$. White means high density, black low density. The vortex positions are indicated by a red dot, such that also the vortices outside the condensate are visible.

depending on Ω , U , and the classical lattice positions. To diagonalize this action along the z axis, we perform a Fourier transformation to obtain

$$S = \sum_k \mathbf{u}_k^* \cdot \{ \mathbf{T} i \partial_t - \mathbf{E} - t[1 - \cos(k\lambda/2)\mathbf{J}] \} \cdot \mathbf{u}_k. \quad (4)$$

Finally, we completely diagonalize this action by the transformation $\mathbf{v}_k = \mathbf{P}_k \mathbf{u}_k$ such that the action becomes $S = \sum_{k,\alpha} \mathbf{v}_{k\alpha}^* [i \partial_t - \omega_\alpha(k)] \mathbf{v}_{k\alpha}$, where $\omega_\alpha(k)$ are the mode frequencies of the vortex lattice. This means that the $\mathbf{v}_{k\alpha}$, where k labels the momentum in the z direction and α labels the mode, correspond to bosonic operators with commutation relation $[\mathbf{v}_{k\alpha}, \mathbf{v}_{k'\alpha'}^\dagger] = \delta_{kk'} \delta_{\alpha\alpha'}$. This allows us to calculate the expectation value for the fluctuations in the vortex positions, but also for the correlations between the various point vortices.

Tkachenko modes.—The Tkachenko modes are almost transverse modes of the vortex lattice. In a harmonic trap with cylindrical symmetry they become modes which are almost angular. In the radial direction their spectrum is discretized, because of the finite lattice size. For 37 vortices 6 Tkachenko modes can be identified. A close comparison with continuum theory for a finite-size system, where also a discrete spectrum was found [25], is possible but beyond the scope of this Letter. Moreover, the Tkachenko modes also have a dispersion in the z direction. Without the optical lattice some aspects of these modes were recently investigated [26]. For typical parameters, this dispersion is plotted in Fig. 3. As is clearly visible, there is one gapless mode, which is linear at long wavelengths, while the other modes are roughly just tight-binding-like. Moreover, various avoided crossings between these modes are clearly visible. The gapless mode is the Goldstone mode associated with the spontaneously broken rotational $O(2)$ symmetry due to the presence of the vortex lattice. When the tunneling rate is very small, the gapped

modes have exactly a tight-binding dispersion and the gapless mode gets a dispersion proportional to $\sin(k\lambda/2)$. This can be understood by observing that in this case the modes are decoupled and the Hamiltonian for the gapless mode reduces to the Josephson Hamiltonian $\mathcal{H} = -E_c \sum_i \partial^2 / \partial \phi_i^2 + E_J \sum \cos^2(\phi_i - \phi_j)$, which indeed has this dispersion upon quadratic expansion.

It is interesting to note that for a small rotation frequency, which implies a small vortex lattice, the Tkachenko modes are not the lowest-lying modes. For $U = 10$, a Tkachenko mode becomes the lowest-lying gapped mode when $\Omega > 0.978$, but there are many modes in between the second and the third Tkachenko mode. This confirms the expectation that increasing the vortex lattice will bring down the Tkachenko spectrum.

Vortex-lattice melting.—Quantum fluctuations of the vortices ultimately result in melting of the vortex lattice. To decide whether or not the lattice is melted, we use the Lindemann criterion. We use the definition that the melting transition takes place if $\langle (\mathbf{u}_{ni} - \mathbf{u}_{mi})^2 \rangle / 2\Delta_{nm}^2 > c_L^2$, where n and m are neighboring vortices at site i , Δ_{nm} is the distance between them, and c_L is the Lindemann parameter. We use the value $c_L^2 = 0.01$, which is appropriate for hexagonal vortex lattices [27]. Looking at correlations between vortices is a more precise way to define the melting transition than only considering the square of the displacement of individual vortices. Moreover, the study of the correlations between vortices also allows to give a more precise description of the crystal and the melted phase. The vortices arrange themselves in shells with almost equal radial distance. Within the crystal phase the radial position of the shells is well defined. However, the angular fluctuations of the shells can be locked together or not. The former phase we call the solid (S), the latter the solid of solid shells (SS). Within the melted phase the radial positions of the shells are no longer well defined, but there can still be long-range order in the angle between vortices on the same shell. This we call a hexatic liquid (HL). When this order is destroyed, we call this the liquid (L). The SS and the HL phase can only be defined in the presence of a harmonic trapping potential and are absent in the homogeneous situation.

Because the coarse-grained particle density decreases with the distance to the origin, the melting criterion has to be applied locally. For a range of parameters, vortices on the outside are already melted, while the inner part of the vortex lattice remains crystalline, and a crystal phase in the inside coexists with a liquid phase on the outside. In Fig. 4 we compute the radius of the crystal phase R_{cr} normalized to the condensate radius R , as a function of the rotation frequency for a fixed number of particles N and an interaction strength U , and for various hopping strengths t . Here we define the condensate radius R as the radius for which the angularly averaged density drops below 0.003. The crystal radius R_{cr} is defined as the radius of the innermost ring of vortices that is melted according to the Lindemann

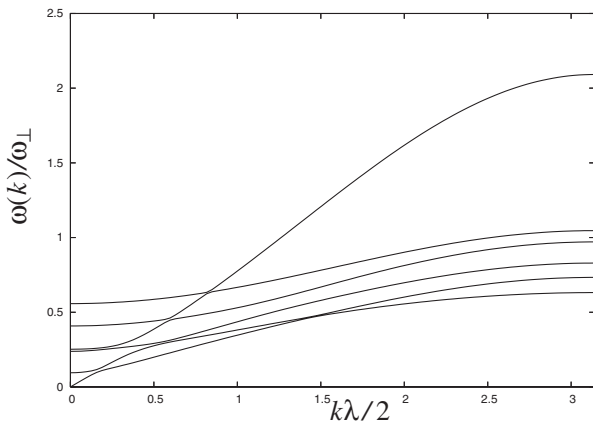


FIG. 3. Dispersion of the Tkachenko modes along the direction of the optical lattice. For this plot the parameters were chosen as: $U = 10$, $t = 1/10$, and $\Omega = 0.97$. One gapless linear mode and five gapped tight-binding-like modes can be identified. Note that these results were obtained in the lowest Landau level approximation, which corresponds to the compressible limit.

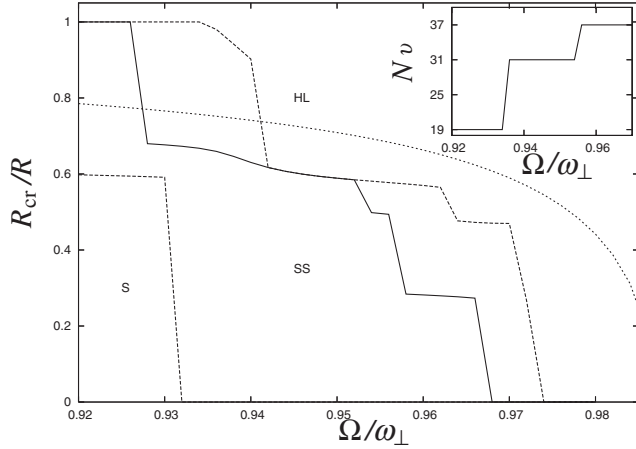


FIG. 4. Phase boundaries between crystalline and melted phases as a function of the rotation frequency for $N = 250$, $U = 10$. The solid and dashed lines are for $t = 0$ and $t = 1/100$, respectively. For $t = 0$ there exist only the solid of solid shells (SS) and the hexatic liquid (HL) phases, whereas for $t = 1/100$ also a solid phase (S) appears. The dotted line is the phase boundary between the crystal and the liquid resulting from a local-density approximation, where the criterion $N/N_v = 6$ [6] is applied locally. In the inset the number of vortices N_v that is within the condensate is plotted as a function of the rotation frequency.

criterion. When according to this definition $R_{cr} > R$, we set the crystal radius equal to the condensate radius, i.e., $R_{cr} = R$. Phase boundaries between the solid, the solid of solid shells, and the hexatic liquid are defined in analogy with the phase boundary between the crystalline and the melted phase above. The phase boundaries show discrete steps because of the shell structure in which the vortices order themselves. For isolated pancakes condensates, it turns out that for our choice of parameters the crystal is always a solid of solid shells, whereas the melted phase is always a hexatic liquid. When the tunneling between pancakes is turned on, the fluctuations are also coupled in the axial direction. This decreases the fluctuations in the vortex displacements because the stiffness of the vortices increases. As a result, melting occurs for higher rotation frequencies, as is visible in Fig. 4. A second result is that now part of the crystal becomes a solid. This possibility to adjust the tunneling parameter is a unique consequence of the one-dimensional optical lattice. By comparing the result to the result of the local-density approximation in Fig. 4 we see that the total melting of the system occurs for significantly lower rotation frequencies than the local-density approximation predicts, which implies a higher critical filling factor than in the homogeneous situation.

The liquid surrounding the crystal has rotational symmetry. Experimentally this can be observed by noting that in the liquid the vortices are no longer individually visible,

while in the crystal they are still visible, although their positions are broadened by the fluctuations. Theoretically it is a challenging problem to describe the coexisting crystal liquid. This will allow us to decide on the occurrence of melting based on energy considerations and thus shed more light on the accuracy of the application of the Lindemann criterion in this inhomogeneous situation. In the future we also want to investigate the melting at nonzero temperatures, which is experimentally relevant, because it appears to be difficult to reach very low temperatures [28].

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