ON THE THEORY OF BETA-RADIOACTIVITY IV

THE POLARIZATION OF BETA-RAYS EMITTED BY ALIGNED NUCLEI IN ALLOWED TRANSITIONS *)

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Synopsis

The consequences of alignment of nuclei, which show allowed β-transitions, are investigated. A general formula is derived for the transition probability of an allowed β-transition, in which the direction of emission of electron and neutrino, the polarization of the electron and the orientation of the nuclear spin are taken into account. The calculations have been made for a Hamiltonian for the β-interaction, which is an arbitrary "mixture" of the five invariants of the Dirac theory. The influence of the nuclear charge has, however, been neglected. From this formula the following results are immediately obtained:

The angular distribution of the β-radiation remains spherically symmetric if the nuclei are aligned, so that the alignment cannot be detected in this way.

The emitted β-radiation is polarized and the degree of polarization follows from the general formula. If we take the special case that the interaction Hamiltonian is of the tensor or the axial vector type and if the β-rays are emitted perpendicular to the direction of the nuclear spin of completely aligned nuclei with nuclear spin j_i, the degree of polarization is given by: *a)* $1/E$, if $j_i = j_f + 1$, *b)* $1/E(j_i + 1)$, if $j_i = j_f$, *c)* $j_i/E(j_i + 1)$, if $j_i = j_f - 1$. (E is the relativistic energy of the electrons, $E \approx 1$ for small kinetic energies; j_f gives the spin of the final nucleus).

§ 1. *Introduction.* Recently G o r t e r [5]) and R o s e [6]) have indicated a method by which it must be possible with the present experimental means, to obtain a considerable alignment for certain nuclei. The nuclei must be contained in paramagnetic ions; the alignment is obtained with very low temperatures and medium magnetic fields. In spite of several attempts [7]), this alignment has

*) Formulae from preceding papers [1]), [2]), [3]) of this series will be quoted as, say, II (12).

not yet been shown by nuclear physical experiments. If the alignment of radioactive nuclei could be detected by methods of nuclear physics, it might be possible to measure the nuclear magnetic moment of very small quantities of these nuclei, by studying the effect as a function of temperature or by destroying the alignment with a radiofrequency field.

In this paper we examine the consequences of such an alignment of radioactive nuclei which show allowed β-transitions. It will be shown that the β-rays emitted by aligned nuclei are polarized (§ 7, preliminary note [4]). Concerning the polarization of electron beams certain notions developed in the two preceding papers are used [2], [3]. The calculations are developed in §§ 2–5. In § 6 we discuss the results of § 5 in relation to a symmetry principle for the Hamiltonian for the β-interaction [1].

§ 2. *The method of calculation of the transition probabilities, taking into account: the direction of emission of electron and neutrino, the polarization of the electron and the orientation of the nuclear spin.* The calculation of this transition probability can be made along the same lines as the calculation of $P(E, \mathbf{p}, \mathbf{q})$ in I § 2; we shall keep the same notations. However, we now take into account the polarization of the electron. We assume that the spin of the emitted electron has an arbitrary direction $\boldsymbol{\zeta}$ (determining the polarization according to II (11) and II (12)). We again take an arbitrary "mixture" of the five invariants of the Dirac theory for the Hamiltonian for the β-interaction. Further we shall neglect the influence of the nuclear charge and we shall restrict ourselves to allowed transitions. The transition probability to a state in which the electron has an energy between E and $E + dE$, the momenta $\mathbf{p}$ and $\mathbf{q}$ of electron and neutrino have directions within $d\omega_e$ and $d\omega_\nu$ respectively, while the polarization of the electron is specified by $\boldsymbol{\zeta}$, is given by

$$P(E, \mathbf{p}, \mathbf{q}, \boldsymbol{\zeta})\, \mathrm{d}E\mathrm{d}\omega_e\mathrm{d}\omega_\nu = (2\pi)^{-5}\, \Sigma_\nu\, |H_\beta|^2\, pEq^2\, \mathrm{d}E\mathrm{d}\omega_e\mathrm{d}\omega_\nu. \qquad (1)$$

The difference with I (7) is only, that in this formula the sum for both states of polarization of the electron had to be taken. The orientation of the nucleus is contained in the nuclear wave functions, occurring in $|H_\beta|^2$ and is not explicitly expressed in $P(E, \mathbf{p}, \mathbf{q}, \boldsymbol{\zeta})$. In the same way as in I § 2 we find the following formula for $\Sigma_\nu |H_\beta|^2$ in case of β^--emission .

$$\Sigma_\nu\, |H_\beta|^2 = G^2\, \Sigma_{k,l=1}^{9}\, \bar{C}_k \bar{C}_l\, (\textstyle\int A^k)\, (\int A^l)^*\, \mathrm{Tr}\, [A^k D_\nu A^l P_e(\boldsymbol{\zeta})], \qquad (2)$$

where D_ν and $P_e(\boldsymbol{\zeta})$ are matrices defined by the following equations, which give their elements

$$(D_\nu)_{\mu\rho} = \Sigma_{spin}\, \varphi_\rho^* \varphi_\mu, \tag{3}$$

$$[P_e(\boldsymbol{\zeta})]_{\mu\rho} = \psi_\rho^*\, \psi_\mu. \tag{4}$$

(2) differs from I (26) only in the respect that D_e was replaced by $P_e(\boldsymbol{\zeta})$. This means a complication, as it will be seen that the expression for $P_e(\boldsymbol{\zeta})$ is more complicated than D_e. In the next section the expression for $P_e(\boldsymbol{\zeta})$ is given. In § 4 we discuss how certain vectors and a tensor, which can be formed with the nuclear matrix elements and which appear in the result for $P(E, \mathbf{p}, \mathbf{q}, \boldsymbol{\zeta})$, depend on the orientation of the nuclei. After these preparations the complete result for $P(E, \mathbf{p}, \mathbf{q}, \boldsymbol{\zeta})$ can be deduced (in § 5). In case of β^+-emission (2) must be replaced by (cf. I (26) and I (29))

$$\Sigma_\nu\, |H_\beta|^2 = G^2\, \Sigma_{k,l=1}^{9}\, \bar{C}_k \bar{C}_l\, (\int A^k)^*\, (\int A^l)\, \mathrm{Tr}\, [A^k D_\nu A^l P_e(\boldsymbol{\zeta})] \tag{5}$$

Except for this change the calculations are analogous for β^+ and β^--emission.

§ 3. *Calculation of the matrices $P^+(\boldsymbol{\zeta})$ and $P^-(\boldsymbol{\zeta})$.* By $P^+(\boldsymbol{\zeta})$ and $P^-(\boldsymbol{\zeta})$ we mean the matrices defined by (4) for the positive energy solution I(12) and the negative energy solution I(14) respectively. The spin orientation for these solutions is determined by II(11) and II(12), from which we get

$$\left.\begin{aligned} A^*A &= \tfrac{1}{2}\,(1 + \cos\chi) &&= \tfrac{1}{2}\,(1 + \zeta_z), \\ B^*B &= \tfrac{1}{2}\,(1 - \cos\chi) &&= \tfrac{1}{2}\,(1 - \zeta_z), \\ A^*B &= \tfrac{1}{2}\sin\chi\, \exp(i\omega) &&= \tfrac{1}{2}\,(\zeta_x + i\zeta_y), \\ B^*A &= \tfrac{1}{2}\sin\chi\, \exp(-i\omega) &&= \tfrac{1}{2}\,(\zeta_x - i\zeta_y). \end{aligned}\right\} \tag{6}$$

We shall give the results for the matrices $P(\boldsymbol{\zeta})$ by writing them as linear combinations of the 16 matrices of the Dirac-theory (cf. I(2); the matrices of (7) are all hermitian; $\gamma_5 = -i\alpha_x\alpha_y\alpha_z$)

$$\beta,\ 1,\ \boldsymbol{\alpha},\ \beta\boldsymbol{\sigma},\ i\beta\boldsymbol{\alpha},\ \boldsymbol{\sigma},\ \gamma_5,\ i\beta\gamma_5. \tag{7}$$

The expressions can be obtained by calculating explicitly the elements of the matrices according to (4) and by application of (6). We obtain for $P^+(\boldsymbol{\zeta})$ (for notations cf. the appendix)

$$P^+(\boldsymbol{\zeta}) = \tfrac{1}{4}\{1 - (m/E)\beta + \boldsymbol{\zeta}\cdot[(m/E)\boldsymbol{\sigma} - \beta\boldsymbol{\sigma}] + (1/E)\,[-\mathbf{p}\cdot\boldsymbol{\alpha} - \\ - \gamma_5(\mathbf{p}\cdot\boldsymbol{\zeta}) + i\beta\boldsymbol{\zeta}\cdot(\mathbf{p}\wedge\boldsymbol{\alpha})] + [1/E(E+m)]\,[(\mathbf{p}\mathbf{p}) : (\boldsymbol{\zeta}\boldsymbol{\sigma})]\,(\beta+1)\}. \tag{8}$$

We write this result in another form

$$P^+(\boldsymbol{\zeta}) = \tfrac{1}{4}\,[K^+(\beta)\,\beta + K^+(1)\,1 + \mathbf{K}^+(\boldsymbol{\alpha})\cdot\boldsymbol{\alpha} + \mathbf{K}^+(\beta\boldsymbol{\sigma})\cdot\beta\boldsymbol{\sigma} + \\ + \mathbf{K}^+(i\beta\boldsymbol{\alpha})\cdot(i\beta\boldsymbol{\alpha}) + \mathbf{K}^+(\boldsymbol{\sigma})\cdot\boldsymbol{\sigma} + \mathbf{K}^+(\gamma_5)\,\gamma_5] \qquad (9)$$

with

$$\left.\begin{aligned} K^+(\beta) &= -\,(m/E),\\ K^+(1) &= 1,\\ \mathbf{K}^+(\boldsymbol{\alpha}) &= -\,(\mathbf{p}/E),\\ \mathbf{K}^+(\beta\boldsymbol{\sigma}) &= -\,\boldsymbol{\zeta} + [1/E(E+m)]\,(\mathbf{pp})\cdot\boldsymbol{\zeta},\\ \mathbf{K}^+(i\beta\boldsymbol{\alpha}) &= -\,(1/E)\,(\mathbf{p}\wedge\boldsymbol{\zeta}),\\ \mathbf{K}^+(\boldsymbol{\sigma}) &= (m/E)\,\boldsymbol{\zeta} + [1/E(E+m)]\,(\mathbf{pp})\cdot\boldsymbol{\zeta},\\ \mathbf{K}^+(\gamma_5) &= -\,(1/E)\,(\mathbf{p}\cdot\boldsymbol{\zeta}). \end{aligned}\right\} \qquad (10)$$

Analogously we find with the negative energy solutions

$$P^-(\boldsymbol{\zeta}) = \tfrac{1}{4}\{1 + (m/E)\beta - \boldsymbol{\zeta}\cdot[(m/E)\boldsymbol{\sigma} + \beta\boldsymbol{\sigma}] + (1/E)\,[-\,\mathbf{p}\cdot\boldsymbol{\alpha} + \\ + \gamma_5(\mathbf{p}\cdot\boldsymbol{\zeta}) + i\beta\boldsymbol{\zeta}\cdot(\mathbf{p}\wedge\boldsymbol{\alpha})] + [1/E(E+m)]\,[(\mathbf{pp}):(\boldsymbol{\zeta}\boldsymbol{\sigma})](\beta-1)\}. \qquad (11)$$

We can write this result also in the form analogous to (9)

$$P^-(\boldsymbol{\zeta}) = \tfrac{1}{4}\,[K^-(\beta)\beta + K^-(1)1 + \mathbf{K}^-(\boldsymbol{\alpha})\cdot\boldsymbol{\alpha} + \mathbf{K}^-(\beta\boldsymbol{\sigma})\cdot\beta\boldsymbol{\sigma} + \\ + \mathbf{K}^-(i\beta\boldsymbol{\alpha})\cdot(i\beta\boldsymbol{\alpha}) + \mathbf{K}^-(\boldsymbol{\sigma})\cdot\boldsymbol{\sigma} + \mathbf{K}^-(\gamma_5)\,\gamma_5] \qquad (12)$$

with

$$\left.\begin{aligned} K^-(\beta) &= (m/E),\\ K^-(1) &= 1,\\ \mathbf{K}^-(\boldsymbol{\alpha}) &= -\,(\mathbf{p}/E),\\ \mathbf{K}^-(\beta\boldsymbol{\sigma}) &= -\,\boldsymbol{\zeta} + [1/E(E+m)]\,(\mathbf{pp})\cdot\boldsymbol{\zeta},\\ \mathbf{K}^-(i\beta\boldsymbol{\alpha}) &= -\,(1/E)\,(\mathbf{p}\wedge\boldsymbol{\zeta}),\\ \mathbf{K}^-(\boldsymbol{\sigma}) &= -\,(m/E)\,\boldsymbol{\zeta} - [1/E(E+m)]\,(\mathbf{pp})\cdot\boldsymbol{\zeta},\\ K^-(\gamma_5) &= (1/E)\,(\mathbf{p}\cdot\boldsymbol{\zeta}). \end{aligned}\right\} \qquad (13)$$

If we consider the matrices $P(\boldsymbol{\zeta})$ in further calculations for electrons, we can put $m = m_e = 1$ if we use relativistic units (cf. I § 2).

§ 4. *Expressions containing the nuclear matrix elements and the orientation of the nuclei.* We consider in this section relations between nuclear matrix elements and their consequences for vectors or tensors that can be formed with these matrix elements. The general expressions for nuclear matrix elements are given by I(18) and I(19). The wave functions Ψ_i and Ψ_f of the initial and final nucleus, occurring in these expressions, are characterized by the nuclear spin j and the magnetic quantum number m belonging to it;

if $j \neq 0$ there will be degeneration: m can take $2j + 1$ values. For Ψ_i we can take wave functions

$$\Psi_i \,(j_i, m_i) \tag{14}$$

with a definite value of m_i, or also a linear combination

$$\Psi_i = \Sigma_{m_i}\, c(m_i)\, \Psi_i \,(j_i, m_i). \tag{15}$$

In this section we consider especially the nuclear matrix elements with the operators 1 and $\boldsymbol{\sigma}$, for which the selection rules are

$$\left.\begin{array}{lll} |\int 1|^2,\ \Delta j=0, & \Delta m = 0, & \text{no change of parity} \\ |\int \boldsymbol{\sigma}|^2,\ \Delta j=0,\ \pm 1\ (\text{no } 0 \rightarrow 0), & \Delta m = 0,\ \pm 1, & \text{no change of parity} \end{array}\right\} \tag{16}$$

Otherwise the matrix elements are $= 0$. The relative magnitude of the matrix elements

$$\int \sigma_x, \int \sigma_y, \int \sigma_z \tag{17}$$

is given in table I [8]).

TABLE I

Relative magnitude of the matrix elements $\int \sigma_x, \int \sigma_y, \int \sigma_z$. The matrix elements that do not occur in the table are 0; a, b and c are constants.

	$j_i = j_f + 1$ ($j_i \geqslant 1$)	$j_i = j_f$ ($j_i \geqslant \frac{1}{2}$, $0 \rightarrow 0$ forbidden)	$j_i = j_f - 1$ ($j_i \geqslant 0$)
$m_f = m_i + 1 \vert \sigma_x + i\sigma_y \vert m_i)$	$a\sqrt{(j_i - m_i)(j_i - m_i - 1)}$	$b\sqrt{(j_i + m_i + 1)(j_i - m_i)}$	$-c\sqrt{(j_i + m_i + 2)(j_i + m_i + 1)}$
$m_f = m_i - 1 \vert \sigma_x - i\sigma_y \vert m_i)$	$-a\sqrt{(j_i + m_i)(j_i + m_i - 1)}$	$b\sqrt{(j_i + m_i)(j_i - m_i + 1)}$	$c\sqrt{(j_i - m_i + 2)(j_i - m_i + 1)}$
$m_f = m_i \vert \sigma_z \vert m_i)$	$a\sqrt{(j_i + m_i)(j_i - m_i)}$	$b\, m_i$	$c\sqrt{(j_i + m_i + 1)(j_i - m_i + 1)}$

The expression

$$|\int \boldsymbol{\sigma}|^2 = |\int \sigma_x|^2 + |\int \sigma_y|^2 + |\int \sigma_z|^2 \tag{18}$$

which is a scalar occurs, e.g., in I(28) and I(30), formulae that give transition probabilities. (In this expression with the nuclear matrix elements the sum Σ_{m_f} must be taken, cf. I(32)).

However, in the results for transition probabilities taking the orientation of the nucleus into account, also other covariant expressions (scalars, vectors, tensors, etc.) occur, which can be formed with the nuclear matrix elements (17) (We must also take the sum Σ_{m_f} in these expressions). According to table I all these expressions can be characterized by one single parameter depending on the nuclear wave functions, e.g., the scalar quantity (18). They depend further only on quantities that describe the orientation of the nucleus, e.g., m_i without referring directly to the exact form of the wave functions Ψ_i and Ψ_f. The covariant, real quantities (of "valence" 0, 1 and 2), which are independent of an arbitrary phase factor in the wave functions Ψ_i and Ψ_f and which can be formed from (17), are

a) the scalar $|\int \boldsymbol{\sigma}|^2$ according to (18),

b) a vector (antisymmetric tensor) defined by

$$\boldsymbol{\Sigma}_\wedge = i(\int \boldsymbol{\sigma}) \wedge (\int \boldsymbol{\sigma})^* = i \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \int \sigma_x & \int \sigma_y & \int \sigma_z \\ (\int \sigma_x)^* & (\int \sigma_y)^* & (\int \sigma_z)^* \end{vmatrix}, \tag{19}$$

c) a symmetric tensor defined by

$$\boldsymbol{\Sigma} = \tfrac{1}{2}[(\int \boldsymbol{\sigma})(\int \boldsymbol{\sigma})^* + c.c.] =$$

$$\begin{bmatrix} |\int \sigma_x|^2 & \tfrac{1}{2}[(\int \sigma_x)^*(\int \sigma_y) + c.c.] & \tfrac{1}{2}[(\int \sigma_x)^*(\int \sigma_z) + c.c.] \\ \tfrac{1}{2}[(\int \sigma_x)^*(\int \sigma_y) + c.c.] & |\int \sigma_y|^2 & \tfrac{1}{2}[(\int \sigma_y)^*(\int \sigma_z) + c.c.] \\ \tfrac{1}{2}[(\int \sigma_x)^*(\int \sigma_z) + c.c.] & \tfrac{1}{2}[(\int \sigma_y)^*(\int \sigma_z) + c.c.] & |\int \sigma_z|^2 \end{bmatrix}. \tag{20}$$

The expressions (18), (19) and (20) must be calculated if the initial nuclei have a certain "orientation". If Ψ_i is given as $\Psi_i(j_i, m_i)$ with a definite m_i, a certain direction in space $\boldsymbol{\eta}$ is related to this m_i, as m_i determines the component of the angular momentum in some direction, which we call $\boldsymbol{\eta}$. In this case we shall say the nuclei are *polarized* with the *axis of polarization* $\boldsymbol{\eta}$. (We take a *unit* vector for $\boldsymbol{\eta}$; often $\boldsymbol{\eta}$ is chosen as the z-direction). In general the "orientation" of an "ensemble" of nuclei cannot be described by a single wave function, but must be described by a density matrix ϱ with the elements $\varrho_{mm'}$ (cf., e.g., [9])). In this general case we define the *axis of polarization* as follows: if the density matrix ϱ is in diagonal form for the fundamental states $\Psi(j, m)$ and if m is related to a direction $\boldsymbol{\eta}$, then $\boldsymbol{\eta}$ is called the axis of polarization of the nuclei described by ϱ. We write in this case

$$[\mathbf{J}_{nucl}] = \boldsymbol{\eta} \tag{21}$$

With these fundamental states, ϱ has the form

$$\varrho_{mm'} = P_m \, \delta_{mm'} \tag{22}$$

The P_m are normalized in such a way that

$$\Sigma_m P_m = 1 \tag{23}$$

The case that the nuclei are described by a single wave function $\Psi_i(j_i, m_i)$ is a special case of (22), for which we can put

$$P_m = \begin{cases} 1, & \text{if } m = m_i. \\ 0, & \text{if } m \neq m_i. \end{cases} \tag{24}$$

The case of random orientation of the nuclei is described by [10])

$$P_m = P, \text{ independent of } m. \tag{25}$$

As a measure for the "alignment" of the nuclei in the direction $\boldsymbol{\eta}$ a quantity f_N the "*degree of polarization of the nuclei*", may be used which is defined by

$$f_N = \Sigma_m \,(m/j)\, P_m \tag{26}$$

and which has the property

$$-1 \leqslant f_N \leqslant 1. \tag{27}$$

(Remark: $|f_N|$, if taken for electrons ($j = \frac{1}{2}$), is the degree of polarization II(7)).

For the calculation of $\boldsymbol{\Sigma}_\wedge$ (19) for the case (24) it is easy to take a special orientation of the coordinate system (e.g., $\boldsymbol{\eta} = \mathbf{k}$) and to use the formula

$$2i[(\textstyle\int \sigma_x)^* (\int \sigma_y) - c.c.] = |\int (\sigma_x + i\sigma_y)|^2 - |\int (\sigma_x - i\sigma_y)|^2. \tag{28}$$

In this way we easily find, using table I

$$\boldsymbol{\Sigma}_\wedge (m_i) = A_\wedge (m_i) \,|\textstyle\int \boldsymbol{\sigma}\,|^2 \,\boldsymbol{\eta} \tag{29}$$

$$\text{with}\quad A_\wedge(m_i) = \begin{cases} m_i/j_i \quad, & \text{if } j_i = j_f + 1 \quad (j_i \geqslant 1). \\ m_i/j_i(j_i+1), & \text{if } j_i = j_f \quad (j_i \geqslant \frac{1}{2}). \\ -m_i/(j_i+1), & \text{if } j_i = j_f - 1 \quad (j_i \geqslant 0). \end{cases} \tag{30}$$

For the general case (22), (19) becomes

$$\boldsymbol{\Sigma}_\wedge = \Sigma_{m_i} P_{m_i} \boldsymbol{\Sigma}_\wedge(m_i) = A_\wedge |\textstyle\int \boldsymbol{\sigma}|^2 \,\boldsymbol{\eta} \tag{31}$$

with

$$A_\wedge = \Sigma_{m_i} P_{m_i} A_\wedge(m_i). \tag{32}$$

It is easily shown that

$$|A_\wedge| \leqslant 1. \tag{33}$$

For random orientation (25) we get

$$\boldsymbol{\Sigma}_\wedge = 0 \;\; (A_\wedge = 0). \tag{34}$$

Analogously we calculate $\boldsymbol{\Sigma}$ (20) for the case (24). We again take a special orientation of the coordinate system, $\boldsymbol{\eta} = \mathbf{k}$. We further use the elementary formula

$$2i[(\textstyle\int\sigma_x)^* (\int\sigma_y) + c.c.] =$$
$$= [\textstyle\int(\sigma_x - i\sigma_y)]^* [\int (\sigma_x + i\sigma_y)] - [\int (\sigma_x + i\sigma_y)]^* [\int (\sigma_x - i\sigma_y)]. \tag{35}$$

If we make use of table I, it is clear that the expression (35) is equal to 0 for every $\Psi_i(j_i, m_i)$, because for a certain m_i and m_f either

$\int(\sigma_x - i\sigma_y)$, either $\int(\sigma_x + i\sigma_y)$, either both are equal to 0. We also have $[(\int\sigma_y)^*(\int\sigma_z) + c.c.] = 0$ and $[(\int\sigma_x)^*(\int\sigma_z) + c.c.] = 0$ for every $\Psi_i(j_i, m_i)$, because we see directly from table I that for a definite m_i and m_f, again at least one factor is 0. Hence all off-diagonal elements of $\boldsymbol{\Sigma}$ are 0. This is not true for any position of the coordinate system, namely not, if the direction to which m_i is related, is not an axis of the coordinate system. For the position $\boldsymbol{\eta} = \mathbf{k}$, $\boldsymbol{\Sigma}$ takes the shape

$$\boldsymbol{\Sigma}(m_i) = \begin{bmatrix} |\int\sigma_x|^2 & 0 & 0 \\ 0 & |\int\sigma_y|^2 & 0 \\ 0 & 0 & |\int\sigma_z|^2 \end{bmatrix} = \tfrac{1}{3}|\int\boldsymbol{\sigma}|^2[1 + A_s(m_i)\,\mathbf{N}] \tag{36}$$

with

$$\mathbf{N} = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{37}$$

For $A_s(m_i)$ we have

$$\begin{aligned} A_s(m_i) &= \{2|\int\sigma_z|^2 - [|\int\sigma_x|^2 + |\int\sigma_y|^2]\}/|\int\boldsymbol{\sigma}|^2 = \\ &= \{2|\int\sigma_z|^2 - \tfrac{1}{2}[|\int(\sigma_x + i\sigma_y)|^2 + |\int(\sigma_x - i\sigma_y)|^2]\}/|\int\boldsymbol{\sigma}|^2. \end{aligned} \tag{38}$$

Hence, with the aid of table I we find

$$A_s(m_i) = \begin{cases} [j_i(j_i+1) - 3m_i^2]/j_i(2j_i-1) & ,\ \text{if } j_i = j_f + 1\ (j_i \geqslant 1). \\ -[j_i(j_i+1) - 3m_i^2]/j_i(j_i+1) & ,\ \text{if } j_i = j_f\ (j_i \geqslant \tfrac{1}{2}). \\ [j_i(j_i+1) - 3m_i^2]/(j_i+1)(2j_i+3), & \text{if } j_i = j_f - 1\ (j_i \geqslant 0). \end{cases} \tag{39}$$

For the general case (22), we get for

$$\boldsymbol{\Sigma} = \Sigma_{m_i} P_{m_i} \boldsymbol{\Sigma}(m_i) = \tfrac{1}{3}|\int\boldsymbol{\sigma}|^2[1 + A_s\,\mathbf{N}] \tag{40}$$

with

$$A_s = \Sigma_{m_i} P_{m_i} A_s(m_i). \tag{41}$$

From (39) and (41), it can be concluded that

$$-1 \leqslant A_s \leqslant 2. \tag{52}$$

For random orientation (25) we get

$$\boldsymbol{\Sigma} = \tfrac{1}{3}|\int\boldsymbol{\sigma}|^2 \quad (A_s = 0). \tag{43}$$

In (37) $\mathbf{N}$ has been given in the coordinate system in which $\boldsymbol{\eta} = \mathbf{k}$. For a different direction of $\boldsymbol{\eta}$, $\mathbf{N}$ of course gets a shape different from (37). For an arbitrary $\boldsymbol{\eta}$ we can write

$$\mathbf{N} = \tfrac{1}{2}[3(\boldsymbol{\eta}\boldsymbol{\eta}) - 1]. \tag{44}$$

Two further real covariant quantities can be formed with the matrix elements (17), if we consider them together with the matrix element $\int 1$. We can then form the following real vectors (independent of an arbitrary phase factor in the wave functions), which can be considered as the *r*eal and *i*maginary part of a complex vector $(\int \boldsymbol{\sigma})(\int 1)^*$, formed as a "*c*ross term"

$$\boldsymbol{\Sigma}^{cr} = \tfrac{1}{2}[(\int \boldsymbol{\sigma})^* (\int 1) + c.c.], \tag{45}$$

$$\boldsymbol{\Sigma}^{ci} = \frac{i}{2}[(\int \boldsymbol{\sigma})^* (\int 1) - c.c.]. \tag{46}$$

From (16) it follows immediately that

$$\boldsymbol{\Sigma}^{cr} = 0 \quad \text{and} \quad \boldsymbol{\Sigma}^{ci} = 0, \quad \text{if} \quad j_i = j_f \pm 1 \quad \text{or} \quad j_i = j_f = 0. \tag{47}$$

Hence we consider further only the case $j_i = j_f$ ($j_i \geqslant \frac{1}{2}$). Using table I we see that we can write for the case (24)

$$\boldsymbol{\Sigma}^{cr}(m_i) = A_c(m_i) M_{cr} \boldsymbol{\eta} \tag{48}$$

and

$$\boldsymbol{\Sigma}^{ci}(m_i) = A_c(m_i) M_{ci} \boldsymbol{\eta} \tag{49}$$

with

$$A_c(m_i) = m_i/\sqrt{j_i(j_i+1)} \quad (j_i = j_f,\ j_i \geqslant \tfrac{1}{2}). \tag{50}$$

M_{cr} and M_{ci} are real quantities, which must be calculated from the nuclear wave functions. They cannot be deduced directly from $|\int \boldsymbol{\sigma}|^2$ and $|\int 1|^2$. However, the following relations exist

$$M_{cr}^2 + M_{ci}^2 = |\int \boldsymbol{\sigma}|^2 |\int 1|^2, \tag{51}$$

hence and

$$\left.\begin{aligned} M_{cr}^2 &\leqslant |\int \boldsymbol{\sigma}|^2 |\int 1|^2 \\ M_{ci}^2 &\leqslant |\int \boldsymbol{\sigma}|^2 |\int 1|^2. \end{aligned}\right\} \tag{52}$$

For the general case (22) $\boldsymbol{\Sigma}^{cr}$ and $\boldsymbol{\Sigma}^{ci}$ become

$$\boldsymbol{\Sigma}^{cr} = A_c M_{cr} \boldsymbol{\eta}, \tag{53}$$

$$\boldsymbol{\Sigma}^{ci} = A_c M_{ci} \boldsymbol{\eta}, \tag{54}$$

with

$$A_c = \Sigma_{m_i} P_{m_i} A_c(m_i). \tag{55}$$

For random orientation of the nuclei (25) we get

$$\boldsymbol{\Sigma}^{cr} = 0 \text{ and } \boldsymbol{\Sigma}^{ci} = 0 \ (A_c = 0). \tag{56}$$

§ 5. *The calculation of the transition probabilities; the results.* $P(E, \mathbf{p}, \mathbf{q}, \boldsymbol{\zeta})$ is calculated according to (1). For β^--emission (2) must be used, while for D_ν and $P_e(\boldsymbol{\zeta})$ the expressions $D_\nu^- = \frac{1}{2}[1 - \boldsymbol{\alpha} \cdot \mathbf{q}/E_\nu]$ according to I(17) and $P_e^+(\boldsymbol{\zeta})$ according to (9) with (10) must be inserted. We have made the calculations by first expressing the matrix

$$R_- = \Sigma_{k,l=1}^{9} \bar{C}_k \bar{C}_l (\int A^k)(\int A^l)^* (A^k D_\nu^- A^l) \tag{57}$$

in terms of Dirac-matrices. After this $\Sigma_\nu |H_\beta|^2$ is obtained as

$$\Sigma_\nu \, |H_\beta|^2 = G^2 \, \mathrm{Tr} \, [R_- P_e^+ \, (\boldsymbol{\zeta})]. \tag{58}$$

For β^+-emission (5) must be used instead of (2). For D_ν and $P_e(\boldsymbol{\zeta})$ the expressions $D_\nu^+ = \frac{1}{2}[1 - \boldsymbol{\alpha} \cdot \mathbf{q}/E_\nu]$ according to I(16) and $P_e^-(\boldsymbol{\zeta})$ according to (12) with (13) must be inserted. After calculating the matrix

$$R_+ = \Sigma_{k,l=1}^{9} \bar{C}_k \bar{C}_l \, (\textstyle\int A^k)^* \, (\int A^l) \, (A^k D_\nu^+ A^l), \tag{59}$$

$\Sigma_\nu |H_\beta|^2$ is obtained according to

$$\Sigma_\nu \, |H_\beta|^2 = G^2 \, \mathrm{Tr} \, [R_+ \, P_e^-(\boldsymbol{\zeta})]. \tag{60}$$

We give the results of the calculations in a combined form for β^+- and β^--emission. The quantities $\boldsymbol{\Sigma}_\wedge$, $\boldsymbol{\Sigma}$, $\boldsymbol{\Sigma}^{cr}$, $\boldsymbol{\Sigma}^{ci}$ introduced in § 4, are used to write the expressions in an invariant way.

We obtain for R_+ and R_-

$$\begin{aligned}
R_\pm = {} & \tfrac{1}{2} C_1^2 \, |\textstyle\int 1|^2 \, [1 + (1/E_\nu) \, \mathbf{q} \cdot \boldsymbol{\alpha}] + \\
& + \tfrac{1}{2} C_2^2 \, |\textstyle\int 1|^2 \, [1 - (1/E_\nu) \, \mathbf{q} \cdot \boldsymbol{\alpha}] + \\
& + \tfrac{1}{2} C_3^2 \, [\, |\textstyle\int \boldsymbol{\sigma}|^2 \mp \boldsymbol{\Sigma}_\wedge \cdot \boldsymbol{\sigma} - (1/E_\nu) \, |\int \boldsymbol{\sigma}|^2 \mathbf{q} \cdot \boldsymbol{\alpha} + \\
& \qquad + (2/E_\nu) \, \boldsymbol{\Sigma} : (\mathbf{q}\boldsymbol{\alpha}) \pm (1/E_\nu) \, (\boldsymbol{\Sigma}_\wedge \cdot \mathbf{q}) \, \gamma_5] + \\
& + \tfrac{1}{2} C_4^2 \, [\, |\textstyle\int \boldsymbol{\sigma}|^2 \mp \boldsymbol{\Sigma}_\wedge \cdot \boldsymbol{\sigma} + (1/E_\nu) \, |\int \boldsymbol{\sigma}|^2 \mathbf{q} \cdot \boldsymbol{\alpha} - \\
& \qquad - (2/E_\nu) \, \boldsymbol{\Sigma} : (\mathbf{q}\boldsymbol{\alpha}) \mp (1/E_\nu) \, (\boldsymbol{\Sigma}_\wedge \cdot \mathbf{q}) \, \gamma_5] + \\
& + \tfrac{1}{2} C_5^2 \, |\textstyle\int \beta\gamma_5|^2 \, [1 + (1/E_\nu) \, \mathbf{q} \cdot \boldsymbol{\alpha}] - \\
& - C_1 C_2 \, |\textstyle\int 1|^2 \, \beta + \\
& + C_1 C_3 \, [\boldsymbol{\Sigma}^{cr} \cdot \boldsymbol{\sigma} \pm (1/E_\nu) \, (\boldsymbol{\Sigma}^{ci} \wedge \mathbf{q}) \cdot \boldsymbol{\alpha} + (1/E_\nu) \, (\boldsymbol{\Sigma}^{cr} \cdot \mathbf{q}) \, \gamma_5] + \\
& + C_1 C_4 \, [- \boldsymbol{\Sigma}^{cr} \cdot \beta\boldsymbol{\sigma} - (1/E_\nu) \, (\boldsymbol{\Sigma}^{cr} \wedge \mathbf{q}) \cdot (i\beta\boldsymbol{\alpha}) \pm (1/E_\nu) \, (\boldsymbol{\Sigma}^{ci} \cdot \mathbf{q}) \, (i\beta\gamma_5)] + \\
& + C_2 C_3 \, [- \boldsymbol{\Sigma}^{cr} \cdot \beta\boldsymbol{\sigma} + (1/E_\nu) \, (\boldsymbol{\Sigma}^{cr} \wedge \mathbf{q}) \cdot (i\beta\boldsymbol{\alpha}) \mp (1/E_\nu) \, (\boldsymbol{\Sigma}^{ci} \cdot \mathbf{q}) \, (i\beta\gamma_5)] + \\
& + C_2 C_4 \, [\boldsymbol{\Sigma}^{cr} \cdot \boldsymbol{\sigma} \mp (1/E_\nu) \, (\boldsymbol{\Sigma}^{ci} \wedge \mathbf{q}) \cdot \boldsymbol{\alpha} - (1/E_\nu) \, (\boldsymbol{\Sigma}^{cr} \cdot \mathbf{q}) \, \gamma_5] + \\
& + C_3 C_4 \, [- |\textstyle\int \boldsymbol{\sigma}|^2 \, \beta \pm \boldsymbol{\Sigma}_\wedge \cdot \beta\boldsymbol{\sigma}].
\end{aligned} \tag{61}$$

With the aid of (61) $P_+(E, \mathbf{p}, \mathbf{q}, \boldsymbol{\zeta})$ and $P_-(E, \mathbf{p}, \mathbf{q}, \boldsymbol{\zeta})$, valid respectively for β^+ and β^--emission, are found to be given by

$$\begin{aligned}
P_\pm \, (E, \mathbf{p}, \mathbf{q}, \boldsymbol{\zeta}) = {} & \tfrac{1}{2}[G^2 / (2\pi)^5] p E q^2 \times \\
& \times \{C_1^2 \, |\textstyle\int 1|^2 \, [1 - \mathbf{p} \cdot \mathbf{q}/EE_\nu] + \\
& + C_2^2 \, |\textstyle\int 1|^2 \, [1 + \mathbf{p} \cdot \mathbf{q}/EE_\nu] + \\
& + C_3^2 \, [|\textstyle\int \boldsymbol{\sigma}|^2 + (1/E) \, (\boldsymbol{\Sigma}_\wedge \cdot \boldsymbol{\zeta}) + (\boldsymbol{\Sigma}_\wedge \boldsymbol{\zeta}) : [\mathbf{pp}/E(E+1)] + \\
& \qquad + |\textstyle\int \boldsymbol{\sigma}|^2 \, (\mathbf{p} \cdot \mathbf{q}/EE_\nu) - 2\boldsymbol{\Sigma} : (\mathbf{pq}/EE_\nu) + (\boldsymbol{\Sigma}_\wedge \boldsymbol{\zeta}) : (\mathbf{pq}/EE_\nu)] + \\
& + C_4^2 \, [\, |\textstyle\int \boldsymbol{\sigma}|^2 + (1/E) \, (\boldsymbol{\Sigma}_\wedge \cdot \boldsymbol{\zeta}) + (\boldsymbol{\Sigma}_\wedge \boldsymbol{\zeta}) : [\mathbf{pp}/E(E+1)] - \\
& \qquad - |\textstyle\int \boldsymbol{\sigma}|^2 \, (\mathbf{p} \cdot \mathbf{q}/EE_\nu) + 2\boldsymbol{\Sigma} : (\mathbf{pq}/EE_\nu) - (\boldsymbol{\Sigma}_\wedge \boldsymbol{\zeta}) : (\mathbf{pq}/EE_\nu)] + \\
& + C_5^2 \, |\textstyle\int \beta\gamma_5|^2 \, [1 - \mathbf{p} \cdot \mathbf{q}/EE_\nu] \mp \\
& \mp 2 \, C_1 C_2 \, |\textstyle\int 1|^2 \, (1/E) \mp
\end{aligned}$$

$$\mp 2C_1C_3[(1/E)(\boldsymbol{\Sigma}^{cr}\cdot\boldsymbol{\zeta})+(\boldsymbol{\Sigma}^{cr}\boldsymbol{\zeta}):[\mathbf{pp}/E(E+1)]+(\boldsymbol{\Sigma}^{ci}\wedge\mathbf{q})\cdot(\mathbf{p}/EE_\nu)-$$
$$-(\boldsymbol{\Sigma}^{cr}\boldsymbol{\zeta}):(\mathbf{pq}/EE_\nu)]+$$
$$+2C_1C_4[(\boldsymbol{\Sigma}^{cr}\cdot\boldsymbol{\zeta})-(\boldsymbol{\Sigma}^{cr}\boldsymbol{\zeta}):[\mathbf{pp}/E(E+1)]+$$
$$+(1/EE_\nu)(\boldsymbol{\Sigma}^{cr}\wedge\mathbf{q})\cdot(\boldsymbol{\zeta}\wedge\mathbf{p})]+$$
$$+2C_2C_3[(\boldsymbol{\Sigma}^{cr}\cdot\boldsymbol{\zeta})-(\boldsymbol{\Sigma}^{cr}\boldsymbol{\zeta}):[\mathbf{pp}/E(E+1)]-$$
$$-(1/EE_\nu)(\boldsymbol{\Sigma}^{cr}\wedge\mathbf{q})\cdot(\boldsymbol{\zeta}\wedge\mathbf{p})]\mp$$
$$\mp 2C_2C_4[(1/E)(\boldsymbol{\Sigma}^{cr}\cdot\boldsymbol{\zeta})+(\boldsymbol{\Sigma}^{cr}\boldsymbol{\zeta}):[\mathbf{pp}/E(E+1)]-(\boldsymbol{\Sigma}^{ci}\wedge\mathbf{q})\cdot(\mathbf{p}/EE_\nu)+$$
$$+(\boldsymbol{\Sigma}^{cr}\boldsymbol{\zeta}):(\mathbf{pq}/EE_\nu)]\mp$$
$$\mp 2C_3C_4[|\textstyle\int\boldsymbol{\sigma}|^2(1/E)+(\boldsymbol{\Sigma}_\wedge\cdot\boldsymbol{\zeta})-(\boldsymbol{\Sigma}_\wedge\boldsymbol{\zeta}):[\mathbf{pp}/E(E+1)]]\}. \quad (62)$$

To get results from this general formula, which describe certain experimental situations, it will generally be necessary to take averages or to integrate over some variables. We mention the following cases:

1) We take random orientation of the nucleus; according to (34), (43) and (56) we have

$$\boldsymbol{\Sigma}_\wedge = 0, \quad \boldsymbol{\Sigma} = \tfrac{1}{3}|\textstyle\int\boldsymbol{\sigma}|^2, \quad \boldsymbol{\Sigma}^{cr} = 0, \quad \boldsymbol{\Sigma}^{ci} = 0. \quad (63)$$

We further take the sum for two orthogonal states of polarization. The result found is given by I(28) and I(30), which are thus special cases of (62).

2) We again take the sum for two states of polarization. We further integrate over all directions of emission of electron and neutrino. The result found has the shape I(31), but averaging over the orientation of the nucleus has not been performed, so that this formula is also valid for an aligned nucleus. We mostly take the sum over m_f in our results. In the deduction of (62) this summation over m_f need not be made; it is then valid for the transition probabilities towards a definite m_f. We can calculate in this way "partial" transition probabilities. The partial transition probabilities can be used to calculate the angular distribution of a γ-radiation that succeeds the β-transition. We shall come back to this point later. Only if we insert the values of $\boldsymbol{\Sigma}_\wedge$, $\boldsymbol{\Sigma}$, $\boldsymbol{\Sigma}^{cr}$, $\boldsymbol{\Sigma}^{ci}$, which were obtained in § 4, the sum over m_f is taken.

3) If we take the sum for two states of polarization and if we integrate over the directions of emission of the neutrino, we obtain the formula for the transition probability of a nucleus with a definite

orientation, taking into account the direction of emission of the electron. The result can be written as

$$P_{\pm}(E, \mathbf{p}) = P_{\pm}(E)/4\pi \tag{64}$$

This means:

The angular distribution of β-rays emitted from aligned nuclei has spherical symmetry, if we have an allowed β-transition.

4) If we are interested in the polarization of β-rays emitted from aligned nuclei, we must integrate over the directions of emission of the neutrino. This will be discussed in § 7.

§ 6. *Consequences of a symmetry principle compared with the results of* § 5. In I § 6 we have proposed a symmetry principle, concerning the complete symmetry of the processes of β^+ and β^--emission. It has a consequence that only two types of combinations for the β-interaction can exist: *a*) combinations of the invariants S, A and P, *b*) combinations of the invariants V and T.

Hence the following products of two constants C_k must be zero

$$C_1C_2,\ C_1C_3,\ C_2C_4,\ C_2C_5,\ C_3C_4,\ C_3C_5 = 0. \tag{65}$$

In I(28) and I(30) this results in the dropping out of the $(1/E)$-term, which is the only difference between the formula I(28) and I(30) for β^- and β^+-emission. Analogously it is seen that as a consequence of (65) all the terms in (62) that are different for β^- and β^+-emission drop out, as is of course demanded by the symmetry principle of I § 6. We see further that the cross-terms with C_1C_2, C_1C_3, C_2C_4 and C_3C_4, which drop out if (65) is satisfied, have different signs for β^+ and β^--emission. It is easy to establish this property of these cross-terms without the explicit calculations of §§ 2—5, with the aid of the formulae I (65)—(69) and I (78), which occurred in the deduction of the consequences of the symmetry principle of I § 6. If we could devise experiments, which can decide the existence of the phenomena corresponding to these cross terms, a check of the symmetry principle could be made.

§ 7. *The polarization of β-rays emitted by aligned nuclei.* We consider in this section the polarization of β-rays that can result if the nuclei are aligned. Before considering the quantitative results, we give some elementary considerations on the conservation of angular momentum, which lead to the conclusion that the β-rays of aligned nuclei can be polarized. For non-relativistic energies we can consider the orbital and spin angular momentum of the electron separately. We know further that for allowed transitions, the electron is emitted without orbital angular momentum, while the neutrino takes the angular momentum $\frac{1}{2}$. From the diagrams (Fig. 1) it is clear, which

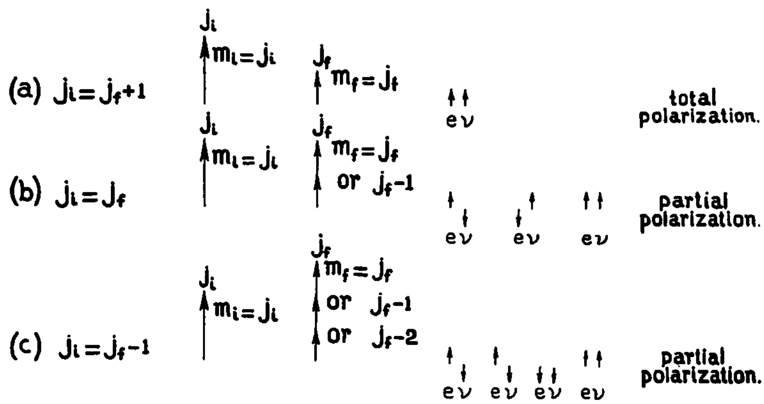

Fig. 1. Polarization of electrons emitted by aligned nuclei. The arrows indicate the possible values of the z-components of the angular momenta of the initial nucleus, the final nucleus, the electron (e) and the neutrino (ν), for the three cases a) $j_i = j_f + 1$, b) $j_i = j_f$, c) $j_i = j_f - 1$, if one assumes $m_i = j_i$. As a consequence of the law of conservation of angular momentum, only one possibility for the orientation of the spin of the electron exists in case a), so that total polarization results.

orientations of the electron and neutrino angular momenta are possible in case of alignment of the nuclear spin. Hence we see that in case a) the emitted β-rays are totally polarized. In order to see that we still have a partial polarization in cases b) and c) the quantitative treatment is necessary. We shall see that even in case a) the polarization is no longer complete, if the electron has a relativistic velocity. The simple scheme of the diagram is then not valid because orbital and spin angular momentum can no longer be separated.

We now pass on to the quantitative treatment of the polarization. If we consider the polarization of β-rays emitted from aligned nuclei, the direction of emission of the neutrino will not interest us in general. Hence we integrate (62) for all directions of emission of the neutrino and obtain

$$\left.\begin{aligned}&P_{\pm}(E,\mathbf{p},\boldsymbol{\zeta})=(G^2/16\pi^4)\,pEq^2\{(C_1^2+C_2^2)\,|\textstyle\int 1|^2+(C_3^2+C_4^2)\,|\textstyle\int\boldsymbol{\sigma}|^2+\\&+C_5^2\,|\textstyle\int\beta\gamma_5|^2\mp(2/E)\,[C_1C_2\,|\textstyle\int 1|^2+C_3C_4\,|\textstyle\int\boldsymbol{\sigma}|^2]+\\&+(C_3^2+C_4^2)\,[(1/E)\,(\boldsymbol{\Sigma}_\wedge\cdot\boldsymbol{\zeta})+[1/E(E+1)]\,(\boldsymbol{\Sigma}_\wedge\boldsymbol{\zeta}):(\mathbf{pp})]\mp\\&\mp 2(C_1C_3+C_2C_4)\,[(1/E)\,(\boldsymbol{\Sigma}^{cr}\cdot\boldsymbol{\zeta})+[1/E(E+1)]\,(\boldsymbol{\Sigma}^{cr}\boldsymbol{\zeta}):(\mathbf{pp})]+\\&+2(C_1C_4+C_2C_3)\,[(\boldsymbol{\Sigma}^{cr}\cdot\boldsymbol{\zeta})-[1/E(E+1)]\,(\boldsymbol{\Sigma}^{cr}\boldsymbol{\zeta}):(\mathbf{pp})]\mp\\&\mp 2C_3C_4\,[(\boldsymbol{\Sigma}_\wedge\cdot\boldsymbol{\zeta})-[1/E(E+1)]\,(\boldsymbol{\Sigma}_\wedge\boldsymbol{\zeta}):(\mathbf{pp})]\}.\end{aligned}\right\}\quad(66)$$

In order to make the physical implications of this still rather complicated formula clearer, we consider some special cases:

1) We first take the case of the pure invariant T for the β-interaction, so that $C_3 = 1$ and $C_1 = C_2 = C_4 = C_5 = 0$. The general formula (66) becomes, using the value (31) for $\boldsymbol{\Sigma}_\wedge$

$$P_{\pm}(E,\mathbf{p},\boldsymbol{\zeta},[\mathbf{J}_{nucl}]=\boldsymbol{\eta})=$$
$$(G^2/16\pi^4)\,pEq^2|\textstyle\int\boldsymbol{\sigma}|^2\{1+(A_\wedge/E)\,(\boldsymbol{\eta}\cdot\boldsymbol{\zeta})+[A_\wedge/E(E+1)]\,(\boldsymbol{\eta}\boldsymbol{\zeta}):(\mathbf{pp})\}.\quad(67)$$

We specialize (67) for two cases, namely that the direction of emission $\mathbf{p}$ of the electron is perpendicular and parallel to $\boldsymbol{\eta}$, the axis of polarization of the nuclei (for which we take the z-direction). We get

$$P_{\pm}(E,\mathbf{p}=p\mathbf{i},\boldsymbol{\zeta},[\mathbf{J}_{nucl}]=\mathbf{k})=(G^2/16\pi^4)\,pEq^2\,|\textstyle\int\boldsymbol{\sigma}|^2\{1+(A_\wedge/E)\,\zeta_z\},\quad(68)$$

and $P_{\pm}(E,\mathbf{p}=p\mathbf{k},\boldsymbol{\zeta},[\mathbf{J}_{nucl}]=\mathbf{k})=(G^2/16\pi^4)pEq^2\,|\int\boldsymbol{\sigma}|^2\{1+A_\wedge\zeta_z\}$. (69)

From the formulae for the transition probabilities like (67), (68) and (69) we must conclude to the state of polarization of the emitted electrons. As explaned in II this polarization can be described by a "degree of polarization" and a specification of the polarized part, which can be given by the "axis of polarization" $\boldsymbol{\zeta}$ according to II(11) and II(12). If we have a beam of electrons with degree of polarization P and axis of polarization $\boldsymbol{\zeta}_0$, then the probability for finding a direction of polarization $\boldsymbol{\zeta}$ is given by

$$\tfrac{1}{2}[1+P(\boldsymbol{\zeta}_0\cdot\boldsymbol{\zeta})].\quad(70)$$

Now the expression between brackets in (67) can be put in this form (apart from the factor $\frac{1}{2}$), if we set

$$\left.\begin{aligned}&\boldsymbol{\xi}=\boldsymbol{\eta}\cdot[1+\mathbf{pp}/(E+1)],\\&\boldsymbol{\zeta}_0=\boldsymbol{\xi}/|\boldsymbol{\xi}|,\\&P=(A_\wedge/E)\,|\boldsymbol{\xi}|.\end{aligned}\right\}\quad(71)$$

For the cases (68) and (69) we get in particular

$$\left.\begin{array}{ll}\mathbf{p} \perp \boldsymbol{\eta}, & \boldsymbol{\zeta}_0 = \boldsymbol{\eta}\ (= \mathbf{k}), \\ & P = A_\Lambda/E,\end{array}\right\} \tag{72}$$

$$\left.\begin{array}{ll}\mathbf{p} \,//\, \boldsymbol{\eta}, & \boldsymbol{\zeta}_0 = \boldsymbol{\eta}\ (= \mathbf{k}), \\ & P = A_\Lambda.\end{array}\right\} \tag{73}$$

Hence $\boldsymbol{\zeta}_0 = \boldsymbol{\eta}$ for these special cases, though we see that for an arbitrary direction of emission the axis of polarization of electrons and nuclei need not be the same according to (71). For $\mathbf{p} \perp \boldsymbol{\eta}$ the electrons have *transverse polarization*, for $\mathbf{p} \,//\, \boldsymbol{\eta}$ there is *longitudinal polarization*. How polarization of electron beams can in principle be completely determined by scattering experiments was discussed in II and III.

We give the result for the case $\mathbf{p} \perp \boldsymbol{\eta}$, which may be the most important, still in another form, namely as the ratio of the intensities of the electrons with spin parallel and antiparallel to $\boldsymbol{\eta}$, the axis of the nuclear polarization. We insert the value (30) for A_Λ and write the result as

$$P(\boldsymbol{\zeta}=\boldsymbol{\eta})/P(\boldsymbol{\zeta}=-\boldsymbol{\eta}) = \begin{cases} \{1+(1/E)\,(m_i/j_i)\}/\{1-(1/E)\,(m_i/j_i)\}, \\ \qquad\qquad \text{if} \quad j_i=j_f+1. \\ \{1+(1/E)\,[m_i/j_i(j_i+1)]\}/\{1-(1/E)[m_i/j_i(j_i+1)]\}, \\ \qquad\qquad \text{if} \quad j_i=j_f. \\ \{1-(1/E)\,[m_i/(j_i+1)]\}/\{1+(1/E)[m_i/(j_i+1)]\}, \\ \qquad\qquad \text{if} \quad j_i=j_f-1. \end{cases} \tag{74}$$

According to Fig. 1 case *a*), we drew the qualitative conclusion that there is complete polarization, if the initial nuclei are fully aligned ($m_i = j_i$) and if we have non-relativistic energies for the electrons ($E \approx 1$). We see that this is confirmed by (74).

2) After considering the polarization of the β-rays emitted in an arbitrary direction if the β-interaction is given by the invariant T, we now take the case of β-rays emitted perpendicular to $\boldsymbol{\eta}$ the axis of the polarization of the nuclei with a β-interaction given by an arbitrary "mixture" of invariants. By specialization we get from (66) using (31) and (54)

$$\begin{aligned} P_\pm\,(E, \mathbf{p} = p\mathbf{i}\ \boldsymbol{\zeta}, [\mathbf{J}_{nucl}] = \mathbf{k}) = \\ = (G^2/16\pi^4)\, pEq^2\{(C_1^2+C_2^2)\,|\textstyle\int 1|^2+(C_3^2+C_4^2)\,|\textstyle\int \boldsymbol{\sigma}|^2+C_5^2|\textstyle\int \beta\gamma_5|^2 \mp \\ \mp (2/E)\,[C_1C_2\,|\textstyle\int 1|^2 + C_3C_4\,|\textstyle\int \boldsymbol{\sigma}|^2] + \\ + [(C_3^2 + C_4^2)/E \mp 2C_3C_4]\,A_\Lambda\zeta_z + \\ +2\,[(C_1C_4+C_2C_3) \mp (C_1C_3 + C_2C_4)/E]\,A_c\,M_{cr}\,\zeta_z\} \end{aligned} \tag{75}$$

From (75) we can calculate the same ratio as in (74) for the more general case. If $j_i = j_f \pm 1$ the result is the same as in (74) (for then $|\int 1|^2 = 0$ and $\boldsymbol{\Sigma}_{cr} = 0$). We give the result for $j_i = j_f$, assuming that $C_1C_2 = 0$, $C_3C_4 = 0$, $C_1C_3 = 0$, $C_2C_4 = 0$ (this would be the case if the symmetry principle of I § 6 is valid, cf. § 6) and that the nucleus does not change its parity so that $|\int \beta\gamma_5|^2 = 0$ (otherwise we would have $|\int 1|^2 = 0$ and $|\int \boldsymbol{\sigma}|^2 = 0$).

$$\begin{aligned} P_{\pm}(\boldsymbol{\zeta}=\boldsymbol{\eta})/P_{\pm}(\boldsymbol{\zeta}=-\boldsymbol{\eta}) = &\{(C_1^2+C_2^2)\,|\int 1|^2+(C_3^2+C_4^2)\,|\int \boldsymbol{\sigma}|^2 + \\ &+[m_i/j_i(j_i+1)]\,[(C_3^2+C_4^2)\,|\int\boldsymbol{\sigma}|^2/E+2(C_1C_4+C_2C_3)\,M_{cr}\sqrt{j_i(j_i+1)}]\}/ \\ &\{(C_1^2+C_2^2)\,|\int 1|^2+(C_3^2+C_4^2)\,|\int \boldsymbol{\sigma}|^2 - \\ &-[m_i/j_i(j_i+1)]\,[(C_3^2+C_4^2)|\int\boldsymbol{\sigma}|^2/E+2(C_1C_4+C_2C_3)\,M_{cr}\sqrt{j_i(j_i+1)}]\} \end{aligned} \tag{76}$$

The formulae (74) and (76) are given on the assumption that the orientation of the nuclei can be described by one single wave function (24). It is, however, easy to deduce from the general formulae (66), (67), (71) and (75), that for the general case (22) the resulting degree of polarization P can directly be expressed with f_N the "degree of polarization of the nuclei", and the maximum polarization $P(m_i=j_i)$ occurring if $f_N = 1$

$$P = f_N\, P(m_i = j_i). \tag{77}$$

As to the experimental possibilities, the polarization of β-rays seems not to be the phenomenon of polarization of nuclei that is easiest for observation. It is favourable in the respect that it depends linearly on f_N, while for example the departure from spherical symmetry of γ-radiation emitted by aligned nuclei depends on f_N^2, which would be important if f_N is rather small compared with unity. The difficulty is that the β-rays cannot leave the cryostate so that they must be measured in its interior. Further the β-radioactive source must be near the surface of the cooled material, where it may be heated up rather soon. If polarization is measured by scattering experiments[2])[3]), very strong sources (several mC) would be nearly indispensable. Other methods of measuring the polarization of β-rays with feebler sources (e.g. transmission by magnetized Fe-foils [11]), [12])) can be imagined, but no successfull experiments have as yet been made along these lines.

A nucleus that can possibly be used in experiments is ^{64}Cu (half life 12,8 h, maximum energy of negatons 0,571 MeV, allowed β-transition, about 35% β^--emission, further β^+-emission and K-capture), which can be aligned according to the method indicated above [5], [6], if it has a nuclear spin (probably $j_i = 1$). A complication is that ^{64}Cu emits both β^+ and β^--rays, of which one wishes to separate the effects.

Appendix. Vector and tensor notations. We have used the following notations for vectors, tensors and their products (the same as, e.g., in [13]):

Vectors: clarendon type, e.g., $\mathbf{a}$.

Tensors (dyads): sanserif type, e.g., A.

Scalar product of two vectors $\mathbf{a} \cdot \mathbf{b}$.

Vector product of two vectors $\mathbf{a} \wedge \mathbf{b}$.

Tensor product of two vectors $\mathbf{ab} = \mathsf{T}$, if $T_{ik} = a_i b_k$.

$\mathsf{T} \cdot \mathbf{a}$ is the vector with components $(\mathsf{T} \cdot \mathbf{a})_i = \Sigma_k T_{ik} a_k$.

By the product $\mathsf{S} : \mathsf{T}$ we mean the scalar $\Sigma_{ik} S_{ik} T_{ki} = \mathsf{S} : \mathsf{T} = \mathsf{T} : \mathsf{S}$.

We have $(\mathbf{ab}) : (\mathbf{pq}) = (\mathbf{a} \cdot \mathbf{q})\,(\mathbf{b} \cdot \mathbf{p}) = (\mathbf{pq}) : (\mathbf{ab})$,
and $\mathsf{S} : (\mathbf{pq}) = (\mathbf{pq}) : \mathsf{S} = \mathbf{q} \cdot \mathsf{S} \cdot \mathbf{p}$.

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