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Fatou's Lemma in Infinite Dimensions

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By a general method, based on weak convergence of transition probabilities, new infinite-dimensional Fatou lemmas are derived. © 1988 Academic Press, Inc.

1. INTRODUCTION

The *tightness approach* to limit problems for ordinary measurable functions under various convergence or boundedness conditions consists of the following three stages:

a. *Establish tightness.* Consider the measurable functions as transition probabilities (this step is called *relaxation*) and prove that in this form they constitute a *tight* set.

b. *Identify the weak limit.* By an analogue of Prohorov's theorem, a tight set of transition probabilities has limit points for a *weak* topology. Establish useful properties of these transition probabilities by means of the original convergence or boundedness conditions.

c. *Replace the weak limit.* Choose a suitable limit point and replace this transition probability by a suitable measurable function so as to solve the limit problem.

The seminal relaxation idea of L. C. Young has been used in optimal control theory to deal with existence problems [18, 16]. The tightness approach to limit problems in the form presented here was started by Berliocchi and Lasry in [4] and expanded by the present author in a series of papers dealing with various applications: cf. [3a, b, c, d] and their references. In [3b, c] the method was used to obtain a very general version of Fatou's lemma in finite dimensions, which generalizes and unifies all previous such lemmas [15, 10, 1b, 7], and also a number of related

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variational existence results [2, 1a, 3a]. Thus, it would appear that the method is very suitable to obtain infinite-dimensional Fatou lemmas as well.

However, in extending the tightness approach to infinite-dimensional Fatou lemmas one is faced with two obstacles. A crucial tool for the approach is formed by an analogue of Prohorov's classical theorem, used to establish the existence of at least one limit point in the above scheme. The versions of this result used in [3a, b, c] can only deal with measurable functions whose range is a metrizable Lusin space. In infinite-dimensional Fatou lemmas this range is nonmetrizable as a rule, since it is a Banach space which is usually equipped with the weak topology (in view of the tightness requirements this may be a natural choice). Thus, the Prohorov theorem of [3b] may not suffice. A stronger version was obtained by the author in [3d]. This result only requires that the range be "approximately" metrizable Lusin, and this requirement turns out to be fulfilled in a number of interesting cases. One application of a similar nature, to an infinite-dimensional lower closure result, was already given in [3d]. A second obstacle, which is of a fundamental nature, is formed by the well-known fact that Lyapunov's theorem is not valid for infinite-dimensional vector measures. As a consequence, only *approximate* Fatou lemmas can be given. Nevertheless, in the course of the tightness approach we shall also come across useful identities which hold exactly. As a whole, the tightness approach to the approximate Fatou lemmas in infinite dimensions is much less subtle than the corresponding approach to the exact finite-dimensional Fatou lemma of [3c], which involves some rather delicate extreme point arguments.

We now briefly describe the contents of this paper. In Section 2 we present four approximate Fatou lemmas in Banach spaces. The first of these, Theorem 2.1, generalizes a recent result of Khan and Majumdar [12, Thm. 2]. After writing the first version of this paper [3f], the author became aware of a quite similar result, obtained independently by N. C. Yannelis [17a]. The present Theorem 2.1 has been improved (in terms of the "dominating" multifunction F being used) so as to incorporate Yannelis' main result as well. Theorem 2.2 is a duplicate of Theorem 2.1, where the weak topology has been systematically replaced by the norm topology. It generalizes a recent result of Yannelis [17b] and underscores the versatility of the tightness approach. Theorem 2.4 is new; it is a Fatou lemma in a reflexive Banach space which can be regarded as the natural infinite-dimensional counterpart to the unifying Fatou lemma of [3b, c] mentioned above. The remaining Fatou lemma is Corollary 2.5 of Theorem 2.4. Proofs of these results are given in Section 3. Some relevant facts concerning the weak convergence theory for transition probabilities have been recapitulated in the Appendix.

2. Main Results

Let X be a separable Banach space with norm $\|\cdot\|$. U will stand for the closed unit ball in X. The norm topology on X will be referred to by the symbol s; thus we speak of s-topology, s-closure, etc. The weak topology $\sigma(X, X^*)$ on X will be indicated similarly by the symbol w. Here X^* stands for the topological dual of X; the duality between X and X^* is denoted by $\langle\cdot,\cdot\rangle$. Since (X, w) is a Lusin space, it follows by [6, III.32] that X^* contains a countable subset $\{x_i^*\}_{i=1}^{\infty}$ which is certainly dense in X^* for the weak topology $\sigma(X^*, X)$. Correspondingly, we define the metric d on X by

$$d(x, y) := \sum_{i=1}^{\infty} 2^{-i} \frac{|\langle x-y, x_i^*\rangle|}{(1+|\langle x-y, x_i^*\rangle|)}.$$

Note that d is weaker than the w-topology on X, so (X, d) is a Lusin space. Note further that the Borel σ-algebras of (X, s), (X, w), and (X, d) coincide, as follows immediately from the fact that $\mathscr{B}((X, s))$ is generated by the balls of X (since X is separable), and from the identity

$$\|x\| = \sup_{i\in\mathbb{N}} \frac{\langle x, x_i^*\rangle}{\|x_i^*\|^*},$$

where $\|\cdot\|^*$ stands for the dual norm of X^*.

For any cone C in X the *polar cone* C^0 of C is defined as the set of all $x^* \in X^*$ such that $\langle c, x^*\rangle \leqslant 0$ for all $c \in C$.

For any sequence $\{A_k\}_{k=1}^{\infty}$ of subsets of X we define the *w-limes superior* $w\text{-}Ls_k\, A_k$ to be the set of all $x \in X$ such that there exists some subsequence $\{k_j\}$ of $\{k\}$ and points $x_{k_j} \in A_{k_j}$ for which $x = w\text{-}\lim_j x_{k_j}$. Similarly, the *w-limes inferior* $w\text{-}Li_k\, A_k$ is defined as the set of all $x \in X$ such that there exists a sequence $\{x_k\}$, $x_k \in A_k$, for which $x = w\text{-}\lim_k x_k$. The definitions of $s\text{-}Ls_k\, A_k$, $d\text{-}Ls_k\, A_k$, $s\text{-}Li_k\, A_k$, and $d\text{-}Li_k\, A_k$ are obvious analogues of the ones above.

Let $(T, \mathscr{T}, \mu)$ be a finite measure space. A multifunction $F\colon T \to 2^X$ is said to have a *measurable graph* if its graph gph F, i.e., the set of all $(t, x) \in T \times X$ with $x \in F(t)$, is $\mathscr{T} \times \mathscr{B}(X)$-measurable. F is said to be *integrably bounded* if there exists $\bar{\varphi} \in \mathscr{L}_1(T; \mathbb{R})$ such that $\sup_{x\in F(t)}\|x\| \leqslant \bar{\varphi}(t)$ for all $t \in T$.

The set of all X-valued Bochner-integrable functions will be denoted by $\mathscr{L}_1(T; X)$. Let $\{f_k\}_{k=1}^{\infty}$ be a given sequence of functions in $\mathscr{L}_1(T; X)$. Our first Fatou lemma in infinite dimensions is as follows.

Theorem 2.1. *Suppose that there exists a w-compact-valued integrably bounded multifunction $F\colon T \to 2^X$, having a measurable graph, such that*

$$\{f_k(t)\} \subset F(t) \quad a.e. \tag{2.1}$$

and suppose that

$$a := w\text{-}\lim_k \int_T f_k \, d\mu \quad \textit{exists.} \tag{2.2}$$

Then there exists a function $f_* \in \mathscr{L}_1(T; X)$ *such that*

$$a = \int_T f_* \, d\mu, \tag{2.3}$$

$$f_*(t) \in \operatorname{cl} \operatorname{co} \bigcap_{p=1}^{\infty} \{f_k(t) : k \geqslant p\} \quad a.e. \tag{2.4}$$

Also, for every $\varepsilon > 0$ *there exists a function* $f_\varepsilon \in \mathscr{L}_1(T; X)$ *such that*

$$\left\| \int_T f_\varepsilon \, d\mu - a \right\| < \varepsilon, \tag{2.5}$$

$$f_\varepsilon(t) \in w\text{-}Ls_k \, \{f_k(t)\} \quad a.e. \tag{2.6}$$

Theorem 2.1 generalizes the infinite-dimensional Fatou lemma of Khan and Majumdar [12, Thm. 2], where $(T, \mathscr{T}, \mu)$ is complete, nonatomic, and the multifunction F has to be a constant. Note that [12, Thm. 1] is already contained in [3d, Remark 3.5]. Theorem 2.1 is also slightly more general than the infinite-dimensional Fatou lemma of Yannelis [17a], where $(T, \mathscr{T}, \mu)$ is complete, nonatomic, and $w\text{-}Ls_k \, \{f_k\}$ is supposed to be measurable (as we shall see, this property follows automatically from the other conditions). The original version of Theorem 2.1 in [3f] was stated for a "dominating" multifunction $F(t) = \bar{\varphi}(t)K$, where $\bar{\varphi}$ is an integrable function on T and K a w-compact subset of X.

Our second result states that Theorem 2.1 continues to hold if everywhere the w-topology is replaced by the s-topology. It slightly improves [17b, Cor. 4.2], which requires $(T, \mathscr{T}, \mu)$ to be complete.

THEOREM 2.2. *Suppose that there exists an s-compact-valued integrably bounded multifunction* $F: T \to 2^X$, *having a measurable graph, such that*

$$\{f_k(t)\} \subset F(t) \quad a.e.$$

and suppose that

$$a := s\text{-}\lim_k \int_T f_k \, d\mu \quad \textit{exists.}$$

Then there exists a function $f_* \in \mathscr{L}_1(T; X)$ *such that*

$$a = \int_T f_* \, d\mu,$$

$$f_*(t) \in \operatorname{cl} \operatorname{co} \bigcap_{p=1}^{\infty} \{f_k(t) \colon k \geqslant p\}.$$

Also, for every $\varepsilon > 0$ *there exists a function* $f_\varepsilon \in \mathscr{L}_1(T; X)$ *such that*

$$\left\| \int_T f_\varepsilon \, d\mu - a \right\| < \varepsilon,$$

$$f_\varepsilon(t) \in s\text{-}Ls_k \, \{f_k(t)\} \quad a.e.$$

Remark 2.3. Using the familiar notion of integrals of multifunctions, the result given in (2.5), (2.6) can be expressed equivalently as

$$w\text{-}Ls_k \left\{ \int_T f_k \, d\mu \right\} \subset s\text{-cl} \int_T w\text{-}Ls_k \, \{f_k\} \, d\mu,$$

and the corresponding result in Theorem 2.2 as

$$s\text{-}Ls_k \left\{ \int_T f_k \, d\mu \right\} \subset s\text{-cl} \int_T s\text{-}Ls_k \, \{f_k\} \, d\mu.$$

Our new, "undominated" Fatou lemmas in infinite dimensions will be stated next. Note that here condition (2.7) is much weaker than (2.1), used before. The price to be paid for this is, of course, that in Theorem 2.4 and Corollary 2.5 the space X has to be reflexive. In the next section we shall see how these two situations correspond to two quite different ways in which tightness can hold.

THEOREM 2.4. *Suppose that in addition* X *is reflexive, that*

$$\sup_{k \in \mathbb{N}} \int_T \|f_k\| \, d\mu < +\infty, \tag{2.7}$$

and that (2.2) *holds. Let* C *be a nonempty closed convex in* X *such that for every* $x^* \in C^0$

$$\{\min(0, \langle x^*, f_k \rangle)\}_{k=1}^{\infty} \quad \textit{is uniformly integrable.} \tag{2.8}$$

Then there exists a function $f_* \in \mathscr{L}_1(T; X)$ *satisfying* (2.4) *and*

$$\int_T f_* \, d\mu - a \in C. \tag{2.9}$$

Also, for every $\varepsilon > 0$ *there exists a function* $f_\varepsilon \in \mathscr{L}_1(T; X)$ *such that*

$$\int_T f_\varepsilon \, d\mu - a \in C + \varepsilon U, \tag{2.10}$$

$$f_\varepsilon(t) \in d\text{-}Ls_k \, \{f_k(t)\} \quad a.e., \tag{2.11}$$

provided that the multifunction $d\text{-}Ls_k \, \{f_k\}$ *has at least one integrable selector.*

COROLLARY 2.5. *Suppose that* X *is as in Theorem* 2.4, *that* (2.2) *holds, and that*

$$\{f_k\}_{k=1}^{\infty} \quad \textit{is uniformly integrable.} \tag{2.12}$$

Then there exists a function $f_* \in \mathscr{L}_1(T; X)$ *satisfying* (2.3), (2.4). *Also, for every* $\varepsilon > 0$ *there exists a function* $f_\varepsilon \in \mathscr{L}_1(T; X)$ *satisfying* (2.5), (2.11).

Proof. Define $C := \{0\}$, and apply Theorem 2.4. Now (2.12) implies (2.13), (2.7). By the choice of C, (2.9) coincides with (2.3) and (2.10) with (2.5). ∎

Remark 2.6. There is an easy way to extend all of the above results to the case where the Banach space X is nonseparable. By strong measurability of all functions f_k, $k \in \mathbb{N}$, there exists a null set $N \in \mathscr{T}$ such that $\bigcup_{k=1}^{\infty} f_k(T \setminus N)$ is separable. Let Y be the closed linear subspace of X generated by this set; Y is clearly a separable Banach space. By the Hahn–Banach theorem the weak topology $\sigma(Y, Y^*)$ on Y coincides with the relative w-topology. If X is in addition reflexive, then Y is also a reflexive Banach space. Note further that in (2.2) and (2.8) both the vectors $\int_T f_k \, d\mu$, $k \in \mathbb{N}$, and the limit a lie in Y. This leads us to conclude that without loss of generality the space X can be taken to be nonseparable in Theorems 2.1, 2.2, and also in Theorem 2.4 and Corollary 2.5, provided that the definition of the metric d is suitably rephrased.

3. PROOFS OF THE MAIN RESULTS

We shall follow the tightness approach explained in Section 1. First we prove Theorem 2.1. Relevant results concerning the weak convergence of transition probabilities can be found in the Appendix.

Proof of Theorem 2.1. a. *Establish tightness.* Observe that (X, d) is a metrizable Lusin space. For the relaxations $\{\varepsilon_{f_k}\}_{k=1}^{\infty}$ of $\{f_k\}_{k=1}^{\infty}$ we have by (2.1)

$$\sup_{k \in \mathbb{N}} I_h(\varepsilon_{f_k}) = 0, \tag{3.1}$$

where h: $T \times T \to \{0, +\infty\}$ is defined by $h(t, x) := 0$ if $x \in F(t)$, and $h(t, x) := +\infty$ if not. By measurability of gph F the function h is $\mathscr{T} \times \mathscr{B}(X)$-measurable, and since F is w-compact-valued, $h(t, \cdot)$ is *a fortiori* d-inf-compact on X for every $t \in T$. So the sequence $\{\varepsilon_{f_k}\}_{k=1}^{\infty}$ is tight.

b. *Identify the weak limit.* By Corollary A.2 there exists a subsequence $\{\ell\}$ of $\{k\}$ and a transition probability δ_* from $(T, \mathscr{T})$ into $(X, \mathscr{B}(X))$ such that

$$\{\varepsilon_{f_k}\}_{k=1}^{\infty} \text{ converges weakly to } \delta_* \text{ in } \mathscr{R}(T, (X, d)). \tag{3.2}$$

Hence it follows by Corollary A.2 and (2.1) that

$$\delta_*\left(t, \bigcap_{p=1}^{\infty} d\text{-cl}\{f_k(t) : k \geqslant p\}\right) = 1 = \delta_*(t, F(t)) \quad \text{a.e.} \tag{3.3}$$

For every $t \in T$ the w-compact set $F(t)$ is metrizable, so

$$w\text{-}Ls_k\ \{f_k(t)\} = \bigcap_{p=1}^{\infty} w\text{-cl}\{f_k(t) : k \geqslant p\},$$

and this set is nonempty; in turn this gives

$$\begin{aligned} w\text{-}Ls_k\ \{f_k(t)\} &= \bigcap_{p=1}^{\infty} d\text{-cl}\{f_k(t) : k \geqslant p\} \\ &= d\text{-}Ls_k\ \{f_k(t)\} =: L(t), \end{aligned} \tag{3.4}$$

since the d-closure and the w-closure of any subset of the w-compact set $F(t)$ are the same. For every $t \in T$, $F(t)$ is also s-bounded. Hence (3.3) implies that

$$f_*(t) := \text{barycenter of } \delta_*(t) := \int_X x\, \delta_*(t, dx) \quad \text{exists a.e.}$$

By setting f_* equal to 0 on the exceptional set, f_* is defined on all of T. Let $\bar{\varphi}$ be as in the definition of integrable boundedness of F. Then it is obvious from (3.3) and the definition of $f_*(t)$ that $\|f_*(t)\| \leqslant \bar{\varphi}(t)$ for all $t \in T$. Thus, f_* belongs to $\mathscr{L}_1(T; X)$. By an elementary property of the barycenter it follows from (3.3) that (2.4) holds. Let $i \in \mathbb{N}$ be arbitrary, and define two normal integrands g: $T \times X \to (-\infty, +\infty]$ by setting $g(t, x) := \pm\langle x, x_i^* \rangle$ if $x \in F(t)$ and $g(t, x) := +\infty$ if not. Then by definition of weak convergence in $\mathscr{R}(T; (X, d))$ it follows that

$$I_g(\delta_*) = \lim_{\ell \to \infty} I_g(\varepsilon_{f_\ell}) = \lim_{\ell \to \infty} \int_T \langle f_\ell, x_i^* \rangle\, d\mu = \langle a, x_i^* \rangle, \tag{3.5}$$

where the last identity follows by (2.2). On the other hand, by (3.3),

$$I_g(\delta_*) := \int_T \left[\int_X \langle x, x_i^* \rangle \, \delta_*(t, dx) \right] \mu(dt)$$
$$= \int_T \langle f_*, x_i^* \rangle \, d\mu = \left\langle \int_T f_* \, d\mu, x_i^* \right\rangle.$$

Since $i \in \mathbb{N}$ was arbitrary, it follows from this identity and (3.5) that (2.3) also holds, since $\{x_i^*\}_{i=1}^{\infty}$ separates the points of X.

c. *Replace the weak limit.* First we shall work under the additional Hypothesis

$$(T, \mathscr{T}, \mu) \text{ is a nonatomic measure space.} \tag{H}$$

Consider the multifunction $L: T \to 2^X$ defined by (3.4). By (2.1) L has nonempty values, and from (3.4) it follows easily by [6, III.14] that the graph of L is $\mathscr{T} \times \mathscr{B}(X)$-measurable. Hence the set $Q \subset X$, defined by (cf. Remark 2.3)

$$Q := s\text{-cl} \int_T w\text{-}Ls_k \, \{f_k\} \, du$$

is nonempty by the von Neumann–Aumann measurable selection theorem [5]. Also, it follows from Lyapunov's theorem that Q is convex (apply [14, Thm. 3.1], noting that the reflexiveness of X in [14] is not essential for the result to hold, as the whole argument is based on a standard approximation by step functions). Now observe that (2.4), (2.5) are equivalent to having $a \in Q$ (see also Remark 2.3). Thus, by the Hahn–Banach theorem it is enough to prove that for every $x^* \in X^*$

$$\inf\left\{ \int_T \langle f, x^* \rangle \, d\mu : f \in \mathscr{L}_1(T; X), f(t) \in L(t) \text{ a.e.} \right\} \leqslant \langle a, x^* \rangle. \tag{3.6}$$

By a standard application of the von Neumann–Aumann measurable selection theorem it follows from the above that the left side of (3.6) equals $\int_T \inf_{x \in L(t)} \langle x, x^* \rangle \, \mu(dt)$ (apply [3e, Thm. B.1]). So (3.6) immediately follows from (2.4).

Let us drop now the additional requirement (H). The space T can be split into a nonatomic and a purely atomic part, T^{na}, $T^{\text{pa}} \in \mathscr{T}$, the latter consisting of at most countably many atoms A_i, $i \in \mathbb{N}$. Of course, for every $i, k \in \mathbb{N}$ the function f_k has a constant value, say f_k^i, a.e. on A_i. By (2.1) we have $f_k^i \in F^i$, $i, k \in \mathbb{N}$, where F^i denotes the w-compact (and sequentially w-compact) set in X that is the a.e.-constant value of the multifunction F

on A_i. By a diagonal extraction procedure we can obtain a preliminary subsequence $\{k'\}$ of $\{k\}$ such that $\{f^i_{k'}\}_{k'=1}^{\infty}$ w-converges to some $f^i_* \in F^i$ for every $i \in \mathbb{N}$. Hence,

$$f_*(t) \in w\text{-}Ls_k \ \{f_k(t)\} \quad \text{a.e. in } T^{\text{pa}}. \tag{3.7}$$

Also, by the dominated convergent theorem it follows from (2.1) that

$$a' := w\text{-}\lim_{k'} \int_{T^{\text{pa}}} f_{k'} \, d\mu \tag{3.8}$$

exists and equals $\int_{T^{\text{pa}}} f_* \, d\mu$. Combining this with (2.2), we see that the previous part of the proof can be applied to the nonatomic part T^{na} of T. This gives the existence of $f_\varepsilon \in \mathscr{L}_1(T^{\text{na}}; X)$ such that

$$\left\| \int_{T^{\text{na}}} f_\varepsilon \, d\mu - (a - a') \right\| < \varepsilon,$$

$$f_\varepsilon(t) \in w\text{-}Ls_k \ \{f_k(t)\} \quad \text{a.e. in } T^{\text{na}}.$$

Clearly, concatenation with $f_* \in \mathscr{L}_1(T^{\text{pa}}; X)$ gives (2.5), (2.6), in view of (3.7), (3.8). ∎

Remark 3.1. In comparison to the original proof of Theorem 2.1 in [3f], a considerable simplification has been reached by working throughout with the d-topology, for which X is a metrizable Lusin space. Thus, for this case the extension of the weak convergence theory found in [3b], which was given in [3d], is not needed. Another simplification is formed by executing a remark made in the original proof in [3f], which dispenses with some delicate extreme point arguments, and is based on applying the Hahn–Banach theorem. We observe that no similar shortcut can be introduced for finite-dimensional Fatou lemmas, as these are *exact* by nature, so that their counterpart of Q need not be closed. (See [1a, Appendix] for a related result involving extreme points.)

Proof of Theorem 2.2. This proof is self-evident, because it is a simpler repetition of the one above, obtained by replacing the metrizable Lusin space (X, d) by the metrizable Lusin space (X, s). ∎

Proof of Theorem 2.3. a. *Establish tightness.* Observe that the unit ball U of X is metrizable and w-compact. Thus, $X = \bigcup_{j=1}^{\infty} jU$ is σ-metrizable Lusin for the w-topology (see [3d] or the Appendix). For the relaxations $\{\varepsilon_{f_k}\}_{k=1}^{\infty}$ of $\{f_k\}_{k=1}^{\infty}$ we have by (2.7)

$$\sup_{k \in \mathbb{N}} I_h(\varepsilon_{f_k}) = \sup_{k \in \mathbb{N}} \int_T \|f_k\| \, d\mu < +\infty, \tag{3.9}$$

where $h: T \times X \to [0, +\infty)$ is defined by $h(t, x) := \|x\|$. By reflexiveness of X, $h(t, *)$ is w-inf-compact on X for every $t \in T$. Also, by Markov's inequality

$$j \sup_{k \in \mathbb{N}} \mu(\{t \in T: \|f_k(t)\| > j\}) \leqslant \sup_{k \in \mathbb{N}} I_h(\varepsilon_{f_k})$$

for every $j \in \mathbb{N}$. So $\{\varepsilon_{f_k}\}_{k=1}^{\infty}$ is σ-tight.

b. *Identify the weak limit.* By Corollary A.4 there exists a subsequence $\{\ell\}$ of $\{k\}$ and a transition probability δ_* from $(T, \mathscr{T})$ into $(X, \mathscr{B}(X))$ such that

$$\{\varepsilon_{f_k}\}_{k=1}^{\infty} \text{ converges weakly to } \delta_* \text{ in } \mathscr{R}(T; (X, w)). \tag{3.10}$$

Hence it follows by Corollary A.4 that

$$\delta_*\left(t, \bigcap_{p=1}^{\infty} w\text{-cl}\{f_k(t): k \geqslant p\}\right) = 1 \quad \text{a.e.} \tag{3.11}$$

By definition of weak convergence in $\mathscr{R}(T; (X, w))$, (3.9), (3.10) imply

$$I_h(\delta_*) = \int_T \left[\int_X \|x\| \, \delta_*(t, dx)\right] \mu(dt) < +\infty. \tag{3.12}$$

This guarantees that

$$f_*(t) := \text{barycenter of } \delta_*(t) := \int_X x \, \delta_*(t, dx) \quad \text{exists a.e.}$$

By setting f_* equal to 0 on the exceptional set, f_* is defined everywhere on T, and it follows directly from (3.12) that f_* belongs to $\mathscr{L}_1(T; X)$. As in the previous proof, (2.4) follows from (3.11). For arbitrary $x^* \in C^0$ we define the normal integrand $g_\alpha: T \times X \to \mathbb{R}$, $\alpha > 0$, by $g(t, x) := \max(\langle x, x^* \rangle, -\alpha)$. By definition of weak convergence it follows from (3.10) that for every $\alpha > 0$

$$\liminf_{\ell \to \infty} I_{g_\alpha}(\varepsilon_{f_\ell}) \geqslant I_{g_\alpha}(\delta_*).$$

By (2.8) it is then standard (cf. [3b, p. 577]) to derive from this

$$\liminf_{\ell \to \infty} I_g(\varepsilon_{f_\ell}) \geqslant I_g(\delta_*),$$

$g\colon T\times X\to\mathbb{R}$ being defined by $g(t,x):=\langle x,x^*\rangle$. This inequality runs equivalently

$$I_g(\delta_*)\leqslant\liminf_{\ell\to\infty}\left\langle\int_T f_\ell\,d\mu,x^*\right\rangle d\mu=\langle a,x^*\rangle,$$

where the last identity follows by (2.2). On the other hand, it is clear that

$$I_g(\delta_*)=\int_T\langle f_*,x^*\rangle\,d\mu=\left\langle x^*,\int_T f_*\,d\mu\right\rangle,$$

so (2.9) follows by applying the bipolar theorem [8, Thm. 22.7].

c. *Replace the weak limit.* Again we adopt for our first step the non-atomicity Hypothesis (H), given in the proof of Theorem 2.1. The difference with that proof is that for every $t\in T$ now only

$$w\text{-}Ls_k\,\{f_k(t)\}\subset\bigcap_{p=1}^{\infty}w\text{-cl}\{f_k(t)\colon k\geqslant p\}\subset d\text{-}Ls_k\,\{f_k(t)\}=:L(t),$$

because the set $\{f_k(t)\}_{k=1}^{\infty}$ no longer need be w-metrizable, and w- and d-closures of subsets of $\{f_k(t)\}_{k=1}^{\infty}$ no longer have to coincide. It follows from [6, III.14] that the graph of the multifunction $L\colon T\to 2^X$, defined above, is $\mathscr{T}\times\mathscr{B}(X)$-measurable, but now L may have empty values. By Lyapunov's theorem the set Q', defined by

$$Q':=s\text{-cl}\left(\int_T d\text{-}Ls_k\,\{f_k\}\,d\mu-C\right),$$

is convex (again this follows by [14, Thm. 3.1], since for $L'(t):=L(t)-C/\mu(T)$, $\int_T L'\,d\mu=\int_T L\,d\mu-C$, by convexity and closedness of C). Also, Q' is nonempty, thanks to the provision made in Theorem 2.4. Observe that (2.10), (2.11) are implied by the statement $\int_T f_*\,d\mu\in Q'$, in view of (2.9). To prove the latter it is enough, by the Hahn–Banach theorem, to show that for every $x^*\in X^*$

$$\inf\left\{\int_T\langle f,x^*\rangle\,d\mu-\langle c,x^*\rangle\colon f\in\mathscr{L}_1(T;X),\,f(t)\in L(t)\ \text{a.e.},\ c\in C\right\}$$
$$\leqslant\left\langle\int_T f_*\,d\mu,x^*\right\rangle. \tag{3.13}$$

Because x^* belongs to be polar cone of C, it is easy to see that in the above infimum we may set $c=0$. As in the proof of Theorem 2.1, we may apply [3e, Thm. B.2] (the nonemptiness provision is also used here). This gives

that the left side of (3.13) equals $\int_T \inf_{x \in L(t)} \langle x, x^* \rangle \, \mu(dt)$, so (3.6) follows immediately from (2.4).

We shall now drop the additional Hypothesis (H), and proceed just as in the proof of Theorem 2.1, splitting T into T^{na} and T^{pa}. This time it follows from (2.7) that for every $i \in \mathbb{N}$

$$\mu(A_i) \sup_{k \in \mathbb{N}} \|f_k^i\| \leqslant \sup_{k \in \mathbb{N}} \int_T \|f_k\| \, d\mu.$$

Here f_k^i again denotes the constant value that f_k has a.e. on the atom A_i, i, $k \in \mathbb{N}$. By reflexiveness of X it follows then from an obvious diagonal extraction argument that there exists a subsequence $\{k'\}$ of $\{k\}$ and $f_* \in \mathscr{L}_1(T^{\mathrm{pa}}; X)$ such that

$$\{f_{k'}(t)\}_{k'=1}^{\infty} \ w\text{-converges to } f_*(t) \text{ a.e. in } T^{\mathrm{pa}}. \tag{3.14}$$

By (2.7) and reflexiveness of X a (sub-)subsequence $\{k''\}$ of $\{k'\}$ exists such that

$$a'' := \underset{k'' \to \infty}{w\text{-lim}} \int_{T^{\mathrm{pa}}} f_{k''} \, d\mu \text{ exists.} \tag{3.15}$$

By Fatou's lemma it follows from (3.14), (3.15), and (2.8) that for every $x^* \in C^0$

$$\left\langle \int_{T^{\mathrm{pa}}} f_* \, d\mu, x^* \right\rangle \leqslant \liminf_{k'' \to \infty} \left\langle \int_{T^{\mathrm{pa}}} f_{k''} \, d\mu, x^* \right\rangle = \langle a'', x^* \rangle,$$

which implies, by the bipolar theorem [8, Thm. 22.7], that

$$\int_{T^{\mathrm{pa}}} f_* \, d\mu - a'' \in C.$$

Combining (3.15) and (2.2), we see that the previous step of the proof applies to the restrictions of $f_{k''}$, $k'' \in \mathbb{N}$, to T^{na}. Then concatenation of f_* on T^{pa} with f_ε on T^{na}, just as in the proof of Theorem 2.1, yields the desired result. ∎

Remark 3.2. In the proofs of Theorems 2.1, 2.4 a function f_* was used in parts b and c: in part c this was only a function defined on T^{pa}. It follows from Corollary A.4 that there is no ambiguity: the f_* of part b coincides a.e. on T^{pa} with the f_* of part c, since on T^{pa} the functions $f_{k'}$, $k' \in \mathbb{N}$, converge pointwise a.e.

Remark 3.3. As was the case for Theorem 2.1, the present proof of Theorem 2.4 improves on the proof given in [3f] by dispensing with some involved extreme point arguments.

Appendix

Here we recapitulate some pertinent facts concerning the weak convergence of transition probabilities; see [3b, c, d, f] and their references for details.

Let $(T, \mathscr{T}, \mu)$ be a finite measure space and S a topological space. The set of all *transition probabilities* from $(T, \mathscr{T})$ into $(S, \mathscr{B}(S))$ is denoted by $\mathscr{R}(T; S)$. Thus, $\mathscr{R}(T; S)$ is the set of all functions $\delta: T \times \mathscr{B}(S) \to [0, 1]$ such that (i) for every $t \in T$ $\delta(t, \cdot)$ is a probability on $(S, \mathscr{B}(S))$, and (ii) for every $B \in \mathscr{B}(S)$ the function $\delta(\cdot, B)$ is $\mathscr{T}$-measurable on T (more details can be found in [13]). A *normal integrand* on $T \times S$ is a function $g: T \times S \to (-\infty, +\infty]$ such that (i) $g(t, \cdot)$ is lower semicontinuous on S for every $t \in T$, and (ii) g is $\mathscr{T} \times \mathscr{B}(S)$-measurable on $T \times S$. The set of all normal integrands on $T \times S$ for which there exists $\varphi \in \mathscr{L}_1(T; \mathbb{R})$ such that $g(t, s) \geqslant \varphi(t)$ for all $t \in T$, $s \in S$, is denoted by $\mathscr{G}_{bb}(T; S)$. The *weak topology* on $\mathscr{R}(T; S)$ is defined as the coarsest topology such that all functionals $I_g: \mathscr{R}(T; S) \to (-\infty, +\infty]$, $g \in \mathscr{G}_{bb}(T; S)$, defined by

$$I_g(\delta) := \int_T \left[\int_S g(t, s)\, \delta(t, ds) \right] \mu(dt),$$

are lower semicontinuous. Let $\mathscr{H}(T; S)$ be the set of all $h \in \mathscr{G}_{bb}(T; S)$ such that $h(t, \cdot)$ is inf-compact on S for every $t \in T$. A subset $\mathscr{R}_0$ of $\mathscr{R}(T; S)$ is defined to be *tight* [3a, b] if there exists $h \in \mathscr{H}(T; S)$ such that

$$\sup_{\delta \in \mathscr{R}_0} I_h(\delta) < +\infty.$$

Equivalently [11], the set $\mathscr{R}_0$ is tight if and only if for every $\varepsilon > 0$ there exists a multifunction $\Gamma_\varepsilon: T \to 2^S$, having a measurable graph and compact values, such that

$$\sup_{\delta \in \mathscr{R}_0} \int_T \delta(t, S \setminus \Gamma_\varepsilon(t))\, \mu(dt) < \varepsilon.$$

In turn, this equivalent definition forms the starting point of [5]. From the latter form it is easy to see that tightness generalizes the classical notion formulated for probability measures [9]. The following result generalizes in an analogous fashion Prohorov's classical criterion for relative compactness [3b, d].

THEOREM A.1 a. *Suppose that S is metrizable Lusin. Then every tight subset of $\mathscr{R}(T; S)$ is relatively weakly compact and relatively weakly sequentially compact.* b. *Suppose that S is a Polish (metrizable, complete, and separable) space. Then every relatively weakly compact or relatively weakly sequentially compact subset of $\mathscr{R}(T; S)$ is tight.*

For every measurable function f from $(T, \mathscr{T})$ into $(S, \mathscr{B}(S))$ we define the *relaxation* $\varepsilon_f \in \mathscr{R}(T; S)$ of f by

$$\varepsilon_f(t, \cdot) := \text{Dirac probability measure at } f(t).$$

Specialized to relaxations, Theorem A.1 runs as follows [3b, Thm. I]:

COROLLARY A.2. *Suppose that S is a metrizable Lusin space. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions from $(T, \mathscr{T})$ into $(S, \mathscr{B}(S))$ such that the sequence $\{\varepsilon_{f_k}\}_{k=1}^{\infty}$ of their relaxations is tight. Then there exists a subsequence $\{\ell\}$ of $\{k\}$ and a transition probability $\delta_* \in \mathscr{R}(T; S)$ such that*

$$\{\varepsilon_{f_\ell}\}_{\ell=1}^{\infty} \text{ converges weakly in } \mathscr{R}(T; S) \text{ to } \delta_*.$$

Moreover,

$$\delta_*\left(t, \bigcap_{p=1}^{\infty} \operatorname{cl}\{f_k(t) : k \geqslant p\}\right) = 1 \quad \text{a.e. in } T.$$

In [3d] the above results were extended to certain nonmetrizable Lusin spaces S. A space S is *σ-metrizable Lusin* if it is the countable union of a nondecreasing sequence $\{S_j\}_{j=1}^{\infty}$ of subsets of S, all of which are metrizable Lusin for the relative topology. A subset $\mathscr{R}_0$ of $\mathscr{R}(T; S)$ is defined to be *σ-tight* with respect to $\{S_j\}_{j=1}^{\infty}$ if for every $j \in \mathbb{N}$ there exists $h_j \in \mathscr{H}(T; S_j)$ such that

$$\sup_{\delta \in \mathscr{R}_0} \int_T \left[\int_{S_j} h_j(t, s)\, \delta(t, ds)\right] \mu(dt) < +\infty,$$

and if

$$\lim_{j \to \infty} \sup_{\delta \in \mathscr{R}_0} (\mu \otimes \delta)(T \times (S \setminus S_j)) = 0.$$

Here $\mu \otimes \delta$ denotes the usual product measure induced by the measure μ and the transition probability δ. Observe that the last condition guarantees that these product measures $\mu \otimes \delta$, $\delta \in \mathscr{R}_0$, live "almost uniformly" on the product of T and a metrizable Lusin space. Also, the first condition guarantees that these same product measures live "almost uniformly" on a

measurable subset of $T \times S$ whose t-sections are compact (in view of the equivalent definition of tightness). Our second analogue of Prohorov's theorem can now be stated [3d, Thm. 2.1].

THEOREM A.3. *Suppose that S is σ-metrizable Lusin. Then every σ-tight subset of $\mathscr{R}(T; S)$ is relatively weakly compact and relatively weakly sequentially compact.*

Specialized to a sequence of relaxations this gives the following [3d, Thm 2.2].

COROLLARY A.4. *Suppose that S is a σ-metrizable Lusin space. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions from $(T, \mathscr{T})$ into $(S, \mathscr{B}(S))$ such that the sequence $\{\varepsilon_{f_k}\}_{k=1}^{\infty}$ of their relaxations is σ-tight. Then there exist a subsequence $\{\ell\}$ of $\{k\}$ and a transition probability $\delta_* \in \mathscr{R}(T; S)$ such that*

$$\{\varepsilon_{f_\ell}\}_{\ell=1}^{\infty} \text{ converges weakly in } \mathscr{R}(T; S) \text{ to } \delta_*.$$

Moreover,

$$\delta_*\left(t, \bigcap_{p=1}^{\infty} \operatorname{cl}\{f_k(t) : k \geqslant p\}\right) = 1 \quad \text{a.e. in } T.$$

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Note added in proof. Recently the author has managed to overcome the metrizability difficulties concerning the extension of Prohorov's theorem. In particular, he has been able to prove that Theorem A.1a of the Appendix continues to hold if S is merely a completely regular Suslin space. In the light of this extension Theorem A.3, Corollary A.4 and the associated notion of σ-tightness become superfluous.

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